A double-pushout approach for modeling pointer redirection

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IFIP WG1.3 meeting, Sierra Nevada, January 18., 2008

Outline

The problem

Data-structure rewriting,
including cyclic data-structures with pointers
such as circular lists, doubly-linked lists, etc.

Required:

- Local pointer redirection,
  for redirecting some specific pointers.
- Global pointer redirection,
  for redirecting all pointers with some specific target
  to another target.

The method

Define a graph rewriting system
for dealing with this pointer redirection issue:

nodes	$\leftrightarrow$	cells
edges	$\leftrightarrow$	pointers

Set-theoretic approach: untractable.

Categorical approach:

YES the DPO (double pushout) method works,
BUT with non-classical assumptions on rules and matchings.

The DPO approach to graph rewriting

In a relevant category of graphs:

- A rewrite rule is a span (ℓ, r):

$$L \xleftarrow{\ell} K \xrightarrow{r} R$$

The DPO approach to graph rewriting

In a relevant category of graphs:

- A rewrite rule is a span (ℓ, r):

$$L \xleftarrow{\ell} K \xrightarrow{r} R$$

- A rewrite step is a double pushout (DPO):

$$\begin{array}{ccccc} L & \xleftarrow{\ell} & K & \xrightarrow{r} & R \\ \downarrow m & & \downarrow d & & \downarrow m' \\ G & \xleftarrow{\ell'} & D & \xrightarrow{r'} & H \end{array}$$

which rewrites G as H,
according to the rule (ℓ, r)
applied to the matching $m : L \to G$.

The classical DPO approach

- Intuitively: K is the intersection of L and R, ie., K is the subgraph common to both handsides.
- Categorically: the morphism $L \stackrel{\ell}{\leftarrow} K$ is a monomorphism.
- Consequence: nice confluence theorems.

A. Corradini et al.
Algebraic approaches to graph transformation
Part I: Basic concepts and double pushout approach.
In Handbook of Graph Grammars, p.163-246 (1997).

Our DPO approach

- Intuitively: (local redirection)
  some vertices of L are disconnected in K,
  then they get reconnected (differently) in R.
- Categorically: the morphism $L \stackrel{\ell}{\leftarrow} K$ is an epimorphism.
- Consequence: NO kind of confluence theorem.
  The intended applications have NO confluence property.

D. Duval, R. Echahed, F. Prost.
Modeling pointer redirection as cyclic term-graph rewriting.
In Proceedings of TERMGRAPH'06. ENTCS 176, p.65-84 (2007).

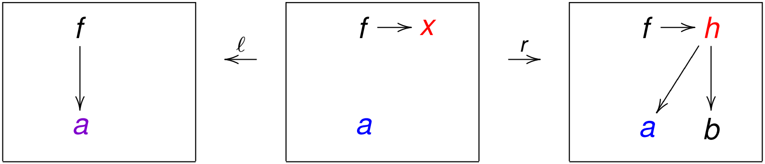
Some related papers

On cyclic term graph rewriting –
some papers use categories, others do not.

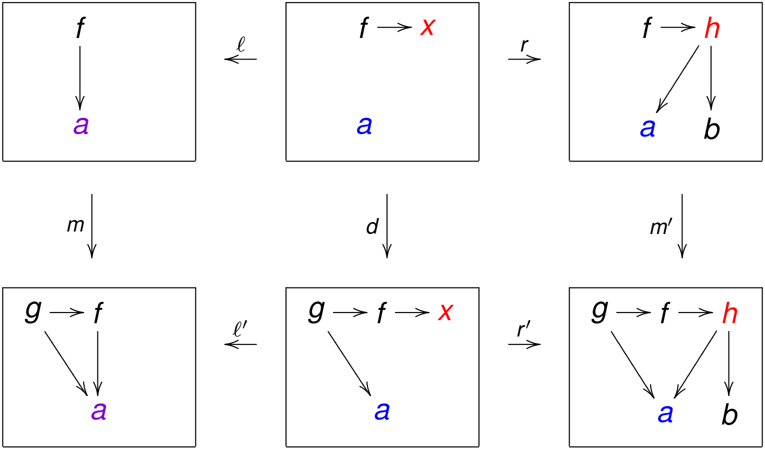
- *H. Barendregt et al.* (1987)
- *J.R. Kennaway et al.* (1994)
- *R. Banach* (1994)
- *R. Echahed, J.-C. Janodet* (1998)
- *E. Barendsen, S. Smetsers* (1999)
- *A. Corradini, F. Gadducci* (1999)
- *S. Antoy et al.* (2000)
- *C. Bertolissi et al.* (2005)
- ......

Outline

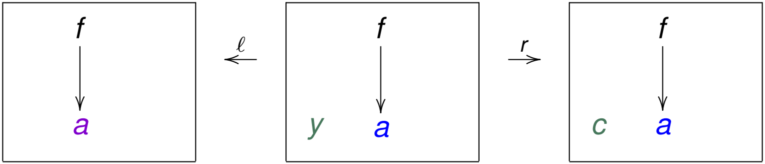
Local redirection



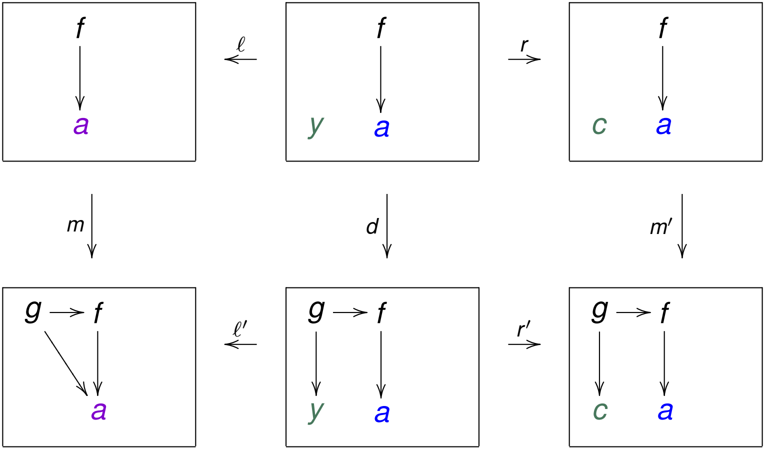
Local redirection



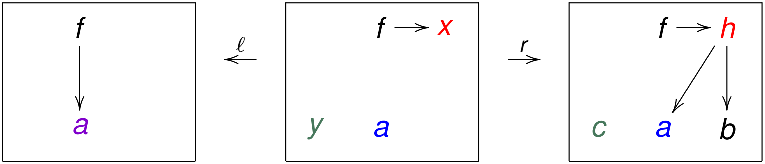
Global redirection



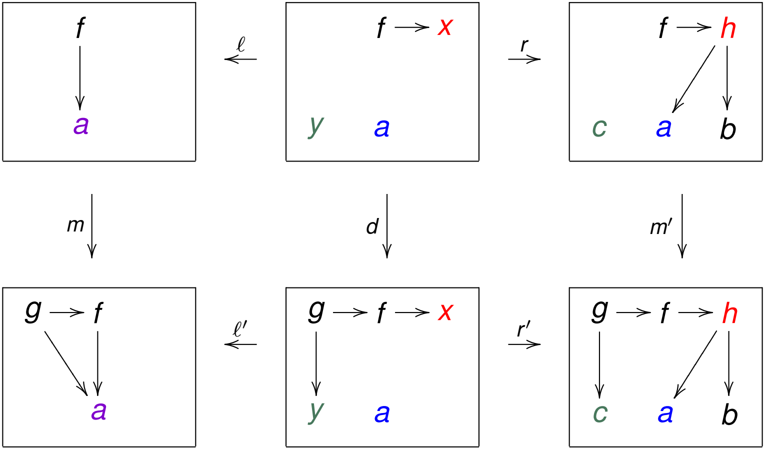
Global redirection



Local and global redirection



Local and global redirection



Outline

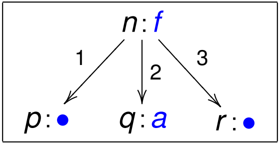
The graphs

A signature Ω is fixed, ie., a set of operations with arity.

A graph G is made of:

- a set of nodes N,
- a subset of labeled nodes $N^\Omega \subseteq N$,
- a labeling function : $N^\Omega \to \Omega$,
  written $n\!:\!f$,
  otherwise $n\!:\!\bullet$,
- a successor function : $N^\Omega \to N^*$,
  written $(\, s(n,1), \ldots, s(n,k) \,)$,
  where $n\!:\!f$ and k is the arity of f.

Example, with a ternary operation f and a constant a:

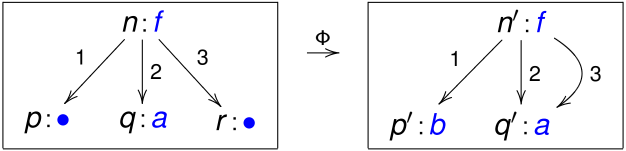


The category of graphs

A graph morphism $\Phi : G \to H$ is:

- a map $\Phi : N_G \to N_H$
- which preserves labeled nodes, labels and successors.

Example:



Hence, the category of graphs $\mathbf{Gr}$.

The node functor is faithful

The node functor $N : \mathrm{Gr} \to \mathrm{Set}$ maps:

- each graph G to its set of nodes,
- each graph morphism $\Phi : G \to H$ to its underlying map.

Proposition.

The node functor $N : \mathrm{Gr} \to \mathrm{Set}$ is faithful.

About proof.
A graph morphism Φ is entirely defined by the map $N(\Phi)$, because the successors of each node are totally ordered.

Pushouts of graphs

In our category of graphs:

- pushouts do not always exist:

$$
\begin{array}{ccc}
\boxed{n : \bullet} & \longrightarrow & \boxed{n : a} \\
\downarrow & & \downarrow \\
\boxed{n : b} & \longrightarrow & \boxed{\textbf{?}}
\end{array}
$$

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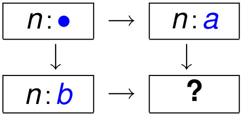
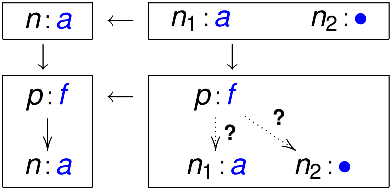
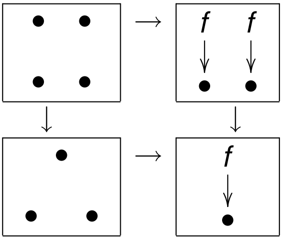


- complement pushouts, when they do exist, are not unique:

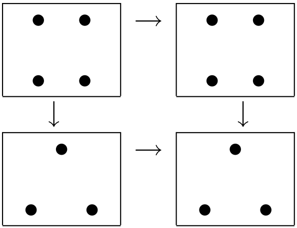


The node functor does not preserve PO's

a PO in Gr:



a PO in Set:



The PO theorem

Our framework relies on the following result.

PO theorem. *Let Γ be a commutative square of graphs:*

$$\begin{array}{ccc} & G_0 & \\ \swarrow & & \searrow \\ G_1 & & G_2 \\ \searrow & & \swarrow \\ & G_3 & \end{array}$$

such that:

- $N(\Gamma)$ *is a pushout of sets,*
- *and each labeled node in* G_3 *is the image of a labeled node in* G_1 *or in* G_2*,*

then Γ *is a pushout of graphs.*

The PO theorem (2)

About proof.

Let Γ' be any commutative square of graphs on the same base as Γ:

$$\begin{array}{ccccc} & & G_0 & & \\ & \swarrow & & \searrow & \\ G_1 & & & & G_2 \\ & \searrow & & \swarrow & \\ & & G_3' & & \end{array}$$

- Since $N(\Gamma)$ is a pushout of sets, there is a unique map $f : N(G_3) \to N(G_3')$ such that $f \circ \cdots = \cdots$ in Set.
- Since each labeled node in G_3 is the image of a labeled node in G_1 or in G_2, $f = N(\Phi)$ for a graph morphism $\Phi : G_3 \to G_3'$.
- Since the functor $N : \mathrm{Gr} \to \mathrm{Set}$ is faithful, we have $\Phi \circ \cdots = \cdots$ and Φ is unique.

Outline

Disconnection of a graph

In a graph L, let:

- E be a set of edges, to be locally disconnected,
- V a set of nodes, to be globally disconnected.

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The disconnection of L with respect to E and V is $\ell : K \to L$:

- K has all the nodes of L, with the same label, and it has additional unlabeled nodes:
  - $n[i]$ for each edge $(n \to s_L(n,i)) \in E$, (with "actual local redirections")
  - $n[0]$ for each node $n \in V$.

  The successors of a node are the same in K and L, except: $s_K(n,i) = n[i]$ for each edge $(n \to s_L(n,i)) \in E$.

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- $\ell : K \to L$ is such that $\ell(n) = n$ for all $n \in N_L$, and:
  - $\ell(n[i]) = s_L(n,i)$ for each edge $(n \to s_L(n,i)) \in E$,
  - $\ell(n[0]) = n$ for each node $n \in V$.

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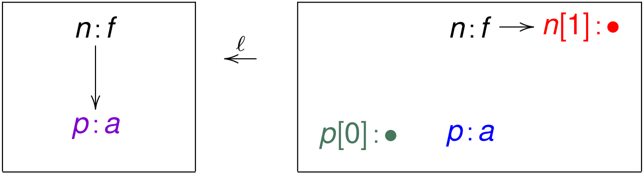
Then, the graph morphism $\ell : K \to L$ is an epimorphism.

A disconnection of graph

Here:

- the locally disconnected edge is $n \to p$,
- the globally disconnected node is p.



Matching

Let us consider the disconnection of L with respect to
a set of locally redirected edges E
and a set of globally redirected nodes V:

$$L \xleftarrow{\ell} K$$

A matching w.r.t. l is a graph morphism:

$$L \xrightarrow{m} G$$

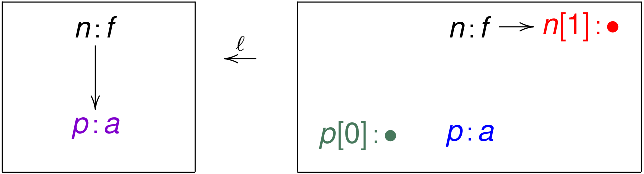
such that m is injective on the union $N_L^\Omega \cup V$
of the labeled nodes and the globally redirected nodes.

Disconnection of a matching

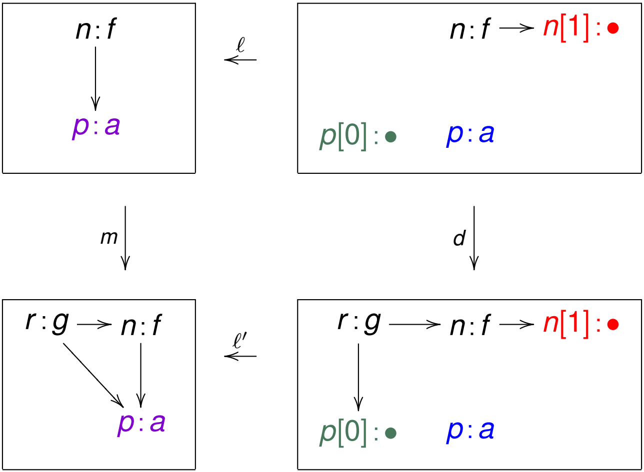
Let $\ell : K \to L$ be the disconnection of L with respect to E and V, and $m : L \to G$ a matching w.r.t. ℓ.
Let $\ell' : D \to G$ be defined in a way similar to $\ell : K \to L$, moreover with "actual global redirections", from:

- the set of locally disconnected edges $m(E)$,
- the set of globally disconnected nodes $m(V)$.

A disconnection of matching



A disconnection of matching



The complement PO, on the left

“Left” theorem.

The disconnection of a matching provides a PO of graphs:

$$\begin{array}{ccc} L & \xleftarrow{\ \ell\ } & K \\ {\scriptstyle m}\downarrow & \lrcorner & \downarrow{\scriptstyle d} \\ G & \xleftarrow{\ \ell'\ } & D \end{array}$$

About proof.

Apply the PO theorem.

Outline

Rewrite rule

A rewrite rule is a span of graphs (ℓ, r):

$$L \xleftarrow{\ell} K \xrightarrow{r} R$$

where $\ell : K \to L$ is a disconnection of L w.r.t. some E and V,

and where $r : K \to R$:

- preserves unlabeled nodes
- and is injective on unlabeled nodes.

The PO, on the right

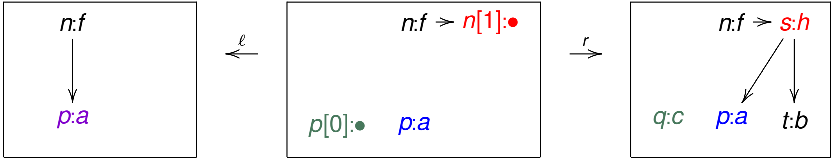
“Right” theorem.
Let (ℓ, r) *be a rewrite rule,* m *a matching w.r.t.* ℓ*,*
thus defining a complement pushout as in the “left” theorem.
Then there is a double pushout of graphs:

$$\begin{array}{ccccc}
L & \xleftarrow{\ \ell\ } & K & \xrightarrow{\ r\ } & R \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle m'} \\
G & \xleftarrow{\ \ell'\ } & D & \xrightarrow{\ r'\ } & H
\end{array}$$

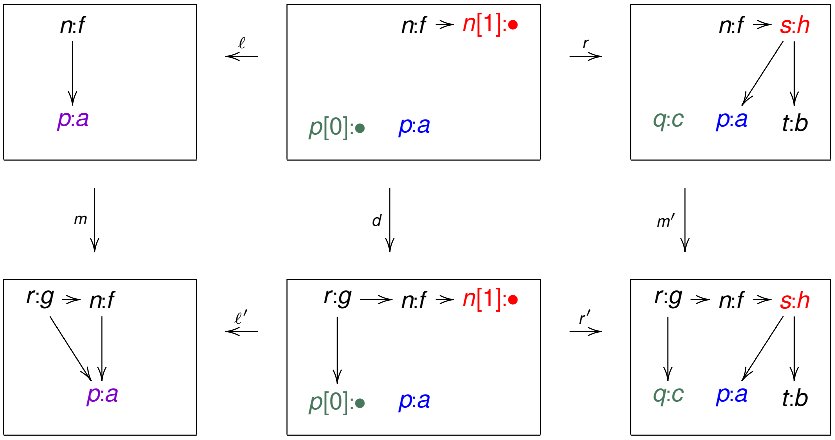
About proof.

- The nodes of H are built as a PO in $\mathbf{Set}$: a node n of H is a class of nodes in D or in R, connected through K.
- For the label of n:
  - If every node (in D or in R) in the class n is unlabeled then n is unlabeled.
  - Otherwise, consider 2 labelled nodes n', n'' (in D or in R) in the class n, and a chain of minimal length connecting n' to n'' (through K). Prove that all nodes in the chain are labelled. Then they all have the same label, say f.
    Let $n : f$ in H.
- For the successors of n: similar.
- To conclude, apply the PO theorem.

A reconnection of matching



A reconnection of matching

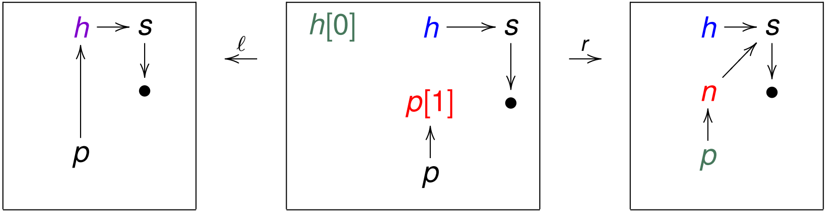


A circular list example (1)

Given a circular list of length ≥ 3,
with head h, preceded by p,
change the cell h for a new cell n,
and move the head of the list to p.

Rewrite G in H, where (labels are omitted):

A circular list example (2)



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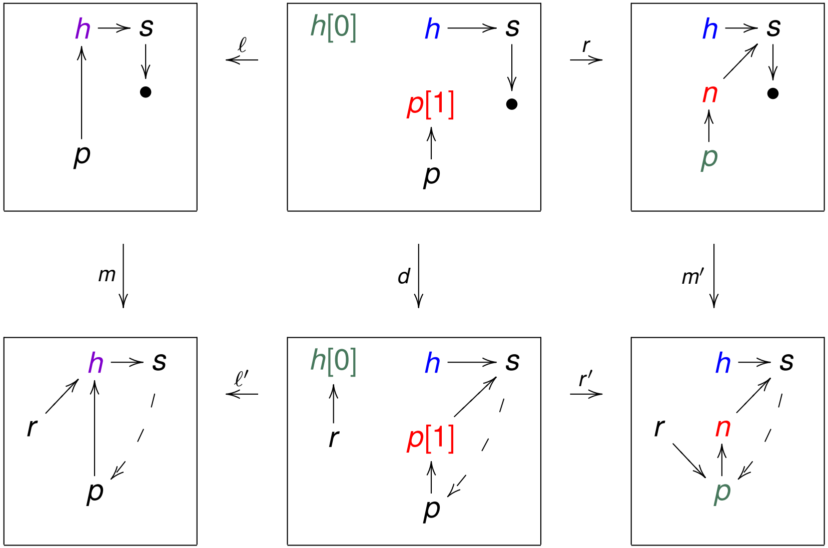

Outline

To sum up

We get:

- a new graph rewriting framework,
- based on the double pushout approach,
- well-suited for local and global pointer redirection.

In our category of graphs, it may happen that:

- pushouts do not exist,
- complement pushouts do not exist,
- complement pushouts are not unique,

Under suitable definitions and assumptions,
we have proved that we get a well-defined double pushout,
which means that:

A rewrite rule can always be fired, once a matching is found.

About garbage collection

In our framework, nodes and edges can be added or identified, but they cannot be destroyed.

So, some kind of garbage collection is required.

- We have investigated garbage collection for rooted graphs from a categorical point of view, using several kinds of adjunction.

  D. Duval, R. Echahed, F. Prost.
  Adjunction for Garbage Collection with Application to Graph Rewriting.
  In Proceedings of RTA'07. LNCS 4533, p.122-136 (2007).

- In addition, the description of a garbage collector dedicated to our DPO framework is in progress.

A personal conclusion

This work has been very pleasant:

- discovering the graph rewriting topic,
- especially the use of categories for graph rewriting,
- finding the right definitions and proofs,

and also:

- giving a talk on a subject (apparently) different from DIAgrammatic LOGics...

...some recent papers on DIAgrammatic LOGics:

– *J.-G. Dumas, D. Duval, J.-C. Reynaud.*
Sequential products in effect categories (2007)

– *D. Duval.*
Diagrammatic inference (2007).