A Semantic PSPACE Criterion for Coalgebraic Modal Logic

Lutz Schröder

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MiS
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Introduction

- Modal logics (ideally) combine
  - the right (taylored) level of expressivity
  - relative computational tractability
- Large zoo of non-normal modal logics
- Upper complexity bounds frequently non-trivial and ad-hoc
- Coalgebraic modal logic provides unifying framework
- Here: semantic PSPACE algorithm for rank-0-1 logics
- Applications:
  - modal logics of quantitative uncertainty
  - conditional logics

Complexity of Modal Logics

- ‘Next-step’ modal logics are typically $PSPACE$, e.g.
  - K (KB, $S4$, ...): witness algorithm for shallow Kripke models (Ladner 77, Halpern/Moses 92)
  - Graded modal logic (GML): constraint set algorithm (Tobies 01)
  - Logic of knowledge and probability: shallow model method based on local small model property (Fagin/Halpern 94)
  - Epistemic logic (Vardi 89), coalition logic (Pauly 02): shallow neighbourhood models.
  - Conditional logic: sequent calculus (Olivetti/Schwindt 01)
  - Presburger and regular modal logic: check local models on the fly (Demri/Lugiez IJCAR 06).

Coalgebraic Modal Logic

Coalgebraic modal logic unifies all these different logics

- Prove meta-theoretic results in the general framework:
  - Hennessy-Milner [Pattinson NDJFL 04, Schröder FOSSACS 05]
  - Completeness [Pattinson TCS 03, Schröder FOSSACS 06]
  - Duality, ultrafilter extensions [Kupke/Kurz/Pattinson CALCO 05]
  - Finite model property [Schröder FOSSACS 06]
- Generic algorithms and upper complexity bounds:
  - Obtain $PSPACE$ bounds by uniform methods [Schröder/Pattinson LICS 06 and here]
  - clarity, reusability, extendability

Examples

Quantitative Uncertainty

(Fagin, Halpern, Megiddo, Pucella)

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \sum_{i=1}^{n} a_i w(\phi_i) \geq b \quad (a_i, b \in \mathbb{Q})$$

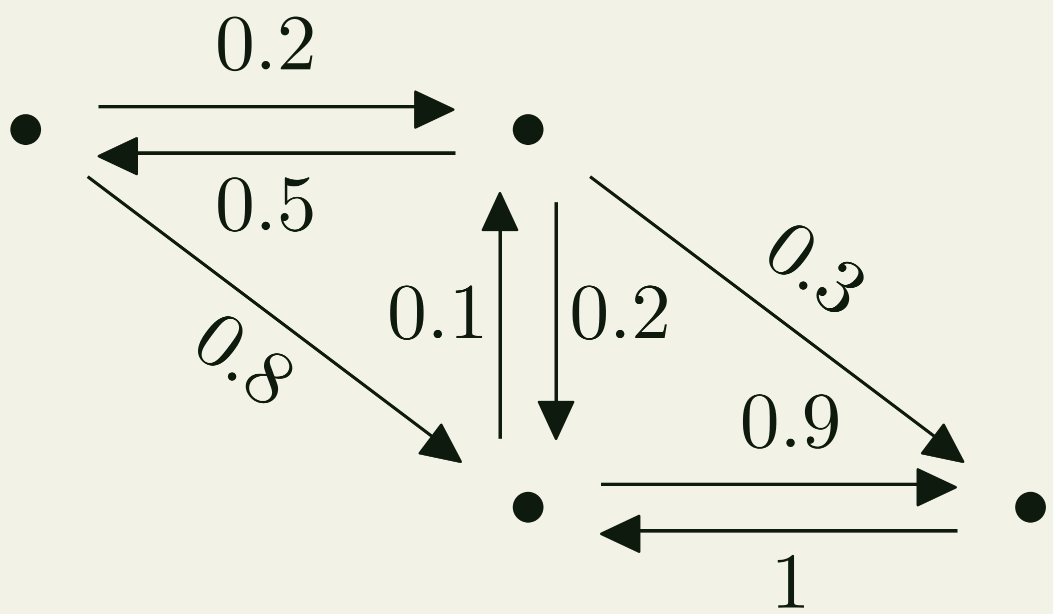
Weights $w(\phi) \in \mathbb{R}$: ‘likelihood’ of ϕ in the next step, e.g.

- Probability (FHM IC 1994)
- Upper probability (HP JAI 2002)
- Dempster-Shafer belief (HP UAI 2002)
- Dubois-Prade possibility (HP UAI 2002)

Extension: Expectations $e(\sum c_i\phi_i)$ (HP UAI 02)

Quantitative Uncertainty: Semantics

- Set X of states
- For $x \in X$, probability distribution (belief function, . . .) P_x on X describing uncertain transitions



- $w(\phi)$ is interpreted in state x as $P_x\{y \mid y \models \phi\}$

Conditional Logics

$$\phi ::= \bot \mid p \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \Rightarrow \phi_2$$

- $\phi \Rightarrow \psi$ reads 'ψ holds under condition ϕ'
- Basic axioms $\rightsquigarrow$ conditional logic CK:
  - Replacement of equivalents on the left
  - Commutation with $\wedge$ on the right
- Relevance logics: $a \Rightarrow a$, $(a \Rightarrow b) \to a \to b$
- Default logics, e.g. Burgess' System C (generalizing KLM)

(REF)	$a \Rightarrow a$
(OR)	$(a \Rightarrow c) \to (b \Rightarrow c) \to (a \vee b \Rightarrow c)$
(CM)	$(a \Rightarrow c) \to (a \Rightarrow b) \to ((a \wedge b) \Rightarrow c)$

Conditional Logic CK: Semantics

Conditional Frames:

- Set X of states
- For $x \in X$, function $f_x : \mathcal{P}(X) \to \mathcal{P}(X)$.
- $x \models \phi \Rightarrow \psi$ iff $f_x[\![\phi]\!] \subseteq [\![\psi]\!]$ (where $[\![\phi]\!] = \{y \mid y \models \phi\}$)

Coalgebra

$T : \mathbf{Set} \to \mathbf{Set}$

T-Coalgebra $(X, \xi) =$ map $\xi : X \to TX$

X: set of states

ξ: transition map

$\xi(x)$: structured collection of observations/successor states

Coalgebraic Modal Logic

[Pattinson NDJFL 04; Schröder FOSSACS 05]
Interpret n-ary modal operator L by predicate lifting

$$[\![L]\!]_X : \mathcal{Q}^n \to \mathcal{Q} \circ T^{op}$$

($\mathcal{Q}$ contravariant powerset).

Semantics of L in T-coalgebra (X, ξ):

$$x \models_{(X,\xi)} L\,(\phi_1, \dots, \phi_n) \iff \xi(x) \in [\![L]\!]_X([\![\phi_1]\!], \dots, [\![\phi_n]\!])$$

(where $[\![\phi]\!] = \{y \mid y \models \phi\}$)

Examples

- Quantitative Uncertainty:
  - TX = likelihood measures over X
  - operators $L(a_1, \dots, a_n; b)(\phi_1, \dots, \phi_n) \equiv \sum a_i w(\phi_i) \geq b$
  - $[\![L(a_1, \dots, a_n; b)]\!]_X(A_1, \dots, A_n) = \{P \in TX \mid \sum a_i P(A_i) \geq b\}$
- Conditional Logic CK:
  - $TX = \mathcal{Q}(X) \to \mathcal{P}(X)$
  - $[\![\_ \Rightarrow \_]\!]_X(A, B) = \{f \in TX \mid f(A) \subseteq B\}$.
- Coalition Logic, majority logic, . . .

The One-Step Logic

No nesting, no transitions

Formally: equivalent to one-step pairs (ϕ, ψ) over V, where

$$\phi \in \mathsf{Prop}(V)$$

$$\psi \text{ conjunctive clause over atoms } L(a_1, \dots, a_n),\ a_i \in V$$

For set X, $\mathcal{P}(X)$-valuation τ:

- $[\![\phi]\!]\tau \subseteq X$
- $X, \tau \models \phi \iff [\![\phi]\!]\tau = X$
- $[\![\psi]\!]\tau \subseteq TX$, with $[\![L(a_1, \dots, a_n)]\!] = [\![L]\!](\tau(a_1), \dots, \tau(a_n))$

The One-Step Polysize Model Property

One-step model (X, τ, t) of (ϕ, ψ):

- $X, \tau \models \phi$
- $t \in [\![\psi]\!]\tau \subseteq TX$

OSPMP: (ϕ, ψ) one-step satisfiable $\implies$
has one-step model of polynomial size in ψ

(Size refers to $|X|$ and $size(t)$)

PSPACE semantically

One-step model checking: $t \in [\![L]\!](A_1, \dots, A_n)$?

Theorem If $\mathcal{L}$ has the OSPMP, then $\mathcal{L}$ has the polynomially branching shallow model property

Corollary If $\mathcal{L}$ has the OSPMP and one-step model-checking is in P, then

- $\mathcal{L}$ is in $PSPACE$
- $\mathcal{L}_k$ (nesting of modal operators bounded by k) is in NP.

Example: Quantitative Uncertainty

- [FHM IC 1994; HP JAI 02; HP UAI 02] prove small model properties for one-step logics ($\to$ NP)
- From these proofs, extract OSPMP
- Obtain polynomially branching shallow models, PSPACE/NP bounds
  - Known for probability [FHM 94]
  - Novel for other cases (only one-step logics considered so far)

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Representation of $f : \mathcal{Q}(X) \to \mathcal{P}(X)$ by lists of maplets,
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- pick $z_i \in f(\tau(a_i)) \setminus \tau(b_i)$ if $\pm_i = \neg$;
- put $Y = \{y_{ij} \mid \dots\} \cup \{z_i \mid \dots\}$;
- put $\tau'(v) = \tau(v) \cap Y$, $f : \tau'(a_i) \mapsto f(\tau(a_i)) \cap Y$

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$\to$ have polysize one-step model (Y, τ', f'), DONE!

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(Olivetti et al. ACM TOCL: sequent calculus, 51 pages)

Rank-0-1 Logics

- Rank-0-1 axioms: clauses over atoms a or $L(a_1, \dots, a_n)$.
- Models: coalgebras satisfying the axioms for all valuations
- Examples:
  - KT: $\Box a \to a$, models: reflexive frames
  - Almost all conditional logics, e.g. CK+MP:

$$(a \Rightarrow b) \to a \to b,$$

models: $x \in A \implies x \in f_x(A)$.

One-Step Models in Rank-0-1

... are quadruples (X, τ, t, x_0), where $x_0 \in X$ present state

Require that (X, t, x_0) satisfies the axioms for all valuations:

- $(X, \kappa, t, x_0) \models L(a_1, \dots, a_n)$ defined as before
- $(X, \kappa, t, x_0) \models a$ iff $x \in \kappa(a)$

OSPMP:

- Require preservation of τ-theory of x_0 in small model.
- Represent pairs $(t, x) \in TX \times X$.

For PSPACE, need 'Is (X, τ, t, x_0) one-step model?'$\in P$.

Example: CK+MP

Everything as for CK, except

- Default $f(A) = A \cap \{x_0\}$
- retain x_0 in small one-step model

Conclusion

- Coalgebraic modal logic subsumes ‘all’ modal logics
- Generic algorithms reproduce tight PSPACE bounds and prove new ones
  - Compact, reusable, and clear proofs
- Bounding the nesting depth reduces complexity to NP
- Can now handle all axioms without nested modalities

Future Work

- Coalgebraic CTL
  - Coalgebra automata (Kupke/Venema)? :-(
  - Pseudomodels (Emerson/Halpern) :-|
- How do we tackle rank n?
- Optimized automatic reasoners?

Coalition Logic

(Pauly 2002)

$$TX = \exists \overbrace{\Sigma_1 \dots \Sigma_n}^{\text{sets of strategies}} . \underbrace{\prod \Sigma_i \to X}_{\text{outcome function}},$$

where $N = \{1, \dots, n\}$ set of agents.

For coalition $C \subset N$, $[C]\phi =$ 'C can force ϕ'.

$$[\![[C]]\!]_X A = \{f : \prod \Sigma_i \to X \mid \exists \sigma_C. \forall \sigma_{N-C}. f(\sigma_C, \sigma_{N-C}) \in A\}.$$