Zooms for Effects

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IFIP W.G.1.3. meeting

Outline

Motivations

Wanted. A framework for the semantics of effects.

Monads. For two kinds of morphisms:

- in general $f : X \to Y$ "stands for" some $f' : X \to T(Y)$
- sometimes $v : X \to Y$ is pure, then $v' = \eta \circ v$

Wanted. Several kinds of objects, of arrows, of equations,...
each kind "stands for" something...

In this talk

A category of logics

- objects: “logics” with models and proofs
- morphisms: “stands for” should be a morphism

“stands for”?

E.g., a monad.

- in general $f : X \to Y$ “stands for” some $f' : X \to T(Y)$

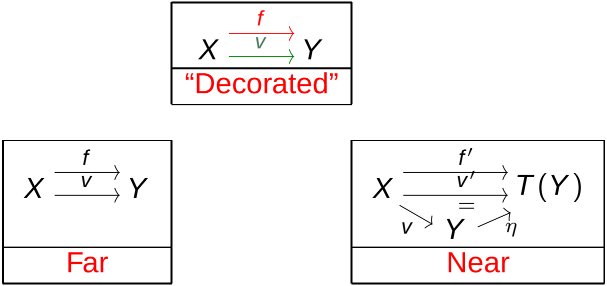
$X \xrightarrow{f} Y$
Far

$X \xrightarrow{f} T(Y)$
Near

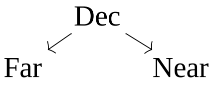
“stands for” is part of a “zoom”

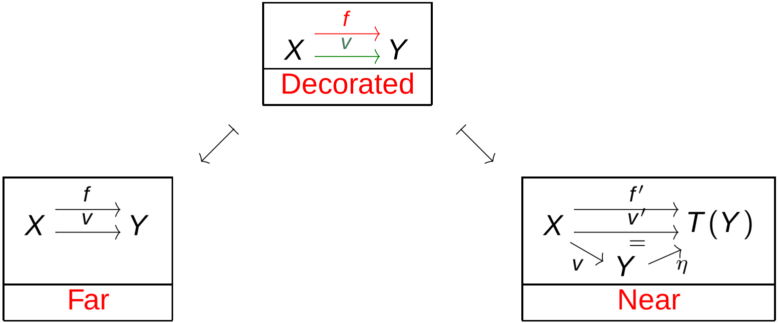
E.g., a monads

- in general $f : X \to Y$ “stands for” some $f' : X \to T(Y)$
- sometimes $v : X \to Y$ is pure, then $v' = \eta \circ v$



“zooms” are spans





Slogan.

First be wrong,

then add corrections,

in order to finally get right.

This talk

- Diagrammatic logics (categories...)
  with *Christian Lair*.

- Zooms for parameterization
  with *César Domínguez*.

- A zoom for sequential product
  with *Jean-Guillaume Dumas* and *Jean-Claude Reynaud*.

Outline

A diagrammatic logic

Definition. A logic L is a functor
with a full and faithful right adjoint R:

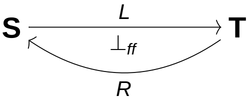
$$\mathbf{S} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{T} \qquad \perp_{ff}$$

In addition, this is induced by a morphism of limit sketches.

Properties.

- R makes $\mathbf{T}$ a full subcategory of $\mathbf{S}$
- $L(R(\Theta)) \cong \Theta$ for each theory Θ
- $\mathbf{S}$ and $\mathbf{T}$ have colimits, and L preserves colimits

Models



Definitions.

- **S** is the category of specifications
- **T** is the category of theories
- Σ presents Θ when $\Theta \cong L(\Sigma)$.
- Σ and Σ' are equivalent when $L(\Sigma) \cong L(\Sigma')$.

Models. $\boxed{\mathrm{Mod}(\Sigma, \Theta) = \mathbf{T}[L(\Sigma), \Theta] \cong \mathbf{S}[\Sigma, R(\Theta)]}$

The models form a category iff **T** is a 2-category.

Proofs

Theorem. [Gabriel-Zisman 1967] (for homotopy theory)
Up to equivalence, L is a localization:
it adds inverses to some morphisms in **S**.

Definition. An entailment is $\tau : \Sigma \to \Sigma'$ in **S**
such that $L(\tau)$ is invertible in **T**.
Then Σ and Σ' are equivalent.

Hence: the bicategory of fractions **S2**.

Definition. A proof is a fraction.

in **S2**:

$$\Sigma \xrightarrow{\ \sigma\ } \Sigma_1' \underset{\tau}{\overset{}{\leftrightarrows}} \Sigma_1$$

in **S**:

$$\Sigma \xrightarrow{\ \sigma\ } \Sigma_1' \xleftarrow[\ \tau\ ]{} \Sigma_1$$

in **T**:

$$L\Sigma \xrightarrow{\ L\sigma\ } L\Sigma_1' \underset{L\tau}{\overset{(L\tau)^{-1}}{\leftrightarrows}} L\Sigma_1$$

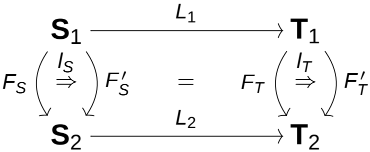
Morphisms of logics

Definition. A **morphism of logics** $F\colon L_1 \to L_2$
is a pair of functors (F_S, F_T) such that:

$$\begin{array}{ccc} \mathbf{S}_1 & \xrightarrow{\quad L_1 \quad} & \mathbf{T}_1 \\ {\scriptstyle F_S}\big\downarrow & \cong & \big\downarrow{\scriptstyle F_T} \\ \mathbf{S}_2 & \xrightarrow[\quad]{\quad L_2 \quad} & \mathbf{T}_2 \end{array}$$

In addition, they are induced by morphisms of limit sketches.

Definition. A **2-morphism of logics** $\ell\colon F \Rightarrow F'\colon L_1 \to L_2$
is a pair of natural transformations (ℓ_S, ℓ_T) such that:



Altogether...

- A 2-category of logics **DiaLog** with a 2-functor that focuses on the theories:

$$\mathbf{DiaLog} \to \mathbf{Cat}$$

$$(L : \mathbf{S} \to \mathbf{T}) \mapsto \mathbf{T}$$

- “Everything” happens in the bicategory of fractions: a specification Σ should be seen up to equivalence.

Outline

Parameterization

Starting point: Sergeraert's software for effective homology.

Goal: formalize the process of:

- adding a parameter to some operations
- then passing a value (an argument) to the parameter

A kind of benchmark, that may be treated with monads ($T(X) = X^A$), hidden algebras, coalgebras, institutions...

Parameterization and diagrammatic logics

- Parameterization: a zoom
- Parameter passing: a zoom and a 2-morphism

Example: Differential monoids

A specification of monoids *Mon*:

- type G
- operations $prd : G^2 \to G$, $e : \ \to G$
- equations $prd(x, prd(y, z)) = prd(prd(x, y), z)$, $prd(x, e) = x$, $prd(e, x) = x$

A specification of differential monoids *DMon*:

- *Mon* with
- operation $dif : G \to G$
- equations $dif(prd(x, y)) = prd(dif(x), dif(y))$, $dif(e) = e$, $dif(dif(x)) = e$

A specification of decorated differential monoids *DecDMon*:

- *Mon* with
- operation $dif : G \to G$
- equations $dif(prd(x, y)) = prd(dif(x), dif(y))$, $dif(e) = e$, $dif(dif(x)) = e$

A specification for monoids with a parameterized differential *ParDMon*:

- *Mon* with
- type A
- operation $dif' : A \times G \to G$
- equations $dif'(p, (prd(x, y))) = prd(dif'(p, x), dif'(p, y))$, $dif'(p, e) = e$, $dif'(p, dif'(p, x)) = e$

A zoom for parametererizing

Dec $\swarrow$ Far, Dec $\searrow$ Near

$$G^2 \xrightarrow{prd} G \qquad G \xrightarrow{dif} G$$

DecDMon

$\swarrow$ $\searrow$

$$G^2 \xrightarrow{prd} G \qquad G \xrightarrow{dif} G$$

DMon

$$A \times G^2 \longrightarrow G^2 \xrightarrow{prd} G \qquad A \times G \xrightarrow{dif'} G$$

ParDMon

Parameter passing

Each parameterized differential monoid PM
together with an argument $\alpha \in PM(A)$
$\Rightarrow$ a differential monoid M_α with:

- the same underlying monoid as PM
- the differential $x \mapsto M_\alpha(dif)(x) = PM(dif')(\alpha, x)$

In the specifications:
Add a constant $a : 1 \to A$ in the “near” logic.

A zoom for parameter passing...

Dec

Far ↙ ↘ Near

$$G \xrightarrow{dif} G$$

DecDMon

↙ ↘

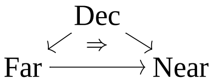
$$G \xrightarrow{dif} G$$

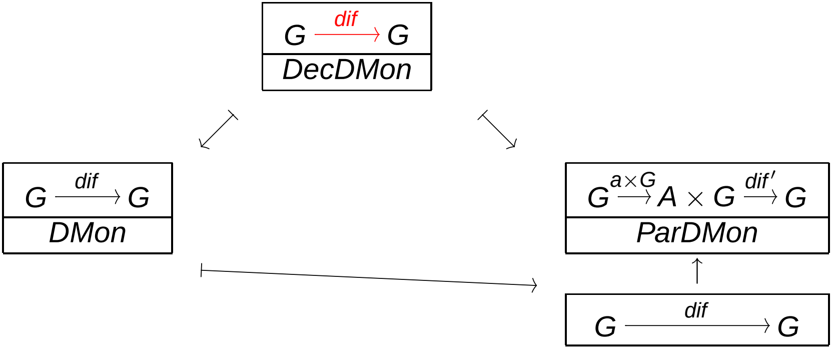
DMon

$$G \xrightarrow{a \times G} A \times G \xrightarrow{dif'} G$$

ParDMon

...with a 2-morphism of logics





Outline

Sequential product

Goal: formalize the fact that the order of evaluation of the arguments does matter when there are effects.

Monads: the strength.

In the framework of diagrammatic logics:
A zoom, from an ordinay product to a sequential product.

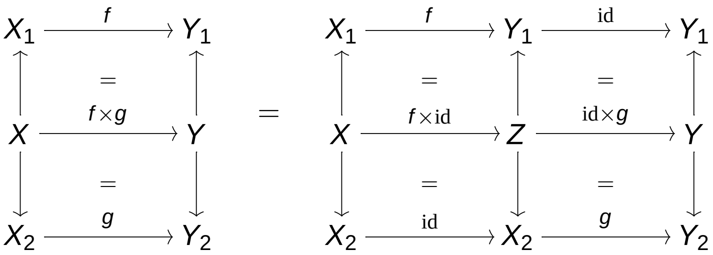
There are two kinds of morphisms
And two kinds of equations!

About products

$X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, $Z = Y_1 \times X_2$.

Without effects:

$$g \times f = (\mathrm{id} \times g) \circ (f \times \mathrm{id})$$



A zoom for the sequential product

$$\begin{array}{ccc} & \text{Dec} & \\ \swarrow & & \searrow \\ \text{Far} & & \text{Near} \end{array}$$

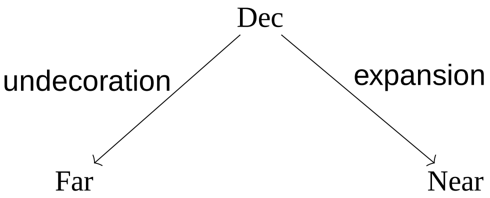
$$\begin{array}{ccc} X_1 & \xrightarrow{f} & Y_1 \\ \uparrow & = & \uparrow \\ X & \xrightarrow{f \times v} & Y \\ \downarrow & \approx & \downarrow \\ X_2 & \xrightarrow{v} & Y_2 \end{array}$$

$$\swarrow \qquad\qquad \searrow$$

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & Y_1 \\ \uparrow & = & \uparrow \\ X & \xrightarrow{f \times v} & Y \\ \downarrow & = & \downarrow \\ X_2 & \xrightarrow{v} & Y_2 \end{array} \qquad\qquad \begin{array}{ccc} S \times X_1 & \xrightarrow{f} & S \times X_2 \\ \uparrow & = & \uparrow \\ S \times X & \xrightarrow{f \times v} & S \times Y \\ \downarrow & = & \downarrow \\ X_2 & \xrightarrow{v} & Y_2 \end{array}$$

Outline

Zooms



PROOFS MODELS

THANK YOU!