SOS, Modal Logic and Compositionality

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Process equivalences and logic

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$$\varnothing = \neg \left(\bigvee_{a \in A} \langle a \rangle \top \right)$$

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Hopefully:

- rules define an LTS in some way,
- our favourite equivalence is a congruence.

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$$\mathtt{a}.(\mathtt{b} + \mathtt{c}) \otimes \mathtt{a}.\mathtt{b} \not\approx_{\mathsf{Ctr}} (\mathtt{a}.\mathtt{b}) + (\mathtt{a}.\mathtt{c}) \otimes \mathtt{a}.\mathtt{b}$$

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$$\mathsf{a} + \mathsf{a}.\mathsf{b} \approx_{\mathsf{Tr}} \mathsf{a}.\mathsf{b} \qquad (\mathsf{a} + \mathsf{a}.\mathsf{b});\mathsf{c} \not\approx_{\mathsf{Tr}} (\mathsf{a}.\mathsf{b});\mathsf{c}$$

GSOS

$$\frac{\{\mathtt{x}_{i_j} \xrightarrow{a_j} \mathtt{y}_j\}_{1\leq j\leq m} \quad \{\mathtt{x}_{i_k} \stackrel{b_k}{\not\longrightarrow}\}_{1\leq k\leq l}}{\mathtt{f}(\mathtt{x}_1,\ldots,\mathtt{x}_n) \xrightarrow{c} \mathtt{t}}$$

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- LTS obtained by induction on the structure of transition sources

- Bisimilarity is a congruence

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How to prove compositionality?

Modal logic decomposition

For each formula ϕ and operator $\texttt{f} \in \Sigma$, $ar(\texttt{f}) = n$, find a tuple $\langle \phi_1, \ldots, \phi_n \rangle$ such that

$$\texttt{f}(x_1, \ldots, x_n) \models \phi \iff x_1 \models \phi_1 \wedge \cdots \wedge x_n \models \phi_n$$

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Decomposition, actually:

For each formula ϕ and operator $\mathtt{f} \in \Sigma$, $ar(\mathtt{f}) = n$, find a family of tuples $(\langle \phi_{i1}, \ldots, \phi_{in} \rangle)_{i \in I}$ such that

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Example:

$$\frac{x \xrightarrow{a} x'}{x || y \xrightarrow{a} x' || y} \qquad \frac{y \xrightarrow{a} y'}{x || y \xrightarrow{a} x || y'}$$

$$x || y \models \langle a \rangle \top \iff (x \models \langle a \rangle \top) \vee (y \models \langle a \rangle \top)$$

$$x || y \models \langle a \rangle \langle b \rangle \top \iff (x \models \langle a \rangle \langle b \rangle \top) \vee \cdots \vee (x \models \langle a \rangle \top \wedge y \models \langle b \rangle \top) \vee \cdots$$

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$$x||y \models \varnothing \iff x \models \varnothing \wedge y \models \varnothing$$

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$$x \otimes y \models \langle a \rangle \varnothing \iff$$

$$\bigvee_{B \cup C = A} \Big((x \models \langle a \rangle \bigwedge_{b \in B} \neg \langle b \rangle \top) \wedge (y \models \langle a \rangle \bigwedge_{c \in C} \neg \langle C \rangle \top)$$

Example

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$$x \otimes y \models \langle a \rangle \varnothing \iff$$ **???**

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$$x \otimes y \models \langle a \rangle \varnothing \iff \text{???}$$

NB. $(a.b + a.c) \otimes a.c \models \langle a \rangle \varnothing$

but $a.(b + c) \otimes a.c \not\models \langle a \rangle \varnothing$

Example 2

$$\frac{x \xrightarrow{a} x'}{x;y \xrightarrow{a} x';y} \qquad \frac{x \not\longrightarrow \quad y \xrightarrow{a} y'}{x;y \xrightarrow{a} y'}$$

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Looking for decomposition

Induction on formulas:

For each logical operator β (of arity n)
and formula variables ψ_i
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define Φ for $\beta(\phi_1, \ldots, \phi_n)$

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LTSs are B-coalgebras for $BX = (\mathcal{P}_\omega X)^A$

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$$\langle a \rangle(-_1 \wedge \cdots \wedge -_n) : B2^n \to 2$$

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etc.

Construction of liftings

Variable renaming:

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Composition:

$$\frac{\beta : B2^n \to 2 \qquad \left(\sigma_i : \Sigma 2^{m_i} \to 2\right)_{i=1,\ldots,n}}{\beta(\sigma_1, \ldots, \sigma_n) : B\Sigma 2^m \to 2}$$

$$m = \sum_{i=1}^m m_i$$

Example

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λ **validates** $\langle a \rangle [\otimes] = [\otimes] \langle a \rangle$

Valid equations

$$\beta(\sigma_1(x_{11}, \ldots, x_{1n_1}), \ldots, \sigma_m(x_{m1}, \ldots, x_{mn_m}))$$
$$=$$
$$\sigma(\beta_1(y_{11}, \ldots, y_{1k_1}), \ldots, \beta_l(y_{l1}, \ldots, y_{lk_l}))$$

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The equation is valid

if the two $B\Sigma$-liftings are equal.

Compositionality

Thm. For a family $(\beta_i)_{i \in I}$ of B-liftings,

- find a family $(\sigma_j)_{j \in J}$ of Σ-liftings,
- for every possible LHS:

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s.t. the equation is valid wrt λ.

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s.t. the equation is valid wrt λ.

Then the logical equivalence defined by $(\beta_i)_{i \in I}$ is a congruence on the coalgebra induced by λ.

Example

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$$\phi ::= \top \mid \langle a \rangle \phi$$

B-liftings:

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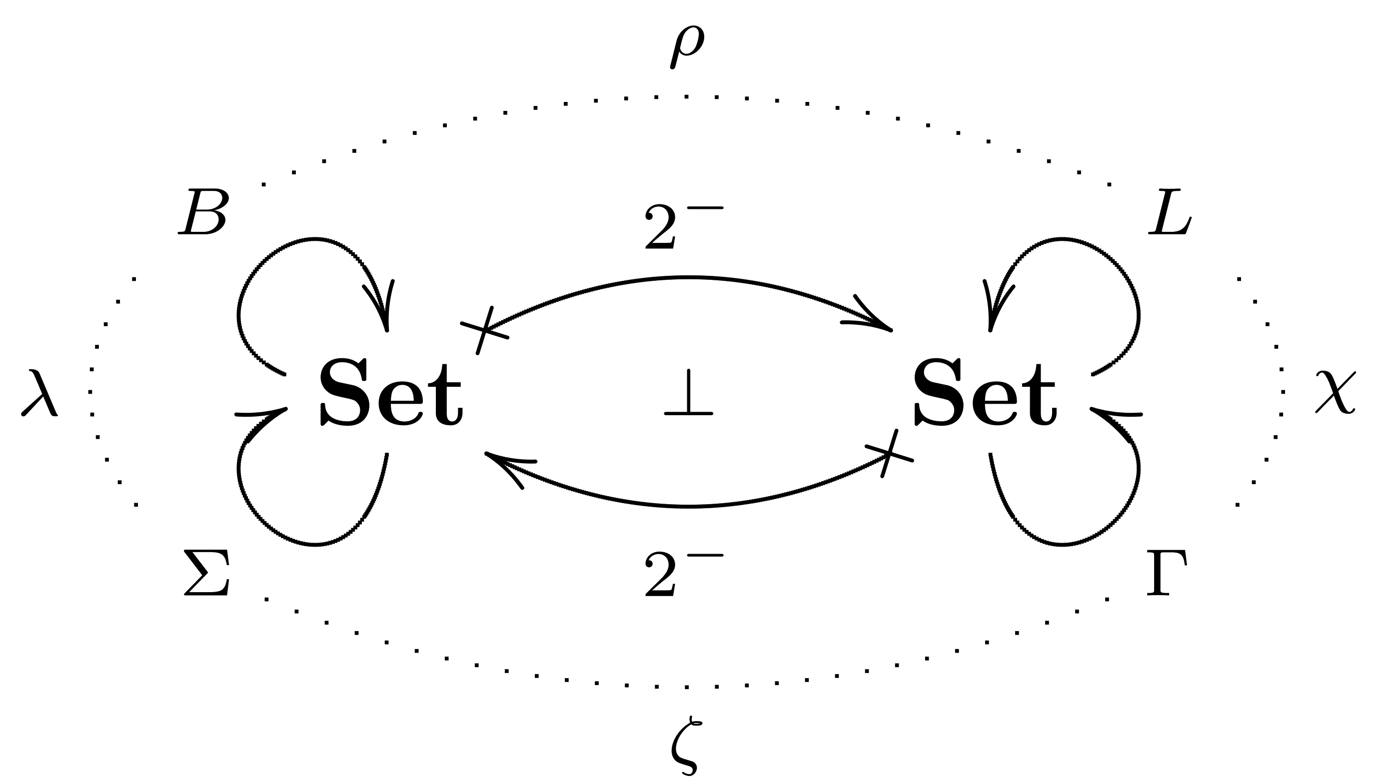
Abstract view

1. Modal logic lifts B to a slice category
2. If λ lifts as well, the logical equiv. is a congruence



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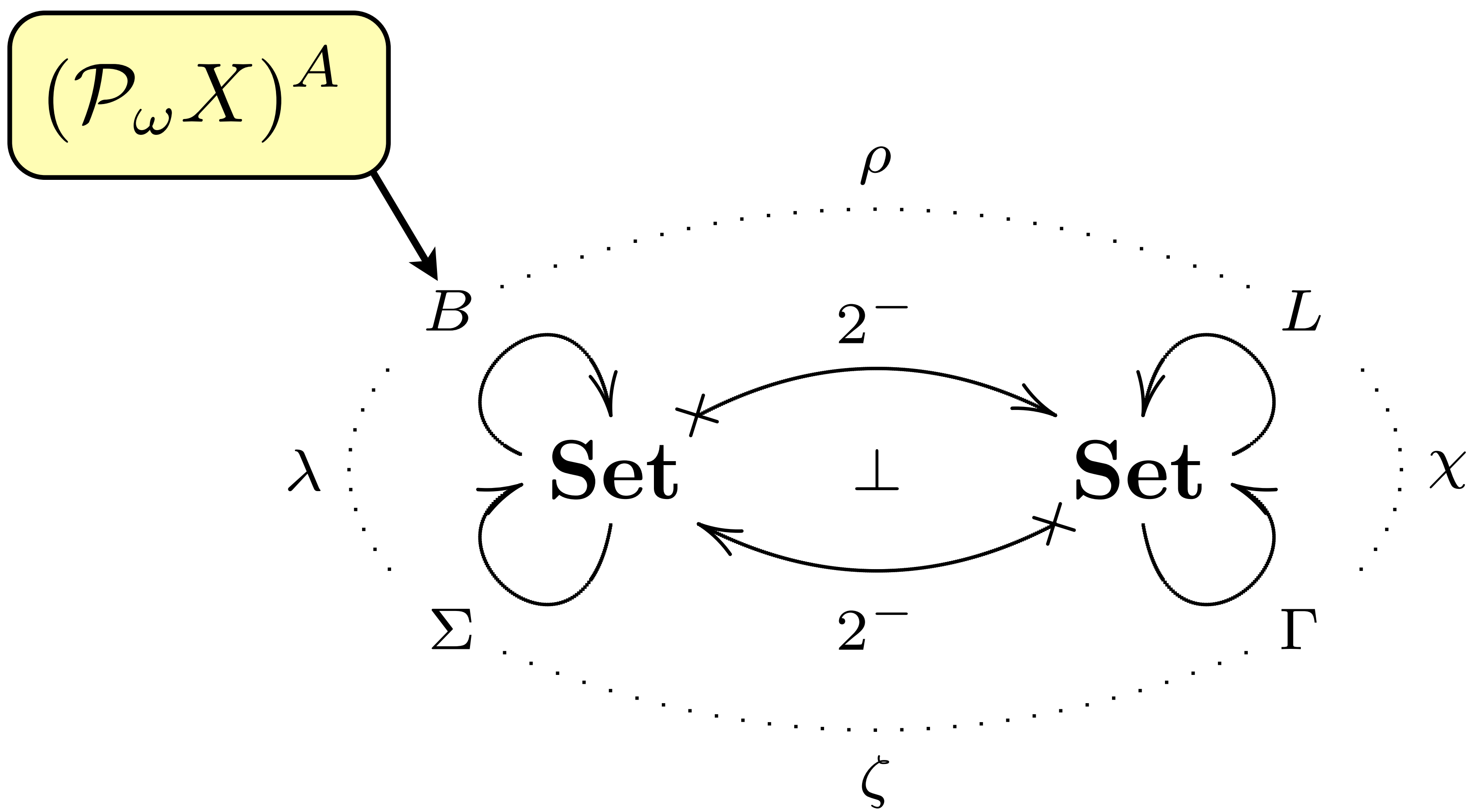

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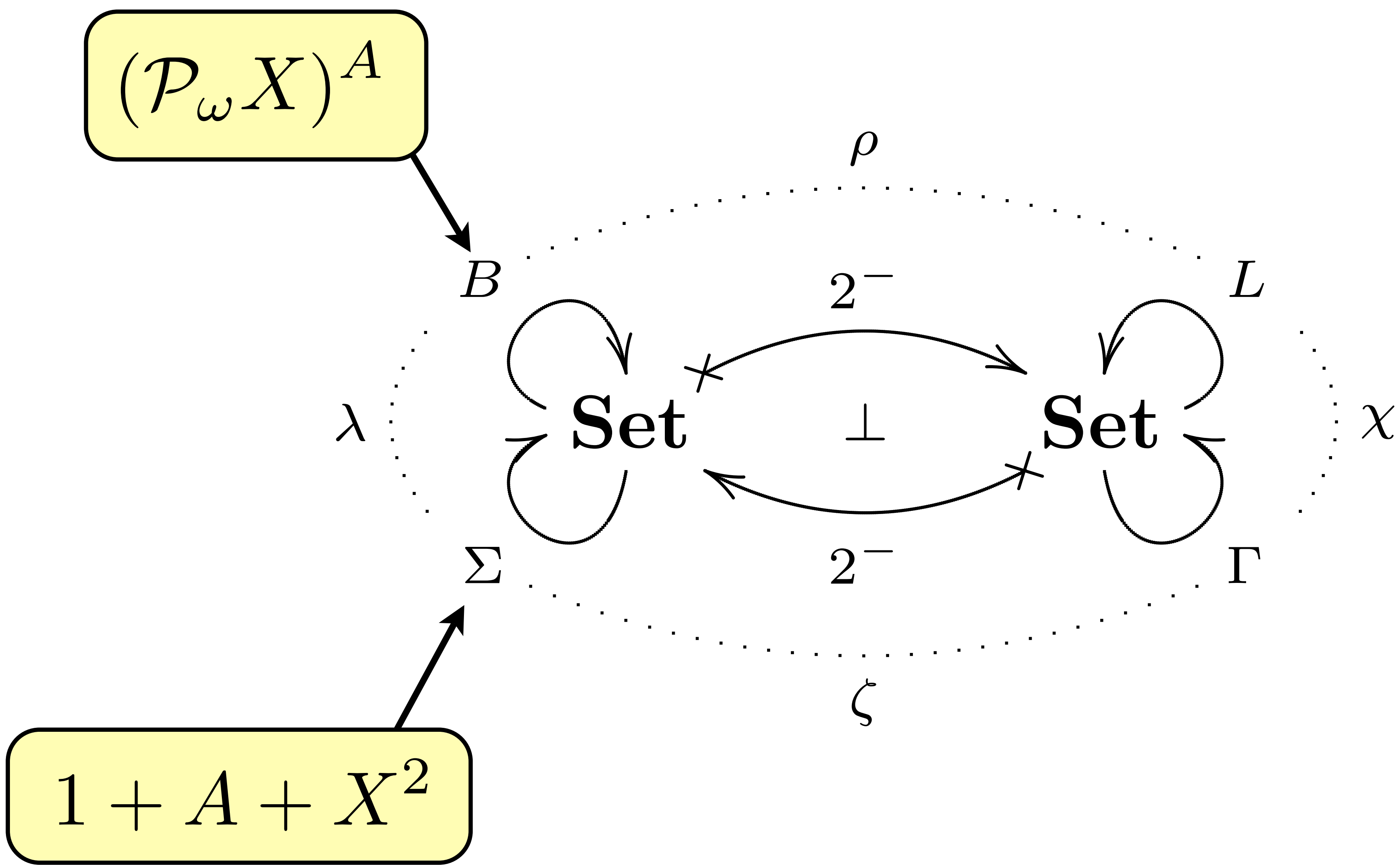

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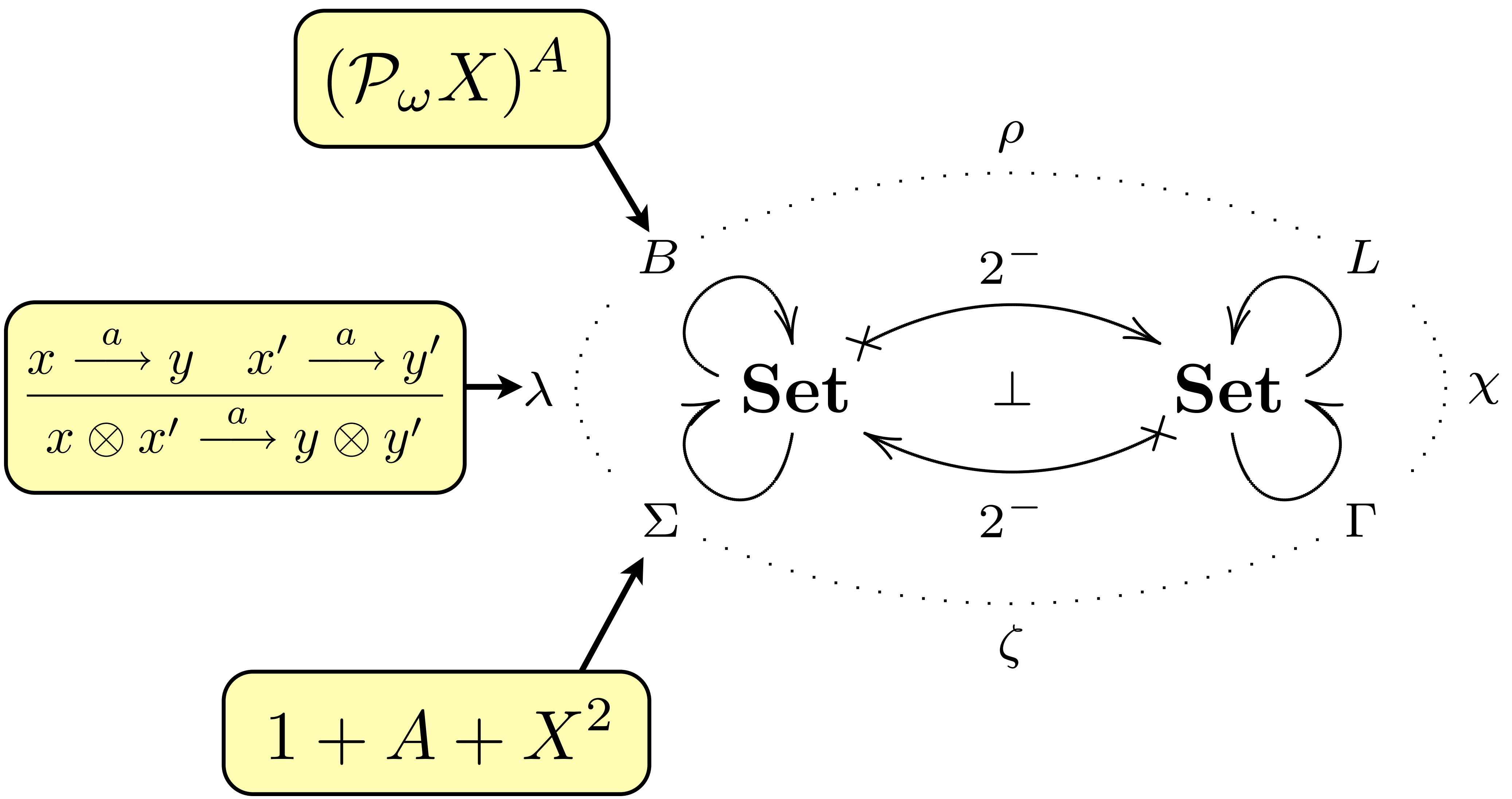

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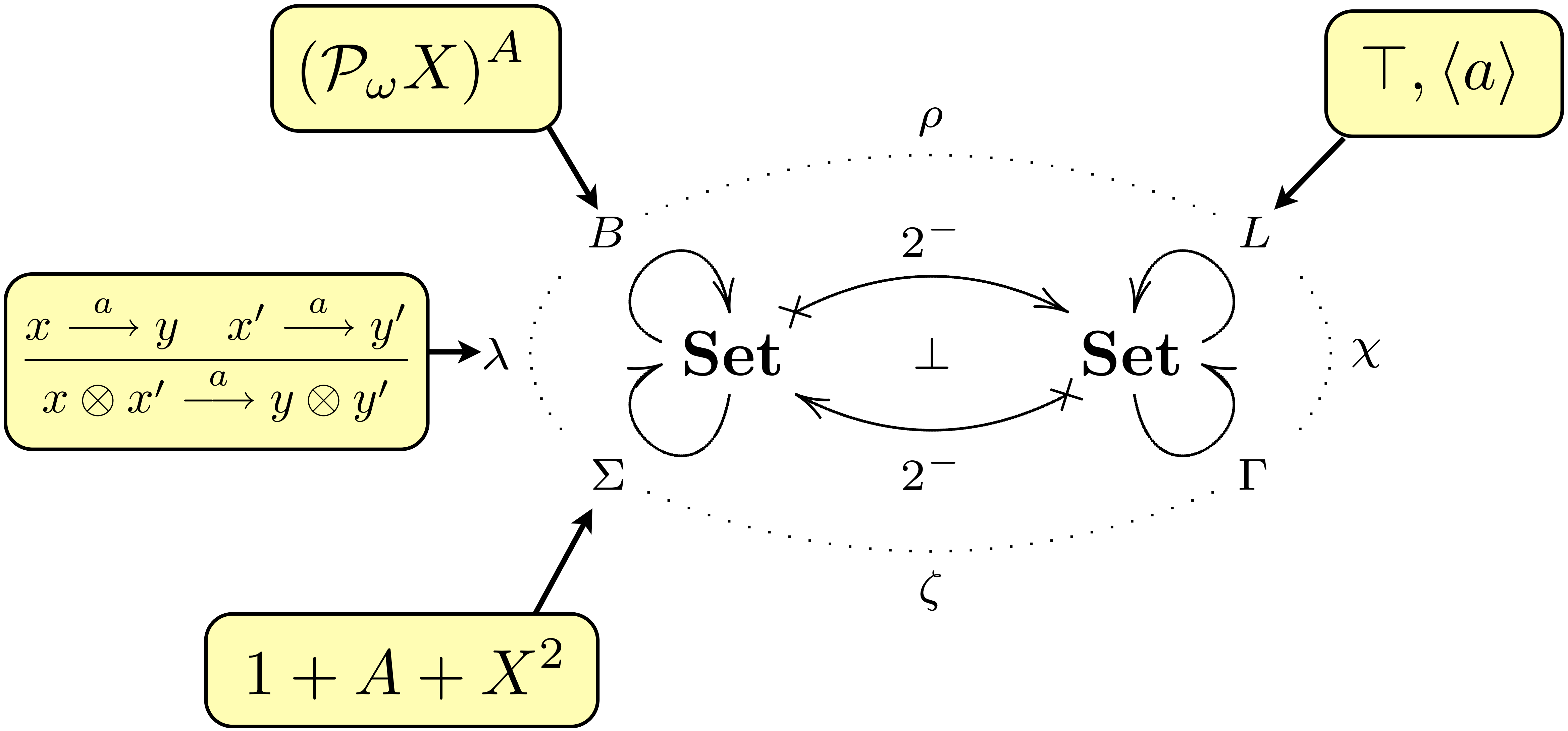

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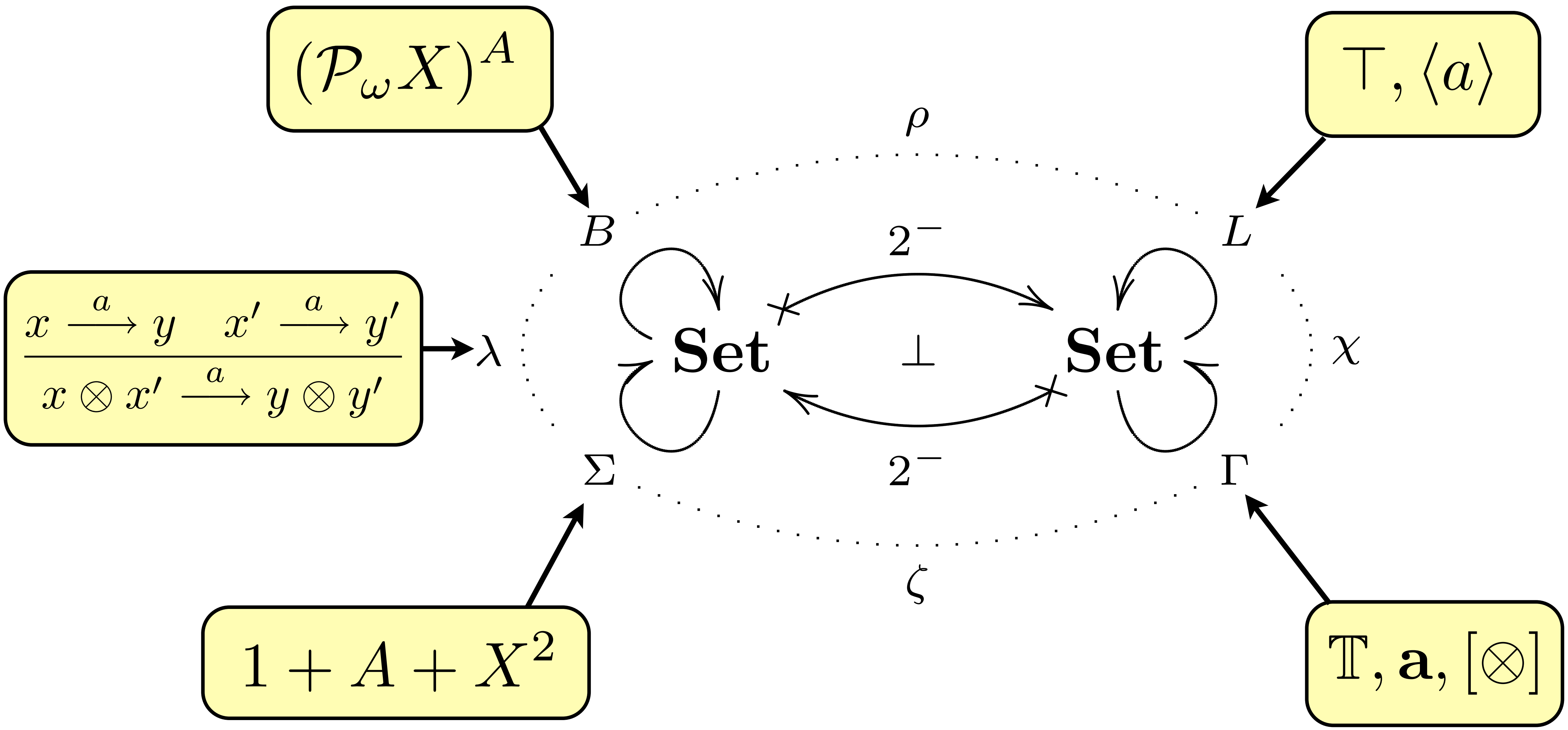

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