Formally Proving and Enhancing a Self-Stabilising Distributed Algorithm

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Abstract. This paper presents the benefits of formal modelling and verification techniques for self-stabilising distributed algorithms. An algorithm is studied, that takes a set of processes connected by a tree topology and converts it to a ring configuration. The Coloured Petri net model not only facilitates the proof that the algorithm is correct and self-stabilising but also easily shows that it enjoys new properties of termination and silentness. Further, the formal results show how the algorithm can be simplified without loss of generality.

1 Introduction

Goals and contributions This paper aims at using a formal model and associated verification techniques in order to prove properties of a self-stabilising distributed algorithm. Although the algorithm considered [8] was shown self-stabilising, the proof was lengthy and cumbersome. Using formal models, in this case Petri nets which are particularly well-suited for such distributed algorithms, provides a much shorter and more elegant proof. It also allows for easily deriving additional properties which were not proven in the past: correctness, termination and silentness. Finally, reasoning on the model leads to a simplification of the algorithm, and a reduction in the number of exchanged messages. This simplification, which now appears straightforward is not obvious at all when considering the distributed algorithm alone.

Context A distributed system consists of a set of processes or processors. Each process k has a local state s_k, and processes communicate with each other via communication channels that have local outgoing and incoming queues. The *configuration*, C, of a distributed system is the set of local states of its processes together with the queues of messages that have yet to be sent or received.

Self-stabilising algorithms are distributed algorithms that, starting from *any initial configuration* and executing the set of possible transitions in *any order*, make the system converge to a *legitimate configuration*. This property makes

self-stabilising algorithms suitable for many fields of application, including *fault tolerant systems*: failures take the system out of its legitimate state and the algorithm is executed to reach a legitimate configuration again. Other properties that may be of interest are the *closure property* which ensures that once it has reached a legitimate configuration, the system remains in it, and the *silent* property which states that once it has reached a legitimate state, inter-process communications are *fixed*. This means that the communication channels hold the same value: processes stop communicating (message queues and communication channels remain empty) or repeatedly send the same message to their neighbours.

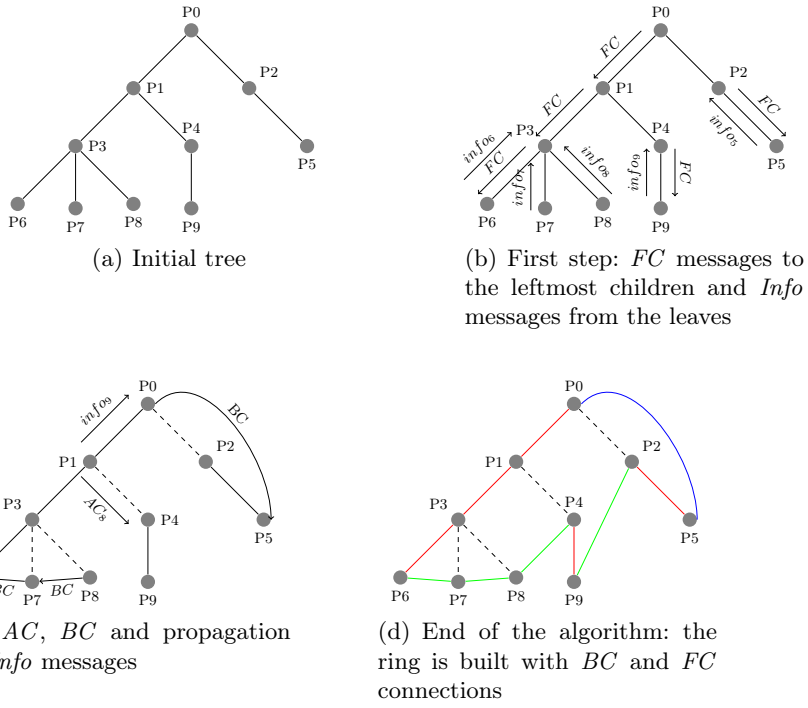
Self-stabilisation must not be mistaken for self-healing. Self-healing algorithms detect a process has failed and re-knit the topology. For instance, if a set of processes are interconnected by a topology and one of them dies, inter-process connections are modified in order to form the topology again. Self-healing is an adaptation to the failure, whereas self-stabilisation considers the post-failure state as a new, non-legitimate state and restarts the algorithm from the beginning. Self-stabilising algorithms are more general than self-healing algorithms and can tolerate *any kind of failure*. When a failure occurs after the system has stabilised, the system gets out of its legitimate configuration and the algorithm executes again to reach a legitimate configuration again [15]. Besides, self-stabilising algorithms ensure self-healing, because after the failure has stopped, they ensure that the system reaches a legitimate state in finite time.

It is not always an easy task to prove the aforementioned properties related to self-stabilisation. It is necessary to prove that these properties hold for any possible execution and starting from any initial state. Some techniques exist for this, coming from traditional distributed algorithm techniques, or specific techniques such as *attraction* and *rewriting* [6]. Some algorithms can also be formalised using a proof assistant [13].

One technique to prove that a system is self-stabilising with respect to a set of legitimate states consists in considering a *norm function* which is representative of the state of the system, and proving that this function is integral, positive and *strictly decreasing* as the algorithm is executed [19].

Outline of the paper In this paper, we examine a distributed, self-stabilising algorithm that, given a set of processes interconnected by any tree topology, builds a ring topology between them. This algorithm is interesting since it is a stepping stone to a binomial graph (BMG) configuration which has several desirable properties, such as performance (low diameter and degree) and robustness (in terms of number of processes and communication links that can fail without disconnecting the graph) [2].

Section 2 presents the algorithm and describes its behaviour. In Section 3 we derive a Coloured Petri Net model of the algorithm. The Coloured Petri Net formalism is ideally suited for this task because of its support for concurrency and the variability of sequencing and timing of distributed processes. Further, the graphical representation of Petri Nets can help to highlight the flow of information in the algorithm. This immediately leads to some simplifications which are presented in Section 4, along with certain invariant and liveness properties



(a) Initial tree

(b) First step: *FC* messages to the leftmost children and *Info* messages from the leaves

(c) *AC*, *BC* and propagation of *Info* messages

(d) End of the algorithm: the ring is built with *BC* and *FC* connections

Fig. 1: Execution of the algorithm

which can be deduced directly from the model. In Section 5 we prove that the algorithm terminates, effectively presenting a norm function. Then, in Section 6 we prove that the algorithm is correct. As a result of the properties proved above, it is possible to simplify the algorithm further, and this is presented in Section 7. In section 8 we validate our results with an automated tool. Finally, the conclusions are presented in Section 9.

2 The algorithm

In this paper, we focus on a distributed, self-stabilising algorithm that, given a set of processes interconnected by any tree topology, builds a ring topology between them. This algorithm was originally described in [8] and we reproduce the pseudocode in algorithm 1.

This algorithm is meant to establish a fault-tolerant, scalable communication infrastructure. For instance, it can be used to support the execution of parallel processes [11] such as the processes of MPI programs [18]. Robust, fault-tolerant run-time environments [17] are necessary to support middleware-level, automatic fault tolerance [7] or application-level fault-tolerance [20, 9, 25]. In order to be efficient at large scale, tree topologies are often prefered [5]. However, trees are not robust and connectivity can be lost if intermediate processes fail. Resilient extensions of trees introduce additional inter-process connections, such

Algorithm 1: self-stabilising algorithm that builds a ring from any tree

Constants:
  $Parent : \mathcal{ID}$ /* empty if I am the root of the tree */
  $Children : List(\mathcal{ID})$ /* empty if I am a leaf process */
  $Id : \mathcal{ID}$ /* my own identifier */
Output:
  $Pred : \mathcal{ID}$ /* both empty start-up time */
  $Succ : \mathcal{ID}$
Initialisation:
1 - $Children \neq \emptyset \rightarrow$
  /* I have children: send a F_Connect message to my leftmost child. */
  $Succ = First(Children)$;
  Send $(F\_Connect, Id)$ **to** $Succ$;

2 - $Children = \emptyset \rightarrow$
  /* I am a leaf process */
  Send $(Info, Id)$ **to** $Parent$;

Run:
3 - **Recv** $(F\_Connect, I)$ **from** $p \rightarrow$
  /* I received a F_Connect; from my parent? If so, here is my predecessor. */
  if $p = Parent$ **then** $Pred = I$;

4 - **Recv** $(Info, I)$ **from** $p \rightarrow$
  /* I have received an Info message. From whom? */
  if $p \in Children$ **then**
    let $q = next(p, Children)$;
    if $q \neq \perp$ **then**
      Send $(Ask\_Connect, I)$ **to** q ;
    else
      if $Parent \neq \perp$ **then**
        Send $(Info, I)$ **to** $Parent$;
      else
        $Pred = I$;
        Send $(B\_Connect, Id)$ **to** I ;
      end if
    end if
  end if

5 - **Recv** $(Ask\_Connect, I)$ **from** $p \rightarrow$
  /* I am being asked to connect to a leaf process. */
  $Pred = I$;
  Send $(B\_Connect, Id)$ **to** I ;

6 - **Recv** $(B\_Connect, I)$ **from** $p \rightarrow$
  /* I received a B_Connect; here is my successor. */
  $Succ = I$;

as k-sibling trees [3] or redundant storage of connection information [4]. The Binomial Graph topology (BMG) is scalable and allows efficient communications [24, 10] while being robust. More precisely, a BMG made of N processes has a diameter in $O(logN)$, therefore a message is routed between two processes in at most $O(logN)$ hops. It has a degree in $O(logN)$, which means that every process needs to handle $O(logN)$ open connections. Every node sees the other nodes along a binomial tree: therefore, efficient (in terms of number of messages) collective communications can be implemented directly on top of a BMG. Its node connectivity and its link connectivity are both in $O(logN)$, which means that $O(logN)$ nodes or $O(logN)$ network connections can fail without its topology becoming disconnected [2].

In this case, the legitimate states of the algorithm are those in which the processes are connected forming a ring topology. The non-legitimate states are all the (connected) topologies, from which a tree can therefore be extracted. In particular, a ring that has lost one process forms a chain, which is a particular form of tree (unary tree). If a new process is spawned, the processes form a tree rooted on the process that spawned the new one, with one long branch (the chain) and a short one (the new process).

Processes are spawned on a set of resources and each process is connected to the process that spawned it. This results in a tree topology that matches the shape of the spawning operation (figure 1(a)).

To connect with another process, a process needs to know the remote process's *communication information* (*e.g.* an IP address and a port). This com-

munication information and the name of a process in a naming system (*e.g.* a rank) form the *identity* of this process. In the initial state, processes know their own identity, that of their parent and of their children. The algorithm allows for *concurrent* or even parallel propagation of the identity of leaf processes in order to establish connections between some leaves and some intermediate processes. The algorithm sets (additional) *Succ* and *Pred* pointers to reflect the node order of depth-first traversal. It achieves this by propagating information bottom-up from the leaves, thereby allowing for *parallel* execution. Besides, inter-process communications are *asynchronous*.

In the algorithm, $\mathcal{ID}$ denotes the set of possible process identities. $List(\mathcal{ID})$ is an ordered list of identities; $First(L)$ returns the first element in the list L, and $next(e, L)$ returns the element following e in L. If there is no such element, these functions return $\perp$, which is also used to denote a non-existing information.

During the first step, each non-leaf process sends an *F_ Connect* (or *FC* for short) message to its oldest (*i.e.* leftmost) child (rule 1). Thus, each process that receives an *FC* message is connected to its predecessor (rule 3). Concurrently, each leaf process sends its identity to its parent in an *Info* message (rule 2 and figure 1(b)). In other words, the *Info* message contains the communications information of the process that built it, *i.e.* how it can be contacted by another process to create a new connection between them.

Then each time a process receives an *Info* message from one of its children (rule 4), it forwards this information to the younger sibling as an *Ask_ Connect* message (or *AC*, see figure 1(c)). The message forwarded here contains the identity of the rightmost leaf process of its older sibling. Therefore, the child that receives the *AC* message will then be able to send a *B_ Connect* (or *BC*) message (rule 5) to the rightmost leaf process of its older (*i.e.* left) sibling. As a consequence, each process that receives a *BC* is connected to its successor (rule 6). The root process receives the identity of the rightmost leaf of the tree from its youngest (*i.e.* rightmost) child and establishes a *BC* connection with it.

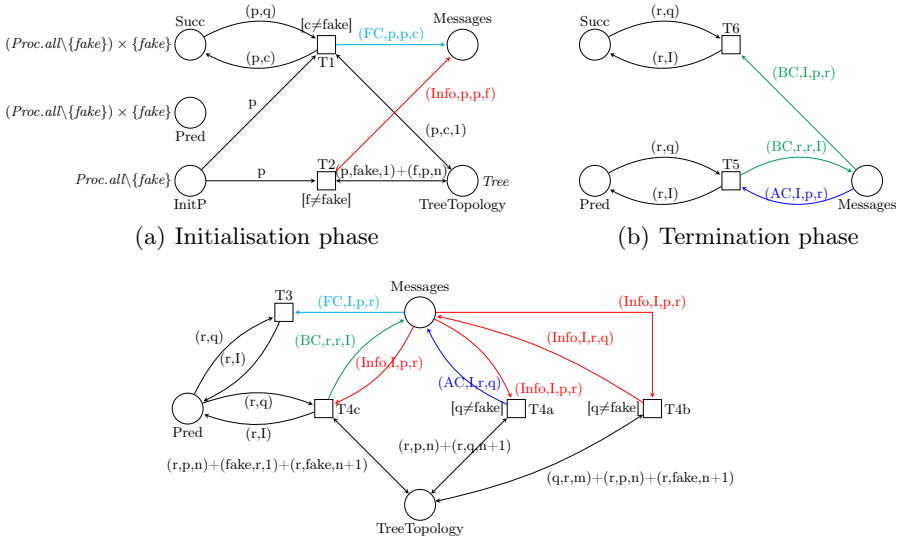
Eventually, all the leaves are connected to another process (figure 1(d), where the colours relate to the different transitions sequences of Section 6), and the set of *FC* and *BC* connections forms the ring.

3 A Coloured Petri Net Model

Algorithm 1 was proved to be self-stabilising in [8], but the proof was lengthy and cumbersome. Moreover, the algorithm enjoys additional properties that were not established before, such as being silent.

Therefore, we build a formal model of the algorithm which provides the following features:

- a graphical representation for an easier and better understanding of the system's behaviour;
- a global view of the system states at a glance, allowing to focus on the flow of messages;
- facilities for property analysis and formal reasoning.



(a) Initialisation phase

(b) Termination phase

(c) Main phase

Fig. 2: Model corresponding to Algorithm 1.

A coloured Petri net (CPN) model [22, 21] is thus designed, capturing the flow of information and the individual actions in its graph, and the actual essential values in the data it manipulates. For the sake of readability, it is presented in 3 parts (Figures 2(a), 2(c) and 2(b)) corresponding to the initialisation phase, the core of the algorithm, and the termination phase, respectively. Should it be presented in a single figure, the places with the same name would be fused together. Also note that the arcs connected to place *Messages* (and the associated arc inscriptions) are coloured according to the messages they handle. This has no formal meaning but enhances readability, which is particularly useful when illustrating message flow in the proofs.

3.1 Data Types Declaration

The type declarations in Figure 3(a) show all data types that are to be used in the CPN. First, the processes are identified as members of a set `Proc` of processes (of the form {`P1,...`}). This set also includes a particular `fake` process that is used to denote that the parent or child of a process does not exist (when it is respectively the root or a leaf). This corresponds to $\perp$ in the algorithm. Type `2Proc` is a pair of process names. Then `MessType` describes all four types of messages: `FC`, `AC`, `BC`, and `Info`. A message also contains a process identifier, its sender and its receiver.

The algorithm we model makes use of the tree topology with parent and child relation plus the next child in a parent's list. To model this, we use triples consisting of the parent, the child, and the number of the child in the list of children. The `fake` child is always the last one in the list, thus denoting its end.

For example, the tree in Figure 1(a) is modelled by the set of triples:

```
Proc = set of processes U {fake};
2Proc = Proc x Proc;
MessType = {FC,AC,BC,Info};
Mess = MessType x Proc x Proc x Proc;
TreeStructure = Proc x Proc x Int;
```

(a) Type declarations

```
InitP: Proc;
Pred, Succ: 2Proc;
Messages: Mess;
TreeTopology: TreeStructure;
```

(b) Places declarations

```
Tree = initial topology
c, f, I, p, q, r: Proc;
m, n : Int;
```

(c) Variables and constants declarations

Fig. 3: CPN declarations

{(fake,P0,1),(fake,fake,2),(P0,P1,1),(P0,P2,2),(P0,fake,3),
(P1,P3,1),(P1,P4,2),(P1,fake,3),(P2,P5,1),(P2,fake,2),(P3,P6,1),
(P3,P7,2),(P3,P8,3),(P3,fake,4),(P4,P9,1),(P4,fake,2),(P5,fake,1),
(P6,fake,1),(P7,fake,1),(P8,fake,1),(P9,fake,1)}.

3.2 Initialisation Phase

The initial phase of the algorithm is modelled by the CPN of Figure 2(a). It describes all the necessary places with their initial marking (in italics) of the type described in the places declarations in Figure 3(b). Figure 3(c) shows the types of variables used in arc expressions as well as the initial `Tree` topology constant.

At the start, there is no message, all processes (except the fake one) are ready to start. No process knows a possible successor or predecessor in the ring, hence is associated with the fake process in the `Pred` and `Succ` places. The tree topology to be processed is described by the constant `Tree`.

The initialisation of the `Pred` and `Succ` places with the specific `fake` process models the fact that, as stated in section 1, self-stabilising algorithms can start from *any arbitrary initial state*. In our case, we are initialising the predecessor and successor of each process in the ring with bogus values.

Transition `T1` models rule 1 in the algorithm. A process `p` with `c` as first child is processed, sending an `FC` message with its identity to this child, and updating its successor information with this child's identity.

Every leaf process `p` executes rule 2 of the algorithm, as depicted by transition `T2`. It is a leaf if described by `(p,fake,1)` in the tree topology, and it is the child number `n` of some (non-fake) parent process `f` (`(f,p,n)` in the tree). It then sends an `Info` message with its identity to its parent.

3.3 Main Phase

The CPN in Figure 2(c) describes the core part of the algorithm, *i.e.* the processing of `FC` and `Info` messages. Transition `T3` handles an `FC` message, as rule 3 of the algorithm by updating the predecessor information of the receiver of the

message. Rule 4 is decomposed into 3 transitions corresponding to the different possible receiver configurations:

`T4a:` relative to receiver `r`, the sending child has a next sibling `q` to whom the received information is forwarded as an `AC` message;

`T4b:` relative to receiver `r`, the sending child is the last in the list (*i.e.* the youngest sibling) and the receiving node is not the root (*i.e.* it has a parent `q` which is not the `fake` one) to which it forwards the `Info` message;

`T4c:` relative to receiver `r`, the sending child is the last in the list (*i.e.* the youngest sibling) and the receiver is the root (*i.e.* it has no parent—its parent is the `fake` one). It updates its predecessor with the information `I` received and sends a `BC` message with its own identity to process `I`.

3.4 Termination Phase

Finally, the termination phase, shown in Figure 2(b) handles the `AC` and `BC` messages, using transitions `T5` and `T6` respectively. In case of an `AC` message, the predecessor information of receiver `r` is updated with the content `I` of the message. It also sends a `BC` message to this process `I` with its identifier `r`. When a `BC` message is handled, only the successor information of the receiver `r` is updated with the identity `I` carried by the message.

4 A Simplified Coloured Petri Net Model

In this section, we first simplify the CPN model, which then makes it easier to exhibit its invariant properties.

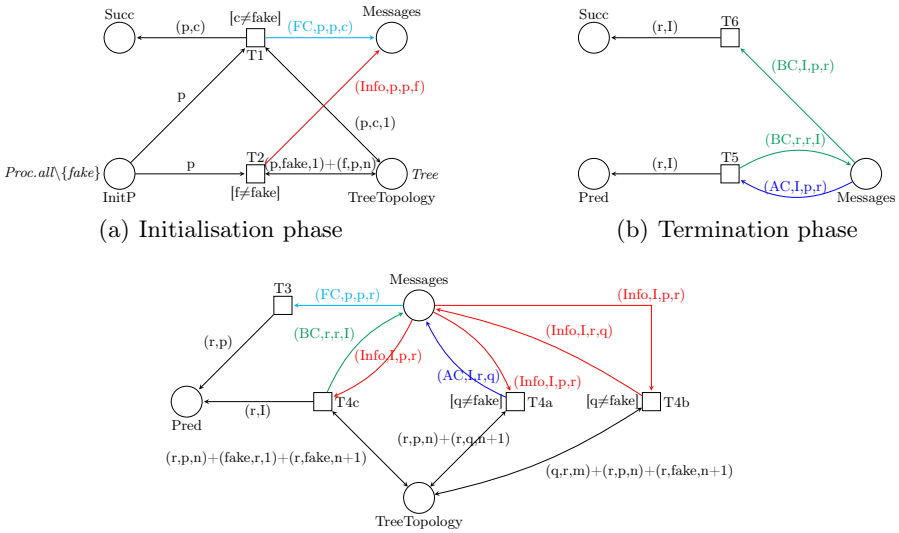
4.1 The Simplified Model

The simplified CPN is given in Figures 4(a), 4(c), 4(b). First, both places `Pred` and `Succ` initially contained bogus values associated with each process (*i.e.* `(p,fake)` for a process `p`), which was then discarded to be replaced by the actual values. In the new net, these two places are initially empty, and the only operation now consists in putting in these places the predecessor and successor values produced.

Second, an `FC` message is only produced by transition `T1`, and thus has the form `(FC,p,p,c)`. Thus the information `I` carried by the message is the identity of the sender process `p`, `I`=`p`. Hence, transition `T3` (the only one for the reception of `FC` messages) is modified by using `p` only.

4.2 Invariants of the simplified CPN

The Petri net model allows us to identify various properties of the system. Of interest here are the *place invariants* [22, 21]. These identify weighted sums of tokens which remain invariant under the firing of any transition. We use projection functions such as π_2 to select the second element of a token which has a



(a) Initialisation phase

(b) Termination phase

(c) Main phase

Fig. 4: Simplified model

tuple value, and $\pi_{2,4}$ to select the second and fourth elements, to form a pair. We also use a function notation to select elements of a particular type, thus *Messages*(`FC`) is the set of `FC` messages in place `Messages`.

It is possible that some of these invariant properties could be extracted automatically from the model by a suitable tool, while others could at least be checked automatically. These properties may then be of assistance in proving more involved results for our system.

When verifying properties of a modelled system, it is important to *validate* the model, *i.e.* show that it is an accurate model. In this regard, we note that the CPN model does not introduce any new information, but it does make explicit certain values (like the sender and receiver of a message) which then make it easier to prove the invariant properties. Another important issue for validating a distributed algorithm is that the model does not have one process inadvertently accessing information that is local to another process. In the case of our model, we need to ensure that each firing of a transition is relevant to only one process and does not access information local to another process. We note the following properties of each transition:

T1 fires for process p and accesses its first child and generates its *Succ* entry.
T2 fires for process p and accesses its parent.
T3 fires for process r and generates its *Pred* entry.
T4a fires for process r and accesses its children.
T4b fires for process r and accesses its parent and its children.
T4c fires for process r and checks that it is the root and generates its *Pred* entry based on the received message.
T5 fires for process r and generates its *Pred* entry based on the received message.
T6 fires for process r and generates its *Succ* entry based on the received message.

Having convinced ourselves that the model accurately captures a distributed system, we now consider the properties of the model.

Property 1 $\texttt{InitP} + \pi_1(\texttt{Succ}) + \pi_2(\mathit{Messages}(\texttt{Info})) + \pi_2(\mathit{Messages}(\texttt{AC})) + \pi_4(\mathit{Messages}(\texttt{BC})) = \texttt{Proc} \setminus \{\texttt{fake}\}$

Proof. Initially, we have no messages and $\texttt{InitP} = \texttt{Proc} \setminus \{\texttt{fake}\}$. Then, we can consider each transition in turn:

T1 The initialisation of a parent removes an item from `InitP` and adds a `Succ` entry with the same identity.

T2 The initialisation of a leaf removes an item from `InitP` and adds an `Info` message with the relevant identity.

T4a This consumes an `Info` message and generates an `AC` message with the same identity.

T4b This consumes one `Info` message and generates another with the same identity.

T4c This consumes an `Info` message and generates a matching `BC` message.

T5 This consumes an `AC` message and generates a matching `BC` message for the destination given by the identity.

T6 This consumes a `BC` message and adds a `Succ` entry for the receiver. □

Property 2 $\texttt{Succ} + \pi_{4,2}(\mathit{Messages}(\texttt{BC})) = \pi_{3,4}(\mathit{Messages}(\texttt{FC})) + \pi_{2,1}(\texttt{Pred})$

Proof. Initially, there are no messages and places `Succ` and `Pred` are empty. Subsequently, we consider the relevant transitions in turn:

T1 The setting of `Succ` is paired with the generation of an `FC` message.

T3 The consumption of an `FC` message is paired with the addition of a `Pred` entry.

T4c The setting of `Pred` is paired with the generation of a `BC` message.

T5 The setting of `Pred` is paired with the generation of a `BC` message.

T6 The consumption of a `BC` message is paired with the addition of a `Succ` entry. □

4.3 Liveness of the simplified CPN

We now summarise some liveness properties of the CPN.

Property 3 *For any tree with at least two nodes, either transition* `T1` *or transition* `T2` *can fire for every node. Thus,* `InitP` *will eventually be empty.*

Proof. A tree with at least two nodes will have a root and at least one leaf. Thus:

1. `T1` can fire for every node which is *not* a leaf, *i.e.* for every node which has a non-fake child.
2. `T2` can fire for every leaf, *i.e.* for every node which has a non-fake parent. □

Property 4 *All messages can eventually be uniquely consumed.*

Proof. We consider the different kinds of messages in turn:

FC The only constraint on the consumption of `FC` messages by transition `T3` is that the identity and the source of the message are the same. This is guaranteed by the generation of `FC` messages in transition `T1`.

Info Every `Info` message can be consumed by one of the transitions `T4a`, `T4b` or `T4c`. Transition `T4a` can consume `Info` messages from node `p` for parent `r` provided `p` has a younger sibling. Transition `T4b` can consume `Info` messages from node `p` for parent `r` provided `p` has no younger sibling and `r` has a (non-fake) parent. Transition `T4c` can consume `Info` messages from node `p` for parent `r` provided `p` has no younger sibling and `r` is the root.

AC There is no constraint on the consumption of `AC` messages by transition `T5`.

BC There is no constraint on the consumption of `BC` messages by transition `T6`.

Note that in each case, exactly one transition can consume each kind of message. □

Property 4 guarantees that the algorithm is *silent*. Once a legitimate configuration has been reached (*i.e.* once the ring has been established), there is no pending message held in the communication channels of the system. A silent algorithm is an algorithm in which, upon a certain point of its execution, the contents of the communication channels remain the same [16]. In our case, no message is sent between processes once the system has stabilised, as mentioned in the introduction of this paper. Also, as stated before, the algorithm is restarted in the event of failure.

5 Algorithm Termination

Definition 1. *We define the weight of the state as follows:*
- *for each node prior to sending its first message:* *weight(node) = 3 + depth(node)*
- *for each node after sending its first message: weight(node) = 0*
- *for each* `FC` *message: weight(*`FC`*) = 1*
- *for each* `BC` *message: weight(*`BC`*) = 1*
- *for each* `AC` *message: weight(*`AC`*) = 2*
- *for each* `Info` *message: weight(*`Info`*) = 3 + depth(target)*

Then the total weight of the state is given by: $Weight = \Sigma_{x \in node \cup msg} weight(x)$.

Note that the weight of a state is always positive if there are any nodes yet to send their first message or any messages to deliver, or else zero when there are none. As a consequence, the *weight* function has separate points and positivity properties. Absolute homogeneity and triangle inequality properties are not relevant in our context. Therefore, the *weight* function is a *norm* on the states (as introduced in Section 1).

Proposition 1. *Given the state of the algorithm, the weight of the state decreases at every step.*

Proof. We consider each possible rule in turn:

rule 1 : The weight of the node is set to zero and the number of `FC` messages is increased by 1. Hence *Weight* is decreased by $2 + depth(node)$.

rule 2 : The weight of the node is set to zero and an `Info` message is generated for the parent. Hence *Weight* is decreased by 1.

rule 3 : The number of `FC` messages is decreased by 1, and hence *Weight* decreases by 1.

rule 6 : The number of `BC` messages is decreased by 1, and hence *Weight* decreases by 1.

rule 5 : The number of `AC` messages is decreased by 1 and the number of `BC` messages is increased by 1. Hence *Weight* decreases by 1.

rule 4a : The `Info` message is received by a parent with other children, and an `AC` message is sent to the next sibling. Consequently, *Weight* is decreased by at least 3, and increased by 2, *i.e.* it is decreased by at least 1.

rule 4b : The `Info` message is passed to the parent, and hence the depth of the target is decreased by 1. Hence $Weight$ is decreased by 1.

rule 4c : The `Info` message received by the root node (with depth 0) is replaced by a `BC` message. Hence *Weight* is decreased by at 3 and increased by 1, *i.e.* it is decreased by 2. □

Property 5 *The algorithm terminates and is self-stabilising.*

Proof. Following initialisation, every execution step of the algorithm involves at least one of the above rules. Thus, *Weight* is *strictly* monotonic, *i.e.* it is decreased by at least one while remaining positive. Consequently, the algorithm terminates. Moreover, as stated in section 1, if the norm function *Weight* is strictly monotonic, then the algorithm is self-stabilising (proof by norm). □

6 Algorithm Correctness

Proposition 2. *The algorithm establishes* `Succ` *and* `Pred` *as mirror images, i.e.* `Succ`$=\pi_{1,2}($`Pred`$)$.

Proof. This follows directly from properties 2 and 4. □

Proposition 3. *The algorithm establishes predecessors of nodes as:*

- *predecessor(node) = parent of node (case 1: node is the oldest child)*
- *predecessor(node) = preceding-leaf of node (case 2: node is not the oldest child and not the root)*
- *predecessor(node) = last-leaf (case 3: node is the root)*

Proof. We consider each possible case in turn. They correspond to the coloured arcs in Figure 1(d) (red, green and blue arcs respectively).

case 1—firing sequence T1T3 : Every non-leaf node generates an `FC` message to its oldest child (T1), which sets the parent to be its predecessor (T3), as required.

case 2—firing sequence T2T4b*T4aT5[T6] : Every leaf generates an `Info` message (T2) which is passed up the tree (T4b) till it finds a sibling. That sibling is sent an `AC` message (T4a) with the identity of the leaf (from the `Info` message). The `AC` message sets the predecessor of the sibling to be the originating leaf (T5), which is the preceding leaf in the tree.

case 3—firing sequence T2T4b*T4c[T6] : Every leaf generates an `Info` message (T2) which is passed up the tree (T4b) till it reaches the root, which

will be the case for the last leaf. In this case, the predecessor of the root is set to the last leaf (T4c).

Thus for each possible firing sequence, the `Pred` values are set as required, and Proposition 2 tells us that the `Succ` values are also set as required. □

Proposition 4. *The algorithm sets the* `Succ` *and* `Pred` *values so there is one connected component.*

Proof. For the purposes of the proof, we consider that a node is connected to the tree until its `Succ` and `Pred` values are set. In this way, the algorithm starts with a tree, which is a single connected component. Then we need to show that every time the `Succ` and `Pred` values are changed, then the node is still connected to the one component. Thus, we have:

- Every oldest child is connected to the parent (by the `Succ` and `Pred` values), which reflects the tree structure, and therefore does not modify the connectedness.
- Every younger sibling is connected to the preceding leaf. Since a leaf has no child, connecting it to another node does not jeopardise the connectivity of the structure. In other words, provided the leaf is connected to the rest of the component, then so is the younger sibling.
- The above items result in a connected structure with the last leaf without a successor. This last leaf is connected to the root.

Thus the algorithm sets the `Succ` and `Pred` values to be a single connected component. □

Property 6 *The algorithm produces a ring topology.*

Proof. It suffices to have a single connected component where all nodes have only one predecessor and one successor. The connectedness is stated by Proposition 4. In the terminal state, there is no message left and `InitP` is empty. Thus, from Property 1, we deduce $\pi_1(\texttt{Succ}) = \texttt{Proc} \setminus \{\texttt{fake}\}$. Similar to the cases in the proof of Proposition 4, the different possibilities entail that $\pi_1(\texttt{Pred}) = \texttt{Proc} \setminus \{\texttt{fake}\}$.

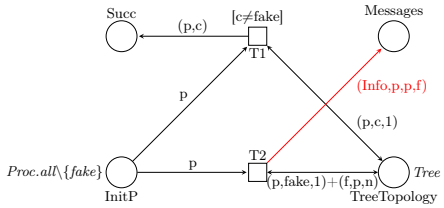
□

7 Slight Simplifications of the Algorithm

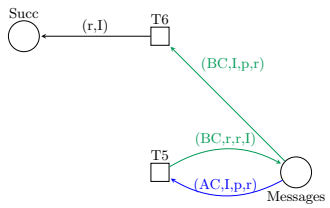
From the properties proved in the previous sections, we can again simplify the model and reflect these simplifications in the algorithm.

7.1 New Models

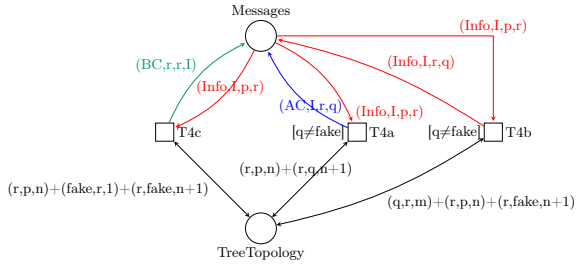
The topology obtained by the predecessor and successor information is a ring, and these are just mirror images of one another. It is thus not necessary to keep them both. So, let us remove place `Pred` from the model of Figures 4(c) and 4(b). In the resulting net, transition `T3` only discards `FC` messages. Since no other transition handles these messages, they are also unnecessary. Therefore, we remove the arc producing `FC` messages from the net in Figure 4(a).



(a) Initialisation phase



(b) Termination phase



(c) Main phase

Fig. 5: Successor-based CPN model

The resulting net is depicted in Figures 5(a), 5(c) and 5(b). The figures are meant to show the modifications on the structure of the net, and the inscriptions are not altered.

Additional simplifications are not possible: even though we could be tempted to get rid of `AC` messages that are immediately transformed into `BC` messages by transition `T5`. This modification would not be sound. Indeed, the successor information for a process `p` must be updated by process `p` itself. This is obviously the case for transition `T1`. The same holds for transition `T6` where process `r` receives a `BC` message and updates its successor information. However, if transition `T4a` were to send immediately a `BC` message, it should be `(BC,q,q,I)` (to be the same as the one generated by `T4aT5`). But then transition `T4a` would handle reception of an `Info` message by process `r`, as well as the sending of a message by its child `q`, hence not the same process and thus not consistent.

We could equally well decide to remove place `Succ` and keep place `Pred`. In this case, `FC` messages remain while `BC` messages are no more necessary. Hence transition `T6` is also deleted. In order to keep this paper compact and to avoid redundancy, it can be found in [12].

7.2 New Algorithms

Algorithm 2 shows the corresponding simplified algorithm, where variable `Pred` has been removed, as well as `FC` messages. Note that rule 3 is not necessary anymore, and that some parts of the algorithm are more balanced.

Indeed, the initialisation part is such that leaf processes send `Info` messages while the others only update their successor value (rules 1 and 2), while in the end (rules 5 and 6) leaf processes update their successor value and the others send

Algorithm 2: Successor-based algorithm

Constants:
$Parent : \mathcal{ID}$
$Children : List(\mathcal{ID})$
$Id : \mathcal{ID}$
Output:
$Succ : \mathcal{ID}$
Initialisation:
1- $Children \neq \emptyset \rightarrow$
$Succ = First(Children)$;

2- $Children = \emptyset \rightarrow$
Send $(Info, Id)$ **to** $Parent$;

Run:
4- **Recv** $(Info, I)$ **from** $p \rightarrow$
if $p \in Children$ **then**
let $q = next(p, Children)$;
if $q \neq \perp$ **then**
Send $(Ask\_Connect, I)$ **to** q ;
else
if $Parent \neq \perp$ **then**
Send $(Info, I)$ **to** $Parent$;
else
Send $(B\_Connect, Id)$ **to** I ;
end if
end if
end if

5- **Recv** $(Ask\_Connect, I)$ **from** $p \rightarrow$
Send $(B\_Connect, Id)$ **to** I ;

6- **Recv** $(B\_Connect, I)$ **from** $p \rightarrow$
$Succ = I$;

a message to a leaf. The main part (rule 4) also features a balanced treatment of `Info` messages: all types of process send a single message only.

The predecessor-based version of this algorithm is similar, for the case where variable `Succ` has been removed, as well as `BC` messages. In order to keep this paper compact and to avoid redundancy, it can be found in [12].

7.3 The algorithms comparison

Both new algorithms send less messages than the original since, in each case, there is one type of message which is no longer used. Let m_k be the number of messages sent by algorithm k, n_l the number of leaf nodes and n the total number of nodes in the tree. We have: $m_2 = m_1 - 2(n - n_l)$ and $m_3 = m_1 - 2n_l$.

Therefore, the algorithm to apply depends on the structure of the tree: if there are more leaf nodes than other nodes, the predecessor-based version of the algorithm is preferred; otherwise, we prefer the successor-based version of the algorithm (Algorithm 2).

8 Experimental confirmation of the Algorithm

While the formal results presented above in the paper can stand on their own, we confirmed the results experimentally. The *CosyVerif* tool [1] provides a graphical front end built on the Eclipse framework [14] for a range of dynamic system formalisms. It also supports a range of backend analysis tools. We prepared a graphical model of the Petri net of Figures 4(a), 4(b) and 4(c) in *CosyVerif* and we analysed the state space using the *prod* reachability analyser [23].

The example of Figure 1(a) was entered as the initial marking and *prod* reported that the state space consisted of 1,275,750 nodes, 9,470,925 arcs with

one terminal node. The terminal node was manually examined to confirm that it represented the appropriate ring structure.

Further validation is considered in the following subsections.

8.1 Exploring different topologies with a pre-initialisation phase

As described in Section 4, the algorithm was modelled as a Petri net with an *Initialisation*, *Main* and *Termination* phases. Rather than just considering one topology, we introduced a pre-initialisation phase to generate an arbitrary tree topology from a given number of nodes. The Petri net segment for this is given in Figure 6.

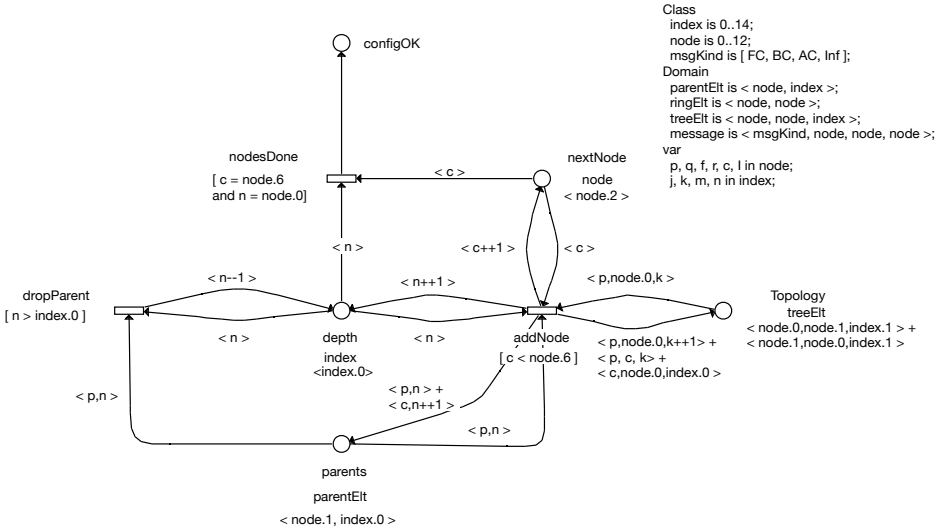


Fig. 6: Pre-initialisation phase

The pre-initialisation phase was designed so that each topology would be generated with a unique labelling of nodes. Further, the labels would be such that a depth-first traversal of the tree would result in a ring with node labels in increasing numeric order. To achieve this, the last node added to the tree is stored in place *parents* together with its parents and their associated depths in the tree. The depth of the last node added is given in place *depth*. In order to add another node — with the numeric label given in place *nextNode* — there are two possible options: either the new node is added as a child of the last node added (with the transition *addNode*) or the last node can be discarded (with transition *dropParent*) and its immediate parent becomes a possible candidate for the parent of the new node. Transition *dropParent* can drop all but the first node, in which case the next node to be added will be a child of the root. Transition *addNode* can fire for all nodes up to a given node number. When that node number is reached, transition *nodesDone* can fire and add a token to

Table 1: State space results for self stabilisation algorithm.

		Initial state space				Reduced state space			
Nodes	Topologies	Pre	Nodes	Arcs	Sec	Pre	Nodes	Arcs	Sec
2	1	0.001	11	17	0.001	0.006	5	5	0.263
3	2	0.001	103	229	0.004	0.014	17	17	1.078
4	5	0.001	1,123	3,314	0.057	0.038	60	60	6.210
5	14	0.002	13,474	49,807	0.983	0.176	217	217	48.607
6	42	0.006	172,248	766,076	17.835	0.976	798	798	381.137
7	132	0.010	2,301,624	11,968,306	328.869	6.220			>1800
8	429	0.063	>10M	>50M	>1800	39.042			
9	1430	0.230				232.136			
10	4862	0.902				1802.274			

place *configOK*. This place then becomes a side condition of the initialisation transitions *T1* and *T2* in Figure 4(a).

With the above pre-initialisation phase, we obtain the state space results of Table 1. The first column indicates the number of nodes in the tree to be processed. The column labelled *Topologies* indicates the number of tree topologies which can be generated with that number of nodes (as given by the pre-initialisation phase above). The column labelled *Pre* gives the time taken to execute the pre-initialisation phase for that number of nodes. The next three columns give the total number of nodes and arcs in the state space for all those topologies (combined) as well as the time to process those topologies in seconds.

With the pre-initialisation phase, each run of the Petri net will consider a number of topologies and that will be the number of terminal states. In order to ensure that the only difference between the terminal states is due to the different starting topologies, we can add a post-termination phase to remove the topology from the state, *i.e.* empty the tokens out of place *Topology*. It is a simple change to the net, and its addition does confirm that irrespective of the starting topology, there is only one terminal node with the correct ring structure.

8.2 State space reduction

In the algorithm we note that there are a number of messages exchanged between a node and what will eventually be its successor and predecessor as well as the intermediate nodes. We hypothesise that this set of messages is independent of the sets of messages exchanged between other pairs of nodes. In other words, the complexity of the algorithm is largely due to the arbitrary interleaving of the message transmission.

Accordingly, we experimented with reducing the state space with the *stubborn set technique* [26] which eliminates much of the interleaving while maintaining the terminal states. This form of reduction can be activated by running *prod* with option *-s*. In Table 1, the last four columns give similar results to the preceding four columns, but this time using the stubborn set technique to reduce the size of the state space. The numbers of nodes and arcs demonstrate that the technique retains only one interleaving, with the state spaces reduced to linear

graphs. Unfortunately, this comes at considerable computational cost which we now consider.

Firstly, we note that the *prod* tool applies the technique to the unfolded net — it unfolds the Coloured Petri Net into a Place Transition Net. The size of the unfolded net is determined not just by the transitions which can fire in the coloured state space but by the possible range of values for the tokens. Initially, we had allowed for up to 12 nodes and up to 14 children per node. Even for, say, 5 nodes, there would be transitions generated in the unfolded net for up to 14 children (even though there can be no more than 4). Consequently, our initial result was that none of our test cases — even the pre-initialisation phase — reached their terminal states in less than 30 minutes! Accordingly, the nets were modified to reduce the ranges of values for node labels and for child indices to be slightly larger than required for the example under consideration. With these modifications, we were able to reach terminal states for some of our test cases. Still, it is unclear whether this was good enough or whether there was still extensive unused net components in the unfolded net.

The second complexity of the stubborn set method is the computational penalty for computing the stubborn sets. This means that this reduction technique may or may not be effective in reducing the state space. The results clearly show that the size of the state space can be reduced but that the computational penalty can be overwhelming. Even the pre-initialisation phase can be very expensive.

9 Conclusions

This paper has demonstrated the benefits of using formal techniques in the analysis of a non-trivial distributed algorithm for converting a tree structure of processes (or processors) into a ring structure.

In such an exercise, the choice of formalism is significant. The Petri Net formalism has proved to be ideal because of its support for concurrency and the variability of sequencing and timing of concurrent processes. In particular, we did not need to make any assumptions about the synchronisation of the communications nor, when several transitions were enabled, their order of firing.

We built a model of the distributed algorithm and then validated it, *i.e.* ensured that it accurately reflected the modelled system. In our case, it was important to ensure that the model faithfully reflected the distributed nature of the algorithm. Thus, we examined each transition to ensure that it only accessed information local to a given process.

Having modelled and validated the system, we observed that without adding any new information, the making explicit of the source and target of each message facilitated the identification of some invariant and liveness properties. These were then utilised to prove termination, correctness of the algorithm, and that it was self-stabilising and silent. These properties could easily be exhibited on the model, but they are far from obvious when considering the algorithm itself.

Further, the above properties helped us to identify non-essential information which then allowed us to simplify the algorithm, leading to a more efficient one.

We also employed automated tools to explore the state space of the system. This validated our earlier results and confirmed that the complexity of the algorithm was due to the level of concurrency, which was reflected in the large state space. While this state space could be significantly reduced using the stubborn set technique, the cost of doing so quickly became prohibitive.

The approach adopted in this paper presents several advantages: first, proving invariant properties induces that the algorithm is correct whatever the initial tree topology; second, the encoding of the network topology is crucial, and the approach can be generalised to other algorithms provided a suitable encoding of the topology they address.

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