Convex circuit free coloration of an oriented graph

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Abstract

We introduce the *convex circuit-free coloration* and *convex circuit-free chromatic number* $\overrightarrow{\chi_a}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ and establish various basic results. We show that the problem of deciding if an oriented graph verifies $\chi_a(\overrightarrow{G}) \leq k$ is NP-complete if $k \geq 3$ and polynomial if $k \leq 2$. We exhibit an algorithm which find the optimal convex circuit-free coloration for tournaments, and characterize the tournaments that are *vertex-critical* for the convex circuit-free coloration.

Keywords: Oriented Chromatic Number, acyclic homomorphism, Vertex-Critical, tournament

1 Introduction

A convex subset is a vertex subset with the property that every 2 directed path beginning and ending inside the convex subset is contained completely within the subset. In this paper we investigate the coloration of an oriented graph $\overrightarrow{G}$ into convex subsets without circuit, referenced in the following by *CCF-coloration* for 'Convex Circuit-Free coloration'. If we color each subset with a different color, such a coloration appears as an extension of the notion of oriented coloring introduced by Sopena [8]. Indeed, within an oriented coloring, each monochromatic subgraph is more than without circuit but without arc (independent set). In the same way, as an oriented coloration may be defined by means of oriented homomorphism ([3, 8]), the CCF-coloration may be equivalently defined by the notion of *circuit-free homomorphism* (called *acyclic homomorphism* in [2]). A *circuit-free homomorphism* of a digraph $\overrightarrow{G}$ into a digraph $\overrightarrow{F}$ is a mapping ϕ from $V(\overrightarrow{G})$ to $V(\overrightarrow{F})$ such that:
(i) for every arc $(u, v) \in A(\overrightarrow{G})$, either $\phi(u) = \phi(v)$ or $(\phi(u), \phi(v))$ is an arc of $\overrightarrow{F}$,
(ii) for every vertex $v \in V(\overrightarrow{F})$, the induced oriented graph $\overrightarrow{G}(\phi^{-1}(v))$ is circuit-free.

An oriented graph $\overrightarrow{G}$ admits a k-CCF coloration if and only if there exists an oriented graph $\overrightarrow{F}$ of order k and a circuit-free homomorphism of $\overrightarrow{G}$ into $\overrightarrow{F}$. Such a minimal k is called *CCF-chromatic number* of $\overrightarrow{G}$ and denoted by $\overrightarrow{\chi_a}(\overrightarrow{G})$. That type of coloration

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was originally motivated by the search of structures in large majority voting tournaments ([5]).

Let us give some notations and definitions. All digraphs considered here are finite and have no loop or multiple edge. A circuit is a directed cycle. An oriented graph is a digraph without circuit of length two. In other words, it is an orientation of a simple graph. An oriented graph T is a tournament if and only if it is complete, ie. for every pair $\{i, j\}$ of vertices, (i, j) or (j, i) is an arc of T. Finally, for a graph having property $\mathcal{P}$ we say that G is vertex-critical for $\mathcal{P}$ if it loses the property $\mathcal{P}$ whenever any vertex is removed. The set of vertices and the set of arcs of a digraph $\overrightarrow{G}$ are respectively denoted by $V(\overrightarrow{G})$ and $A(\overrightarrow{G})$. If (x, y) is an arc of $\overrightarrow{G}$, then we say that x dominates y or y is a successor of x and that x is a predecessor of y. We shall use the notation $x \to y$ to denote this. We respectively denote by $\Gamma^+(x)$ and $\Gamma^-(x)$ the set of successors and the set of predecessors of x. The in-degree of a vertex x is the cardinal of $\Gamma^-(x)$, and the out-degree of x is the cardinal of $\Gamma^+(x)$. If A and B are disjoint subsets of $V(\overrightarrow{G})$ such that all arcs between A and B are directed toward B, then we use the notation $A \to B$ and said that the sets A and B verify the *unidirection property* or are *in unidirection*. For a subset B of $V(\overrightarrow{G})$, $\overrightarrow{G} \setminus B$ denotes the subdigraph of $\overrightarrow{G}$ obtained after removing the vertices of B and all the arcs with at least one extremity in B. The subdigraph induced by a vertex subset B of $\overrightarrow{G}$ is define as $\overrightarrow{G} \setminus \left(V(\overrightarrow{G}) \setminus B\right)$ and is denoted by $\overrightarrow{G}(B)$.

The paper is organized in two parts. In the first one, we prove that the minimization problem of finding the smallest integer k such that G has a CCF-coloration in k classes is of polynomial complexity if G is a tournament and NP-complet in the general case. In a second part we focus on the CCF-indecomposable tournaments, that is tournaments for which the previous k is equal to the number of vertices. That class is large since the probability that a tournament belongs to it tends toward one when the number of its vertices goes to the infinity. Here, we caraterize tournaments that are CCF-indecomposable and critical for that property.
Questions related to the minimum subsets of a CCF-coloration are also closed in their formulation to those of the dichromatic number [1]. The dichromatic number is calculated to avoid monochromatic circuits since a CCF-coloration is caraterized by the absence of dichromatic circuits. In fact the CCF-coloration may be seen as the satisfaction of two properties on the subsets : circuit-free and convexity. In the particular case of tournaments both of these properties have been studied separately by several authors. In the case of tournaments circuit-free subsets are the transitive ones and [6] characterizes some critically r-dichromatic tournaments. Such tournaments have a partition of its vertex set in at least r transitive subsets and are critical for that property. Concerning the convexity, in the case of tournaments, convex subsets are also called intervals or modules. In [7] are studied indecomposable tournaments that is tournaments which convex subsets are the singletons, the empty set and the whole vertex set. Indecomposable tournaments are CCF-indecomposable. In [4], the author characterizes indecomposable tournaments that are critical.

2 Complexity of the CCF-chromatic number problem

For the oriented chromatic number, the threshold between the "easy" and the "hard" computable oriented chromatic number is between 3 and 4. For the CCF-coloration, deciding whether the CCF-chromatic number is less or equal to 3 is already NP-complete.

Let k be a fixed positive integer. The k-**CCF Col** problem is the following decision problem:

> k-**CCF Col** (CCF-chromatic number $\leq k$).
> *Instance:* An oriented graph $\overrightarrow{G}$.
> *Question:* Does $\overrightarrow{G}$ admit a k-CCF coloration ?

We first note that an oriented graph $\overrightarrow{G}$ admits a 1-CCF coloration if and only if $\overrightarrow{G}$ is circuit-free. Moreover, if $\overrightarrow{G}$ admits a 2-CCF coloration then $\overrightarrow{G}$ is circuit-free and admits a 1-CCF coloration. Hence 1-**CCF Col** and 2-**CCF Col** can be solved in polynomial time.

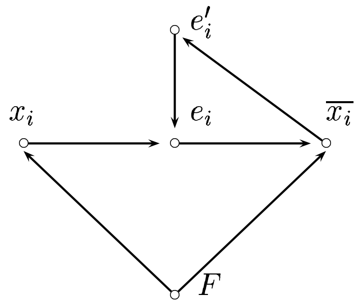
Theorem 1 *The decision problem* 3-***CCF Col*** *is NP-complete, even if the input is restricted to connected oriented graphs.*

Proof: It is clear that the 3-**CCF Col** problem belongs to NP. To show its NP-completness, we shall describe a polynomial-time reduction from 3-**Sat** to 3-**CCF Col**.

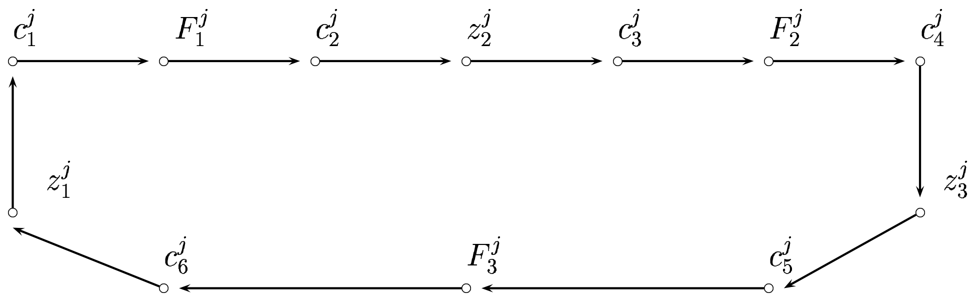
Let us consider an instance $(X, \mathcal{C})$ of 3-**Sat**, where $X = \{x_1, x_2, ..., x_n\}$ is a set of boolean variables and $\mathcal{C} = \{C_1, C_2, ..., C_m\}$ contains m clauses of 3 literals (the set of literals is denoted by $\mathcal{L} = \bigcup_{1 \leq i \leq n}\{x_i, \overline{x_i}\}$). The clause C_j is denote by $z_1^j \vee z_2^j \vee z_3^j$, where $\{z_1^j, z_2^j, z_3^j\} \subset \mathcal{L}$. Since we may assume that no clause is a tautology (ie. contains x_i and $\overline{x_i}$), we will consider that the indexes of literals of any clause are strictly increasing.

To such an instance of 3-**Sat**, we associate the following oriented graph $\overrightarrow{G}$:
$V(\overrightarrow{G}) = \bigcup_{1 \leq i \leq n}\{x_i, e_i, e'_i, \overline{x_i}\} \cup \bigcup_{1 \leq j \leq m}\{c_1^j, c_2^j, c_3^j, c_4^j, c_5^j, c_6^j, F_1^j, F_2^j, F_3^j\} \cup \{T, F, I\}$.
The arc set of $\overrightarrow{G}$ is the union of four types of arcs:
First type: For all integer $i \in \{1, 2, ..., n\}$, we have the set of arcs

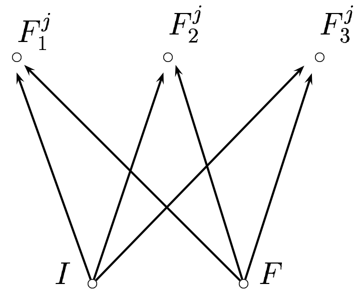
$$\{(e'_i, e_i), (e_i, \overline{x_i}), (\overline{x_i}, e'_i), (x_i, e_i), (F, x_i), (F, \overline{x_i})\}.$$



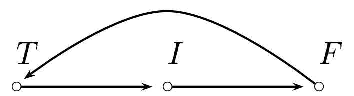
Second type: For all $j \in \{1, 2, ..., m\}$, for $C_j = z_1^j \vee z_2^j \vee z_3^j$, we get a copy of the oriented graph $\overrightarrow{K_j}$, identifying the vertices z_1^j, z_2^j, z_3^j to vertices in $\bigcup_{1 \leq i \leq n} \{x_i, \overline{x_i}\}$:



Third type: For all $j \in \{1, 2, ..., m\}$, we have: $\{I, F\} \rightarrow \{F_1^j, F_2^j, F_3^j\}$. Then, we obtain a copy of the following oriented graph:



Fourth type: The induced oriented graph $\overrightarrow{G}(\{V, F, I\})$ is isomorphic to:



The construction of $\overrightarrow{G}$ may be carried out in polynomial time. We claim that $\overrightarrow{G}$ is 3 CCF-decomposable if and only if the clauses $C_1, C_2, ..., C_m$ are simultaneously satisfiable.

Let us suppose that the oriented graph $\overrightarrow{G}$ admits a 3-CCF-coloration. The arcs of the fourth type imply that there exists a circuit-free homomorphism ϕ from $\overrightarrow{G}$ to the 3-circuit $(1, 2, 3)$. Without loss of generality, we may assume that $\phi(T) = 1$, $\phi(I) = 2$ and $\phi(F) = 3$. The arcs of the first type imply that, for each i in $\{1, 2, ..., n\}$, $\{\phi(x_i), \phi(\overline{x_i})\} = \{1, 3\}$. Since the vertices $\{F_l^j\}_{\substack{1 \leq j \leq m \\ 1 \leq l \leq 3}}$ are successors of I and F, then $\forall j \in \{1, ..., m\}, \forall l \in \{1, 2, 3\}$, $\phi(F_l^j) = 3$. Given an integer j in $\{1, 2, ..., m\}$, let us suppose that $\phi(z_1^j) = \phi(z_2^j) = \phi(z_3^j) = 3$, then, for all $l \in \{1, 2, ..., 6\}$, $\phi(c_l^j) = 3$. Then, $K_j \subset \phi^{-1}(3)$, which contradicts the fact that ϕ is a circuit-free homomorphism.

Then, at least one of the vertices $\{z_1^j, z_2^j, z_3^j\}$ is in the monochromatic class $\phi^{-1}(1)$.

The truth distribution $\mathcal{T} : X \to \{\texttt{True}, \texttt{False}\}$ define by

$$\begin{cases} \mathcal{T}(x_i) = \texttt{True} & \textit{if} \quad \phi(x_i) = 1, \\ \mathcal{T}(x_i) = \texttt{False} & \textit{if} \quad \phi(x_i) = 3 \end{cases}$$

satisfies all the clauses $\{C_j\}_{1 \leq j \leq m}$ of the **3-Sat** instance.
Conversely, suppose that $\mathcal{T} : X \to \{\texttt{True}, \texttt{False}\}$ is a satisfying truth assignment for the clauses $C_1, C_2, ..., C_m$. Then, we define the circuit-free homomorphism ϕ from $V(\overrightarrow{G})$ into the set of vertices of the 3-circuit $(1, 2, 3)$ by $\phi(T) = 1$, $\phi(I) = 2$ and $\phi(F) = 3$.

$$\begin{cases} \text{if } \phi(x_i) = \texttt{True} \quad \text{then } \phi(x_i) = 1 \text{ and } \phi(\overline{x_i}) = 3; \\ \qquad\qquad\qquad\qquad\quad \text{else } \phi(x_i) = 3 \text{ and } \phi(\overline{x_i}) = 1 \end{cases}$$

For every integer $j \in \{1, 2, ..., m\}$, $\phi(F_1^j) = \phi(F_2^j) = \phi(F_3^j) = 3$, and, for $k \in \{1, 2, 3\}$, if $\phi(z_k^j) = 3$ then $\phi(c_k^j) = 3$ else $\phi(c_k^j) = 1$.

Such a mapping is a 3 circuit-free homomorphism from $\overrightarrow{G}$ to the 3-circuit $(1, 2, 3)$, and then $\overrightarrow{G}$ admits a 3-CCF coloration. □

3 The case of tournaments

In this section, we investigate the complexity of the k-**CCF Col** problem over the family of tournaments. Let $T = (V, A)$ be a tournament and x a vertex of T. We define, if it exists, the *highest successor* of x as the vertex of $\Gamma^+(x)$ which dominates all other vertices of $\Gamma^+(x)$. Such a vertex, denoted by x^+ is characterized by the equality $\Gamma^+(x^+) = \Gamma^+(x) \setminus \{x^+\}$. Given a tournament T and a vertex x, we can compute x^+ in polynomial time by the following greedy algorithm:

Down(x)
Input: A tournament T and a vertex x of T.
Output: A vertex y such that $y = x^+$ if it exists, $\emptyset$ if not.

We denote by $\{y_1, y_2, ..., y_k\}$ the set $\Gamma^+(x)$, $i = 1$ and $x^+ = \emptyset$.
While $i \leq k$ Do:
  If y_i verifies $\Gamma^+(y_i) = \Gamma^+(x) \setminus \{y_i\}$, then $x^+ = y_i$ and $i = k + 1$;
  Else $i = i + 1$.
Return(x^+).

Proposition 1 *Let T be a tournament of order n with $\overrightarrow{\chi_a}(T) = k$ and x a vertex of T.*
(i) If there is a k-CCF coloration c of T such that x is not the smallest vertex in its monochromatic class then x^+ exists.
(ii) Conversely, if x^+ exists then for all convexe circuit-free k-coloration c of T, $c(x) = c(x^+)$.

Proof: (i) Let $C_x = c^{-1}(c(x))$ be the CCF-monochromatic class of x, and let us suppose that x is not the smallest vertex of C_x. The intersection of the induced subdigraphs $\overrightarrow{G}(C_x)$ and $\overrightarrow{G}(\Gamma^+(x))$ is a non-empty order. Let y be the highest vertex of this order, since any other CCF-monochromatic class is in unidirection with C_x, we have $\Gamma^+(y) = \Gamma^+(x) \setminus \{y\}$. Then y is the highest successor of x and $y = x^+$.
(ii) Let c be a k-CCF coloration of T and let us suppose, for contradiction, that $y = x^+$ and $c(x) \neq c(y)$. We denote by C_1 and C_2 the color classes of x and y respectively, and by $\{C_j\}_{3 \leq j \leq k}$ the other CCF-monochromatic classes of T. We have $x \to y$ and by convexity $C_1 \to C_2$. Moreover, since $y = x^+$, we have: $\forall j \in \{3, 4, ..., k\}$, $[C_1 \to C_j] \Leftrightarrow [C_2 \to C_j]$. Consequently, the $(k-1)$-partition $\{C_1 \cup C_2, C_3, ..., C_k\}$ is a $(k-1)$-CCF-coloration of T, which contradicts the equality $\overrightarrow{\chi_a}(T) = k$. $\square$

Previous results are also true if we consider predecessors instead of successors. We then define the *smallest predecessor* of x as, if it exists, the vertex of $\Gamma^-(x)$ which is dominated by all other vertices of $\Gamma^-(x)$. Such a vertex, denoted by x^-, verifies the condition: $\Gamma^-(x^-) = \Gamma^-(x) \setminus \{x^-\}$. It could be computed in polynomial time by the following greedy algorithm:

`Up(x)`
Input: A tournament T and a vertex x of T.
Output: A vertex y such that $y = x^-$ if it exists, $\emptyset$ if not.

We denote by $\{y_1, y_2, ..., y_k\}$ the set $\Gamma^-(x)$, $i = 1$ and $x^- = \emptyset$.
While $i \leq k$ Do:
  If y_i verifies $\Gamma^-(y_i) = \Gamma^-(x) \setminus \{y_i\}$, then $x^- = y_i$ and $i = k+1$;
  Else $i = i + 1$.
Return(x^-).

Proposition 2 *Let T be a tournament with $\overrightarrow{\chi_a}(T) = k$ and let x be a vertex of T.*
(i) *If it exists a k-CCF coloration such that x does not dominate all vertices of its CCF-monochromatic class then x^- exists.*
(ii) *Conversely, if x^- exists then for every k-CCF coloration c of T, $c(x) = c(x^-)$.*

Corollary 1 *Let T be a tournament such that $\overrightarrow{\chi_a}(T) = k$. The k-CCF coloration of T is unique.*

Proof: Let c be a k-CCF-coloration of T. We then have the following equivalence:

$$\left[c(x) = c(y) \textit{ and } y \textit{ is the direct successor of } x \textit{ whithin the order } c^{-1}(c(x))\right] \Leftrightarrow y = x^+$$

Then, as the highest successor and the smallest predecessor are unique (if there exist), we deduce the unicity of the optimal convex circuit-free coloration. $\square$

The following algorithm `OptDec` computes in polynomial time the optimal CCF-coloration of a tournament T.

```
Algorithm OptDec
Input:  A tournament T
Output:  The optimal CCF-coloration of T

For every vertex x ∈ V(T), let M(x) denote a mark.
Initialization:  ∀x ∈ V(T), M(x) = 0 and k = 0.
While a vertex x such that M(x) = 0 exists, DO:
        k ← k + 1
        v ← x
        M(x) ← k
    While Up(v) ≠ ∅, DO:
     v ← Up(v)
     M(v) = k end.
        v ← x
    While Down(v) ≠ ∅, DO:
        v ← Down(v),
        M(v) = k end.
  End.
```

Proposition 3 *Given a tournament T with $\overrightarrow{\chi_a}(T) = k$, the optimal $CCF - k$ coloration is computed in polynomial time by the algorithm* `OptDec`*. The optimal CCF-coloration c of T is given by $\forall x \in V(T)$, $c(x) = \mathcal{M}(x)$.*

4 CCF-Indecomposable oriented graph

The aim of this part is to introduce the notion of CCF-indecomposable oriented graph and to characterize vertex-critical CCF-indecomposable tournament. An oriented graph $\overrightarrow{G}$ with n vertices is *CCF-indecomposable* if $\overrightarrow{\chi_a}(\overrightarrow{G}) = n$. In other words, any convex subset of T with at least 2 vertices contains a circuit. Remark that such an indecomposable tournament does not contain convex subset of size two. If $\overrightarrow{G}$ is not CCF-indecomposable then $\overrightarrow{G}$ is called CCF-decomposable. For the following probabilistic proof, we need the notion of *random tournament*, constructed by picking uniformly at random and independently the orientation of every edge of the complete graph K_n (i.e. if $\{x, y\}$ is an edge of K_n, $P((x, y) \in A(T)) = P((y, x) \in A(T)) = \frac{1}{2}$). We denote by $\mathcal{T}_n$ the set of such random tournaments with n vertices.

Proposition 4 *The probability for a tournament $T \in \mathcal{T}_n$ to be CCF-indecomposable tends to 1 when $n \to \infty$.*

Proof: Let $\mathcal{A}$ the event "T is CCF-indecomposable". The event A^c is realized when there exist two vertices x and y such that $\forall z \in V(T) \setminus \{x, y\}$, $(x, z) \in A(T) \Leftrightarrow (y, z) \in A(T)$. We then obtain: $P(A) = 1 - P(A^c) \leq 1 - \binom{n}{2}(\frac{1}{2})^{n-2}$, and so $\lim_{n\to\infty} P(A) = 1$. $\square$

We could easily exhibit a family of CCF-indecomposable tournaments. Let us recall that a tournament is regular if the in and out-degrees of its vertices are equals. Regulars tournaments are CCF-indecomposable, otherwise the existence of both vertices x and x^+ implies $d^+(x^+) = d^+(x) - 1$ (where $d^+(x)$ denotes the outdegree of x). Given a CCF-indecomposable tournament, the following proposition shows that we can add a vertex in order to obtain another CCF-indecomposable tournament. Let us remind that a vertex is a *source* if it has no predecessor and a *sink* if it has no successor.

Proposition 5 *Let T be a CCF-indecomposable tournament without source and sink.*

- *Tournament T' obtained by adding a source s to T is CCF-indecomposable.*
- *Tournament T'' obtained by adding a sink p to T' is CCF-indecomposable.*
- *Tournament T''' obtained by reversing the arc (s,p) in T'' is CCF-indecomposable.*

Indeed, we also obtain a CCF-indecomposable tournament by the converse operations (deleting a source or a sink from a CCF-indecomposable tournament).

We now characterize tournament that are vertex-critical for the $CCF-indecomposable$ property. Tournament T is *vertex-critical CCF-indecomposable* if T is CCF-indecomposable and, for every vertex u of T, $T \setminus \{u\}$ is CCF-decomposable. Given such a tournament, for every vertex u, there exists a pair of vertices $\{i_u, j_u\}$ which verifies the unidirection property with every set $\{x\}$ for x in $V(T) \setminus \{u, i_u, j_u\}$. Such a pair is said to be *associated* to vertex u, which is denoted by $u \sim \{i_u, j_u\}$.

Remark 1 *For every vertex u of a vertex-critical CCF-indecomposable tournament, there exists at least one pair $\{i_u, j_u\}$ of vertices such that (i_u, u, j_u) is the only 2-directed path between the vertices i_u and j_u.*

Lemma 1 *Let u be a vertex of a vertex-critical CCF-indecomposable tournament T, and let $\{i_u, j_u\}$ be a pair associated to u.*
a. $i_u \sim \{u, v\}$ with $v \in V(T) \setminus \{u, i_u\}$.
b. Let u and v be two vertices of a vertex-critical CCF-indecomposable tournament, we have:

$$u = v \quad \Leftrightarrow \quad \{i_u, j_u\} = \{i_v, j_v\}.$$

Proof: a. Since $u \sim \{i_u, j_u\}$ then (i_u, u, j_u) is the only 2-directed path between vertices i_u and j_u. Hence, for every vertex $z \in V(T) \setminus \{i_u, u, j_u\}$, the subsets $\{i_u, j_u\}$ and $\{z\}$ verify the unidirection property in T. Let z' be a vertex of T such that $i_u \sim \{z, z'\}$. Then (z, i_u, z') and (z, j_u, z') are two different 2-directed path in T between z and z' which contradicts the previous remark unless $z' = u$.

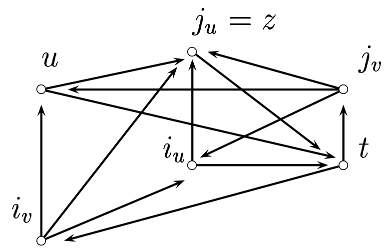
b. We may easily verify that such a proposition is true for tournaments with less than four vertices. Let us now consider that $|V(T)| \geq 5$. Let us suppose that $u \neq v$ and $i_u = i_v$, $j_u = j_v$. Then, there exist two different path between i_u and j_u, which contradicts the remark.

Let us suppose now that the pairs $\{i_u, j_u\}$ and $\{i_v, j_v\}$ are different but $u = v$. We may then suppose that $j_u \neq j_v$. Without loss of generality, we could suppose that (i_u, u, j_u) and (i_v, u, j_v) are 2-directed paths of T. There exists a vertex z in $V(T) \setminus \{j_v, u\}$ such that $j_v \sim \{u, z\}$.

First case: $j_u = z$. In that case, there exists a fifth vertex t such that $z \sim \{u, t\}$. Because of the unidirections due to the critical property of T, we obtain: $\{i_u\} \to \{u, t\}$, $\{j_v\} \to \{u, t\}$, and $\{u, z\} \to t$.

The subcase $i_u = i_v$ is impossible because of the existence of two 2-directed paths (i_v, t, j_v) and (i_v, u, j_v).

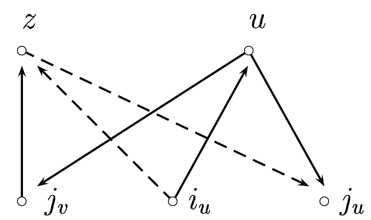
We then have $i_u \neq i_v$. We are in the following configuration:



There exist two 2-directed paths (t, i_v, u) and (u, z, t) between u and t, which contradicts $z \sim \{u, t\}$.

Second Case: $j_u \neq z$.

As $\{i_u, j_u\}$ and $\{j_v\}$ verify the unidirection property, we deduce that $z \notin \{i_u, j_u\}$.



By the unidirection property, $\{u, z\} \to \{j_u\}$ and $\{i_u\} \to \{u, z\}$. That configuration leads to a contradiction because there are two 2-directed path between i_u and j_u. $\square$

Let T be a vertex-critical CCF-indecomposable tournament of order n. We define the graph G_T associate to T by: $V(G_T) = V(T)$ and $\{i, j\} \in E(G_T)$ if it exists $u \in V(T)$ such that $u \sim \{i, j\}$.

Lemma 2 *Let T be a vertex-critical CCF-indecomposable tournament of order n and G_T its associated graph. Then, we have the following properties:*

- *a. The degree of any vertex of G_T is less or equal to 2.*
- *b. Connected components of G_T are cycles (without chord).*
- *c. Let u be a vertex of T and $\{i_u, j_u\}$ be the edge of G_T associate to u. We denote by $\mathcal{C}$ the cycle of G_T which contains $\{i_u, j_u\}$. Then, $u \in \mathcal{C}$.*

- *d. The cardinal of any circuit of G_T is odd.*

Proof: a. Suppose, for a contradiction, that a vertex u has three distinct neighbourgs (x, y and z) in G_T. We denote by $\{i_u, j_u\}$ the edges associate to u. By lemma 1 a., edges $\{u, x\}$, $\{u, y\}$ and $\{u, z\}$ are associate to one of the vertices i_u or j_u, which contradicts the proposition b. of lemma 1.
b. The equivalence of lemma 1b. implies that $|V(G_T)| = |E(G_T)|$. Such equality implies that G_T contains at least one cycle. As the degree of every vertex of G_T is bounded by 2, it follows that the connected components of G_T are cycles, and that every vertex of G_T belongs to exactly one cycle.
c. We denote by $(i_u, j_u, a_1, a_2, ..., a_k)$ the cycle $\mathcal{C}$ and suppose that $u \notin \mathcal{C}$. Then (i_u, u, j_u) is a 2-directed path in T. As $\{j_u, a_1\}$ is not associated to u, $u \rightarrow \{j_u, a_1\}$. For the same reason, $u \rightarrow \{a_i, a_{i+1}\}$, for $i \in \{1, 2, ..., k\}$. This implies that (i_u, u, a_k) is a 2-directed path of T which contradicts $u \sim \{i_u, j_u\}$ (by lemma 1b.). Then, there exists at least two different 2-directed path in T between vertices i_u and a_k, which contradicts $\{i_u, a_k\} \in G_T$.
d. Let $\mathcal{C} = (x_0, ..., x_k)$ be a cycle in G_T. We suppose that $x_0 \sim \{x_l, x_{l+1}\}$. By lemma 1a, $\{x_0, x_1\}$ is associated to x_l or x_{l+1}.
First case: Suppose that $x_{l+1} \sim \{x_0, x_1\}$. Using lemma 1a., we iterate the process: from $[x_0 \sim \{x_l, x_{l+1}\}$ *and* $x_{l+1} \sim \{x_0, x_1\}]$, we obtain $[x_1 \sim \{x_{l+1}, x_{l+2}\}$ *and* $x_{l+2} \sim \{x_1, x_2\}]$ and $[x_2 \sim \{x_{l+2}, x_{l+3}\}$ *and* $x_{l+3} \sim \{x_2, x_3\}]$ ... . As every vertex of $\mathcal{C}$ must be associated to a unique edge of $\mathcal{C}$, the iterated process ends with $[x_l \sim \{x_k, x_0\}$ *and* $x_k \sim \{x_{l-1}, x_l\}]$. Then, $k = 2l$ is even, and so the cycle is odd.
Second case: Suppose that $x_l \sim \{x_0, x_1\}$. Previous iterated process lead to a contradiction, because a vertex must be associate to an edge incident to it, which is impossible. □

For any integer k, the *circular tournament* $\overrightarrow{C_k}$ is the tournament of order $2k+1$ defined by: $V(\overrightarrow{C_k}) = \{0, 1, 2, ..., 2k\}$ and $(i, j) \in A(\overrightarrow{C_k})$ if $1 \leq j - i \leq k$, where $j - i$ is considered modulo $2k + 1$.

Proposition 6 *Let $\mathcal{C}$ be a cycle of G_T of length $2k + 1$. The induced oriented graph $T(V(\mathcal{C}))$ is isomorphic to $\overrightarrow{C_k}$.*

Proof: Lemma 2 shows that if $\mathcal{C} = (x_0, x_1, ..., x_{2k})$ then for all i in $\{0, ..., 2k\}$, $x_i \sim \{x_{k+i}, x_{k+i+1}\}$, where the indexes are considered modulo $2k + 1$. □

Let T be a tournament which vertex set is $V(T) = \{1, 2, ..., n\}$ and let $T_1, ..., T_n$ be tournaments. We denote by $T[T_1, ..., T_n]$ the tournament obtained from T by substituting each vertex i of T by the tournament T_i. If $(i, j) \in A(T)$, then $T_i \rightarrow T_j$. Such a definition allows us to characterize the vertex-critical CCF-tournament.

Theorem 2 *Every vertex-critical CCF-indecomposable tournament is isomorphic to $T'[\overrightarrow{C_{k_1}}, \overrightarrow{C_{k_2}}, ..., \overrightarrow{C_{k_p}}]$ where T' is a tournament of order p and where $(k_1, k_2, ..., k_p) \in (\mathbb{N}^*)^p$.*

Proof: Let T be a vertex-critical CCF-tournament and G_T the graph associated to T. We denote by p the number of cycles in G_T. For $1 \leq i < j \leq p$, if $\mathcal{C}_i$ and $\mathcal{C}_j$ are two disjoint cycles of G_T then the subtournaments of T induced by the vertices of $\mathcal{C}_i$ and $\mathcal{C}_j$ verify the unidirection property in T. We define the tournament T' by $V(T') = \{1, 2, ..., p\}$ and $(i, j) \in A(T')$ if and only if the subtournament induced by $V(\mathcal{C}_i)$ dominates the subtournament induced by $V(\mathcal{C}_j)$.

For every i in $\{1, 2, ..., p\}$, k_i is the integer such that the number of vertices of cycle $\mathcal{C}_i$ is $2k_i + 1$, and by proposition 6, we deduce that T is isomorphic to $T'[\overrightarrow{C}_{k_1}, \overrightarrow{C}_{k_2}, ..., \overrightarrow{C}_{k_p}]$.

Conversely, let X be a convex subset of $T'[\overrightarrow{C_{k_1}}, \overrightarrow{C_{k_2}}, ..., \overrightarrow{C_{k_p}}]$ with at least two vertices. If every vertex of X belongs to the same $\overrightarrow{C_i}$, then $\overrightarrow{C_i} \subset X$ because circulant tournaments are CCF-indecomposable. If $\{x, y\} \subset X$ such that x belong to $\overrightarrow{C_i}$ and y belong to $\overrightarrow{C_j}$ (with $i \neq j$), then $\overrightarrow{C_i} \cup \overrightarrow{C_j} \subset X$. We conclude that X contains at least a circulant tournament and then a circuit, and so such tournament are CCF-indecomposable. It is easy to see that $T'[\overrightarrow{C_{k_1}}, \overrightarrow{C_{k_2}}, ..., \overrightarrow{C_{k_p}}] \setminus \{x\}$ is CCF-decomposable. □

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