Coalgebraic Modal Logic: Forays Beyond Rank 1

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Introduction

- Coalgebraic Modal Logic serves as a generic semantic framework for modal logic, covering e.g.
  - Kripke frames/normal modal logics
  - Game frames/coalition logic
  - Conditional frames/conditional logics
  - Multigraphs/graded modal logics
  - Probabilistic transition system/probabilistic modal logics
- General results include
  - Expressiveness
  - Soundness and (weak) completeness
  - Decidability, finite model property
  - Complexity (finite models, tree models)

Coalgebra

$T : \mathbf{Set} \to \mathbf{Set}$

T-Coalgebra (X, ξ) = map $\xi : X \to TX$

X: set of states

ξ: transition map

$\xi(x)$: structured collection of observations/successor states

Modal Syntax

Modal signature Λ: set of finitary modal operators $L, \dots$

$$\phi ::= \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid L(\phi_1, \dots, \phi_n) \quad (L \in \Lambda \text{ } n\text{-ary})$$

(N.B.: Variables can be regarded as nullary operators)

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Λ-structure:

- Functor $T : \mathbf{Set} \to \mathbf{Set}$;
- For $L \in \Lambda$ n-ary predicate lifting

$$[\![L]\!]_X : \mathcal{Q}^n \to \mathcal{Q} \circ T^{op}$$

($\mathcal{Q}$ contravariant powerset).

For (X, ξ) T-coalgebra, $x \in X$,

$$x \models L(\phi_1, \dots, \phi_n) \iff \xi(x) \in [\![L]\!]([\![\phi_1]\!], \dots, [\![\phi_n]\!])$$

$([\![\phi]\!] = \{x \mid x \models \phi\})$

Example: Kripke Frames

(Covariant) powerset functor $\mathscr{P}$: **Set** $\to$ **Set**

$\mathscr{P}$-coalgebras $X \to \mathscr{P}(X)$ are Kripke frames (X, R), $R \subseteq X \times X$.

Standard modal signature $\{\Box\}$,

$$[\![\Box]\!]_X(A) = \{B \in \mathscr{P}(X) \mid B \subseteq A\}$$

$\to$ normal modal logic K,

$$x \models_{(X,R)} \Box\phi \quad \text{iff} \quad \forall y.\, xRy \Rightarrow y \models \phi$$

Example: Multigraphs

Infinitary multiset functor $\mathscr{B}_\infty(X) = X \to \mathbb{N} \cup \{\infty\}$

$\mathscr{B}_\infty$-coalgebras $B : X \to \mathscr{B}_\infty(X)$ are multigraphs,
i.e. graphs with weighted edges

$$x \xrightarrow{n} y, \quad n = B(x)(y) \in \mathbb{N} \cup \{\infty\}.$$

Modal signature $\{\geq n. \mid n \in \mathbb{N}\}$ (qualified number restrictions),

$$[\![\geq n.]\!]_X(A) = \{B \in \mathscr{B}_\infty(X) \mid \underbrace{B(A)}_{=\sum_{x \in A} B(x)} \geq n\}$$

Variant: $\mathscr{B}_\mathbb{N}(X) \subseteq \mathscr{B}_\infty(X)$ finite multisets.

Example: Conditional Frames/Conditional Logic

Functor $Cf(X) = \underbrace{\mathscr{Q}(X)}_{\text{contravar.}} \to \underbrace{\mathscr{P}(X)}_{\text{covar.}}$

Cf-coalgebras are conditional frames $(X, (R_A \subseteq X \times X)_{A \subseteq X})$.

Modal signature $\{\Rightarrow\}$, $\Rightarrow$ binary,
with non-monotonic conditional $a \Rightarrow b$ read e.g.

- a relevantly implies b (relevance logic)
- if a then normally b (default logic)

$$[\![\Rightarrow]\!]_X(A, B) = \{f \in Cf(X) = \mathscr{Q}(X) \to \mathscr{P}(X) \mid f(A) \subseteq B\},$$

i.e.

$$x \models_{(X,(R_A))} \phi \Rightarrow \psi \quad \text{iff} \quad \forall y.\, xR_{[\![\phi]\!]}y \Rightarrow y \models \psi.$$

$$\text{Prop}(Z) = \text{ propositional formulas over } Z$$
$$\Lambda(Z) = \{L(x_1, \dots, x_n) \mid L \in \Lambda \ n\text{-ary}, x_i \in Z\}$$

Rank-1 formulas ψ over V: $\psi \in \text{Prop}(\Lambda(\text{Prop}(V)))$, e.g.

$$(K) \qquad \Box(a \to b) \to \Box a \to \Box b.$$

Given $\tau : V \to \mathscr{P}(X)$, have $[\![\psi]\!]\tau \subseteq TX$.

ψ is one-step sound if $[\![\psi]\!]\tau = TX$ for all X, τ.

Definition

A set $\mathscr{A}$ of rank-1 formulas is one-step complete if, whenever $[\![\chi]\!]\tau = TX$ for $\chi \in \text{Prop}(\Lambda(\text{Prop}(V)))$, then χ is derivable from axiom instances and replacement of equivalents over (X, τ).

Finite Models

Set Σ of formulas closed (under subformulas and negation)

Σ-atom = maximally consistent subset of Σ.

Idea: construct for consistent ϕ a small canonical model (S,ξ) on the set S of $\Sigma(\phi)$-atoms, $\Sigma(\phi)$=closure of $\{\phi\}$.

A coalgebra (S,ξ) is coherent if

$$L(\psi_1,\dots,\psi_n) \in A \iff \xi(A) \in [\![L]\!](\psi_1\uparrow,\dots,\psi_1\uparrow) \quad (L(\psi_1,\dots,\psi_n) \in \Sigma),$$

where $\psi\uparrow = \{B \in S \mid \psi \in B\}$.

Lemma (Existence)

If $\mathscr{A}$ is one-step complete, then there exists a coherent coalgebra on S.

Corollary (Finite model property)

If $\mathscr{A}$ is one-step complete, then every consistent ϕ is satisfiable in a coalgebra of size $\leq 2^{|\phi|}$.

So far, so good.

Rank 1

Lemma

The set of all one-step sound rank-1 formulas is one-step complete.

Corollary

Every coalgebraic modal logic

- *has the finite model property (every satisfiable formula is satisfiable in a finite model)*
- *is completely axiomatizable by rank-1 formulas*

The latter result is nice but limitative (what about $S4$, $K4$, ...)

Beyond Rank 1

$\mathscr{L}$ coalgebraic modal logic with frame conditions, i.e. arbitrary axioms (e.g. (4) $\Box a \to \Box\Box a$)
$\mathscr{L}$-frame =
coalgebra (X, ξ) satisfying the frame conditions for all $\mathscr{P}(X)$-valuations

Basic idea: equip the set S of Σ-atoms for $\mathscr{L}$ with a sieve ν, where

$$A \subseteq \nu(A) \subseteq \bar{\Sigma} \subseteq \Lambda(\mathrm{Prop}(\Sigma)) \cup \Sigma \text{ max. } \mathscr{L}\text{-cons.} \qquad (A \in S)$$

adds one more (finite) layer of modalities.

(S, ξ) is ν-coherent if

$$L(\phi_1, \dots, \phi_n) \in \nu(A) \iff \xi(A) \in [\![L]\!](\phi_1\uparrow, \dots, \phi_n\uparrow) \quad (L(\phi_1, \dots, \phi_n) \in \bar{\Sigma})$$

where $\phi\uparrow = \{B \in S \mid B \vdash_{PL} \phi\}$.

As before, a ν-coherent coalgebra (S, ξ) exists under one-step completeness.

From Models to Frames

$\mathscr{L}(\Sigma) = \{\phi \in \mathsf{Prop}(\bar{\Sigma}) \mid \ \vdash_{\mathscr{L}} \phi\}$ shallow instances of the frame conditions.

Lemma

A ν-coherent coalgebra (S, ξ) satisfies $\mathscr{L}(\Sigma)$.

Definition

$\mathscr{L}$ is $K4$-like if for every Σ-filtered $\mathscr{L}(\Sigma)$-model (X, ξ) there exists an $\mathscr{L}$-frame on X that satisfies the same Σ-formulas.

Theorem (Completeness, finite model property)

If $\mathscr{L}$ is $K4$-like, then $\mathscr{L}$ is weakly complete over finite models.

Obtain moreover under suitable sanity conditions

- Decidability
- Upper bound *NEXPTIME*, or even
- Upper bound *EXPTIME*

Example: S4

Recall: $S4$ is K plus $\Box a \to \Box\Box a$, $\Box a \to a$
$S4$-frames are transitive reflexive Kripke-frames

$\bar{\Sigma} = \{\Box\}\Sigma \cup \Sigma$.

Let $(X, R) \models S4(\Sigma) \supseteq \{\Box\phi \to \Box\Box\phi \mid \Box\phi \in \Sigma\}$.

(X, R^*) is $S4$-frame.

Show $x \models_{(X,R)} \Box\phi \in \Sigma$ iff $x \models_{(X,R^*)} \Box\phi$.

'If' trivial, 'only if': Show $xR^n y \Rightarrow y \models \Box\phi$ by induction:

$$
\begin{aligned}
xR^n zRy &\implies z \models \Box\phi && \text{(induction)} \\
&\implies z \models \Box\Box\phi && (X, R) \models S4(\Sigma) \\
&\implies y \models \Box\phi && (zRy)
\end{aligned}
$$

Example: Conditional Logic CK+CMi

Conditional logic CK plus frame condition

$$(CMi) \quad ((a \wedge b) \Rightarrow c) \rightarrow (a \Rightarrow (a \wedge b \Rightarrow c)).$$

Resembles cautious monotony

$$(a \Rightarrow b) \rightarrow (a \Rightarrow c) \rightarrow ((a \wedge b) \Rightarrow c),$$

lies between duplication

$$(a \Rightarrow c) \rightarrow (a \Rightarrow (a \Rightarrow c))$$

and currying (B. Skyrms)

$$(a \wedge b) \Rightarrow c \rightarrow a \Rightarrow (b \Rightarrow c).$$

$(X, (R_A)_{A \subseteq X})$ satisfies *CMi* iff for all $B \subseteq A \subseteq X$,

$$R_A ; R_B \subseteq R_B,$$

CK+CMi is K4-like

Turn conditional frame $C = (X, (R_A)_{A \subseteq X})$ into CK+CMi-frame $\bar{C} = (X, (\bar{R}_A))$ by

$$\bar{R}_B = \bigcup_{B \subseteq A_i} R_{A_1}; \dots; R_{A_n}; R_B.$$

For C Σ-filtered, $A_i = [\![\chi_i]\!]_C$, $\chi_i \in \mathrm{Prop}(\Sigma)$.

Then prove invariance of $\rho \Rightarrow \psi \in \Sigma$ using shallow instances

$$(\rho \Rightarrow \psi) \to (\rho \vee \chi_i \Rightarrow (\rho \Rightarrow \psi)).$$

Number Restrictions on Transitive Roles

Problem in Description Logic: qualified number restrictions

$$\geq nR.\phi$$

on transitive roles R quickly lead to undecidability
when combined with role hierarchies.

As closure operations tend to be monotone, it is improbable that our methods work for qualified number restrictions on transitive roles.

Technical problem: transitive closure of multigraph $B : X \to \mathscr{B}_\infty(X)$ is

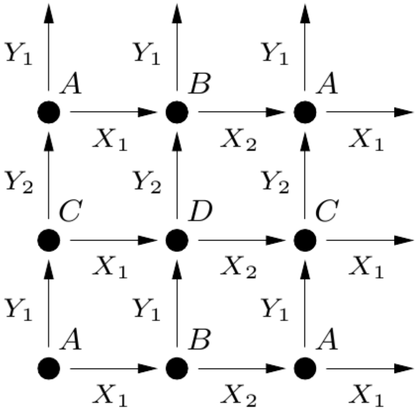
$$\bar{B}(x)(z) = \max(B(x)(z), \sup_{B(x)(y)>0} B(y)(z)),$$

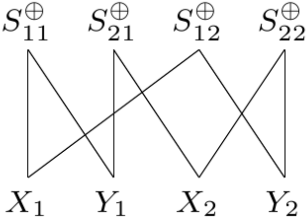
but one does not have

$$\bar{B}(x)(A) = \max(B(x)(A), \sup_{B(x)(y)>0} B(y)(A)) \qquad (A \subseteq X),$$

as $\sum$ and sup do not commute.

Undecidability (Horrocks et al. 1999)





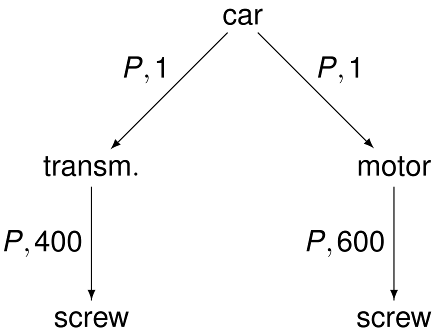
Formulas $\leq 3S_{ij}.\top$

Transitivity and Parthoods

Want

$$\text{car} \models \geq 1000\, P.\,\text{screw}$$

from



→ parthoods are often expected to be treelike

Example: Description Logic CD

Solution:

CD Kripke frame = Kripke frame with relations C (child), D (descendant) such that

- Siblings never have common descendants
- $D = C \cup C; D$.

Correspondingly, CD-multiframe satisfies

$$D(x)(z) = C(x)(z) + \sum_{C(x)(y)>0} C(x)(y)D(y)(z).$$

and then also

$$D(x)(A) = C(x)(A) + \sum_{C(x)(y)>0} C(x)(A)D(y)(A) \qquad (A \subseteq X).$$

Axiomatisation of CD

Lower bound on descendant numbers:

$$\bigwedge_{i=1}^{m} \geq n_i C.(\phi_i \wedge \geq k_i D.b) \rightarrow \geq \textstyle\sum_{i=1}^{q} n_i + \sum_{i=1}^{m} n_i k_i D.b,$$

ϕ_i mutually exclusive, $\phi_i \vdash b$ for $1 \leq i \leq q \leq m$

Upper bound on descendant numbers:

$$\geq rD.b \rightarrow \geq 1C. \geq rD.b \vee \bigvee_{(n_i,k_i),q} \bigwedge_{i=1}^{m} \geq n_i C.(\phi_i \wedge = k_i D.b),$$

$\phi_i = b$ for $i \leq q \leq m$, $\phi_i = \neg b$ otherwise,
$\sum_{i=1}^{q} n_i + \sum_{i=1}^{m} n_i k_i \geq r$

CD is (slightly non-trivially) K4-like.

Decidability and Complexity

Algorithms:

- Guess set of atoms and sieve, check existence of coherent coalgebra
  - ideally *NEXPTIME*
- Elimination of Hintikka sets:
  - Start with all Hintikka sets
  - Apply monotone operator
    'eliminate Hintikka sets A that do not admit a coherent $\xi(A)$'
  - ideally *EXPTIME*

Examples:

- Description logics with role hierarchies and reflexive/symmetric/transitive roles are in *EXPTIME* (hence *EXPTIME*-complete)
- CK+CMi is decidable
- CD is in *NEXPTIME*
  (hardness open, but not bad for number restrictions)

Conclusions

- Coalgebraic modal logic has broken free of rank 1
- Crucial: spikes =
  axiom instances one level of modal operators beyond finite scope
- Example applications:
  - Description logic for descendants in trees
  - Description logic with reflexive/symmetric/transitive roles
  - Conditional logic
- Examples are designed but natural
  - K4-likeness as a design principle
- Obtain high but often tight upper complexity bounds (EXPTIME/NEXPTIME)

Future Work

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- PSPACE for coalgebraic modal logics with frame conditions
  - K4, S4 are in PSPACE
- Implementation
  - Under way: CoLoSS (M4M 2007)
- Optimized algorithms:
  - propositional tableaux/ BDDS
  - integer linear programming
  - heuristic strategies