Oriented coloring: some classes of oriented graphs

Jean-François Culus [a], Marc Demange [b]

[a] *UTM, équipe Grimm - SMASH, 5 Allée Antonio Machado 31000 Toulouse cedex 9 France, e-mail: culus@univ-tlse2.fr*

[b] *ESSEC, département SID, Avenue Bernard HIRSH, BP 105, 95021 Cergy Pontoise cedex France, e-mail: demange@essec.fr*

Abstract

This paper [1] is devoted to an oriented coloring problem motivated by a task assignment model. A recent result established the NP-completeness of deciding whether a digraph is k-oriented colorable; we extend this result to the classes of planar bipartite circuit-free oriented graphs. Finally, we investigate the approximation of this problem: both positive and negative results are devised.

Key words: Oriented Coloring, Complexity, Differential Approximation.

1 Introduction

1.1 The problem

In this paper, $G = (V(G), E(G))$ will denote a simple graph and $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ a digraph (i.e. a directed graph). An oriented graph is a anti-symmetric digraph, i.e. without 2-circuit, and a *tournament* is a complete anti-symmetric digraph. A *mixed graph* $\mathcal{G} = (V(\mathcal{G}), A(\mathcal{G}), E(\mathcal{G}))$ contains both arcs $(A(\mathcal{G}))$ and edges $(E(\mathcal{G}))$. We do not allow loops or parallel arcs or edges, but mixed graph may have an edge and an arc with the same end-vertices. Graphs and oriented graphs are particular case of mixed graphs, so every notion defined for mixed graph can be considered for graph and oriented graph. If S is a subset of $V(\mathcal{G})$, we denote by $\mathcal{G}[S]$ the sub-mixed graph of $\mathcal{G}$ induced by S. If $v \in V(\mathcal{G})$, $\Gamma^+(v) = \{w|(v,w) \in A(\mathcal{G})\}$ and by $\Gamma^-(v) = \{w|(w,v) \in A(\mathcal{G})\}$.

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A vertex x of an oriented graph $\overrightarrow{G}$ is a *source* if $\Gamma^{-}x = \emptyset$, and is a *sink* if $\Gamma^{+}(x) = \emptyset$. Given a subset U of $V(\mathcal{G})$, $\Gamma^{+}(U)$ denotes the subset $\bigcup_{v \in U} \Gamma^{+}(v)$ of vertices of $\mathcal{G}$ and $\Gamma^{-}(U)$ the denotes the subset $\bigcup_{v \in U} \Gamma^{-}(v)$. A k-vector $(x_1, x_2, ..., x_k)$ of vertices of mixed graph $\mathcal{G}$ is a k-path of $\mathcal{G}$ if, for all i in $\{1, \ldots, k\}$, $\{x_i, x_{i+1}\}$ is an edge of $\mathcal{G}$. The length of such path is $k-1$ and P_k will denote a path of lenght $k-1$. A cycle of $\mathcal{G}$ is a k-path such that $x_1 = x_k$. A directed k-path of $\mathcal{G}$ is a k vector $(x_1, \ldots, x_k)$ of vertices of $\mathcal{G}$ such that (x_i, x_{i+1}) is an arc of $\mathcal{G}$. A circuit of $\mathcal{G}$ is a directed k-path such that $x_1 = x_k$.

Let G, G' be graphs, and $\overrightarrow{G}, \overrightarrow{G'}$ be oriented graphs. An *homomorphism* of G to G' [resp. of $\overrightarrow{G}$ to $\overrightarrow{G'}$] is a mapping $f : V(G) \rightarrow V(G')$ [resp. $f : V(\overrightarrow{G}) \rightarrow V(\overrightarrow{G'})$] which preserves the edges [resp. the arcs]: i.e. $\{x, y\} \in E(G)$ [resp. $(x, y) \in A(G)$] implies $\{f(x), f(y)\} \in E(G')$ [resp. $(f(x), f(y)) \in A(G')$]. Homomorphisms of undirected and directed graphs have been studied as a generalization of graph coloring in the literature [8,9].

A *k-coloring* of a graph G is equivalent to an homomorphism of G to the complete graph K_k. Therefore, the chromatic number $\chi(G)$ of a graph G is equal to the smallest integer k such that there exists an homomorphism of G to K_k. Then, Min Coloring is equivalent to the problem of finding such an homomorphism.

Generalizing previous definition, an *oriented k-coloring* of $\overrightarrow{G}$ is an homomorphism of $\overrightarrow{G}$ to an oriented graph $\overrightarrow{G'}$ on k vertices. The *oriented chromatic number* of an oriented graph $\overrightarrow{G}$, denoted by $\chi_o(\overrightarrow{G})$, is the smallest integer k such that there is an oriented k-coloring of $\overrightarrow{G}$. This problem will be called `Min Oriented Coloring`. Since an oriented coloring of $\overrightarrow{G}$ is also an oriented homomorphism c of $\overrightarrow{G}$ to an oriented graph $\overrightarrow{G'}$, we will call $\overrightarrow{G'}$ the *color oriented graph* of $\overrightarrow{G}$. For $i \in \{1, 2, ..., |V(\overrightarrow{G'})|\}$, subsets $c^{-1}(i)$ of $V(\overrightarrow{G})$ are independent sets of $V(\overrightarrow{G'})$. We call those sets monochromatic classes (for c) of oriented graph $\overrightarrow{G}$. If no possible confusion arise, we omit the reference to homomorphism c.

An oriented coloring of $\overrightarrow{G}$ can also be define as follows: Given two independent sets S and S' in a graph G, we say that they don't respect the *unidirection-property* if two arcs (i, i') and (j', j) exist such that $\{i, j\} \subset S$ and $\{i', j'\} \subset S'$ (we may have $i = j$ or $i' = j'$); in the opposite case, the unidirection-property holds (and we then note $S \rightarrow S'$). An oriented k-coloring is a partition of the vertex set into k independent sets such that, all pairs of independent sets in this family respect the unidirection-property. The oriented chromatic number of an oriented graph $\overrightarrow{G}$ is the minimum k such that $\overrightarrow{G}$ admits an oriented k-coloring.

The notion of oriented chromatic number has been first introduced by Nesetril and Sopena ([16,14]) and has been also studied in [15,17,11,13]. Most of these works focus on upper and lower bounds of the oriented chromatic number. Recently, Klostermeyer and MacGillivray [12] studied its complexity, but to our knowledge, its approximation behavior has not been studied until now. Deciding if the oriented chromatic number of a given oriented graph is at most k is stated to be **NP**-complete for every $k \geq 4$ in [12]. In section 2, we extend this result to the case of bipartite planar circuit-free oriented graphs and sudy the complexity of some class of digraph. In section 3, we are interested in polynomial time algorithm providing guarantees on the number of colors. Two kinds of approximation ratios are usually used to characterize the performance guarantees of an approximation algorithm $\mathcal{A}$. The most classical one is, for a given instance G, the ratio between the minimum number $\chi(G)$ of colors required and the number of colors used by the algorithm, denoted by $m_{\mathcal{A}}(G)$. Algorithm $\mathcal{A}$ is said to guarantee a ratio of $\rho(G)$ if, for every instance, the related ratio is bounded below by $\rho(G)$. $\mathcal{A}(G)$ will denote the solution computed by $\mathcal{A}$ for G. The analysis of approximation algorithms for Min Coloring started with Johnson [10] who shown that an algorithm which greedily computes independent sets until all the graph is colored lead to a classical approximation ratio of $O(n/\log n)$. So far, the best known approximation algorithm achieves a $O(n(\log\log(n)^2/(\log(n))^3$-approximation [5].

Another framework, called differential ratio or also z-approximation, is also widely used [6,2,18], particularly for coloring problems [7,3,1]: Min Coloring is known to be constant approximable under this ratio although it is hard to approximate in the usual sense. Given an instance G, the differential ratio of an algorithm $\mathcal{A}$ is defined by $[w(G)-m_{\mathcal{A}}(G)]/[w(G)-\beta(G)]$, where $m_{\mathcal{A}}(G), \beta(G)$ and $w(G)$ respectively denote the value of the computed solution, the optimal value of instance G and its worse value. $w(G)$ is obtained by maximizing (minimizing) the same objective under the same constraints for a minimization (maximization) problem. In the frame of Min Oriented Coloring, the worst value of an instance $\overrightarrow{G}$ is the number n of vertices and the related ratio for $\overrightarrow{G}$ is $[n - m_{\mathcal{A}}(\overrightarrow{G})]/[n - \chi_o(\overrightarrow{G})]$. For this problem, we can see the differential framework as maximizing the number of unused colors among n potential colors.

1.2 Motivation

The first motivation of oriented coloring is a theoretical one, since it is a natural extension of Min Coloring seen as an homomorphism problem. Nevertheless, such problems may arise in scheduling models. Min coloring models some simple tasks assignment problems. Let us consider a set $V = \{T_1, T_2, ..., T_n\}$ of different tasks to be handled on n identical processors when no preemption is possible. Every processor can perform only one task at a time and every task is supposed to have a unit processing time on any processor. Let $E \subset \{\{t, t'\}/t \in V, t' \in V, t \neq t'\}$ be a set of incompatibilities: two incompatible tasks cannot be performed during the same time by (different) processors. On the other hand, a set of p tasks without incompatibility can be performed at a time by using p processors. Let us consider the incompatibility graph $G = (V, E)$; it is well known that the minimum time required to handle all tasks in V is the chromatic number of G, denoted by $\chi(G)$. Color classes correspond to the tasks which have to be performed simultaneously.

Let us now consider a similar model where incompatibilities are oriented and defined by $\overrightarrow{E} \subset \{(t, t')/t \in V, t' \in V, t \neq t'\}$; an incompatibility $(t, t') \in \overrightarrow{E}$ means that t' cannot be neither performed (on any processor) at the same time as t, nor during the following time unit. So, if t and t' are performed consecutively, then t must be performed after t'. Consider then a scheduling problem organized in two phases. During the first step (corresponding to middle-term decisions), batches of compatible tasks are performed. During the second step, (short-term step) only a subset of p batches is selected and one wants to perform all these selected batches in p time units (without break). The aim is to find a minimum number of batches covering all the tasks in a such a way that for every p batches selected during the second phase, it is possible to schedule it in exactly p time units. The batches defined during the first step correspond to independent sets in the incompatibility graph; a family of p such independent sets corresponds to batches that can be handled in p time units if they can be numbered $S_1, \dots S_p$ in such a way that is no arc from S_i to S_{i+1}. It is easily shown that such a numbering can be found for every p and every p selected batches if and only if every two batches satisfy the unidirection-property. So this scheduling problem can be seen as an oriented coloring problem.

2 The complexity of oriented chromatic number

The oriented k-chromatic number problem OCN_k is formally defined as follows: *an instance is an oriented graph* $\overrightarrow{G}$ *and the question is: "does* $\overrightarrow{G}$ *admits an oriented* k*-coloring ?"*

Theorem 1. *([12]) Let k be a fixed positive integer. If $k \leq 3$, then OCN_k can be decided in polynomial time. If $k \geq 4$, then OCN_k is* **NP***-complete, even if the input is restricted to connected oriented graphs.*

In what follows, we study the complexity of Min Oriented Coloring for particular classes of oriented graphs.

Proposition 1. *Let G be a tree (with at least one edge), and let $\overrightarrow{G}$ be an orientation of G. If every vertex of $V(\overrightarrow{G}$ is a source or a sink, then $\chi_o(\overrightarrow{G}) = 2$. Else, $\chi_o(\overrightarrow{G}) = 3$ and there exists an oriented homomorphism of $\overrightarrow{G}$ to the 3-circuit $(1, 2, 3)$.*
Consequently OCN_k can be polynomialy solved when the input is restricted to an oriented tree.

Scketch of proof We prove by induction on the order n of the oriented tree $\overrightarrow{G} = (V, A)$ that there exists an oriented homomorphism of $\overrightarrow{G}$ to the 3-circuit $(1, 2, 3)$. Let x be a leaf of $\overrightarrow{G}$, and let us consider the oriented tree $\overrightarrow{G}[V \backslash \{x\}]$. By induction, there exists an oriented homomorphism of $\overrightarrow{G}[V \backslash \{x\}]$ to $(1, 2, 3)$ which could be completed into an oriented homomorphism of $\overrightarrow{G}$ to $(1, 2, 3)$. □

A simple greedy algorithm allows us to find an optimal oriented coloring for oriented tree. Such algorithm may also find an optimal oriented coloring for the following family of oriented graph:
Proposition 2. *If $\overrightarrow{G}$ is an oriented graph such that, for every vertices x and y, every directed path from x to y have the same length modulo 3, then $\chi_o(\overrightarrow{G}) = 3$ and `Min Oriented Coloring` can be solved in polynomial time.*

Every bipartite (undirected) graph admits a 2-coloring. Then, the family of bipartite graph is polynomial for the (classical) coloring problem. Given an integer n, we could exibit a bipartite oriented graph of order $2n$ such that its oriented chromatic number is equal to n. Moreover, we could find such oriented bipartite graph such that its colors digraph is a given tournament of order n.
Proposition 3. *Let $\overrightarrow{G}$ be a tournament of order n. We define the oriented bipartite graph $B(\overrightarrow{G})$ by:*

- $V(B(\overrightarrow{G})) = \{x_i, y_i / i \in V(\overrightarrow{G})\}$,
- $A(B(\overrightarrow{G})) = \{(x_i, y_j), (y_i, x_j) / (i, j) \in A(\overrightarrow{G})\}$.

Then, $\chi_o(B(\overrightarrow{G})) = n$ and the only optimal oriented coloring of $B(\overrightarrow{G})$ is given by :

$$\forall i \in V(\overrightarrow{G}), \quad c(x_i) = c(y_i) = i.$$

Nevertheless, we could find the oriented chromatic number of a complete multipartite oriented graphs in polynomial time.

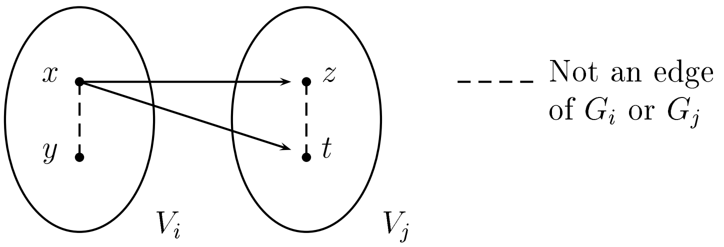
Proposition 4. *Let* $\overrightarrow{G} = (V_1 \cup V_2 \cup ... \cup V_l, A(\overrightarrow{G}))$ *be a complete multipartite oriented graph. We consider the (simple) graph* G *defined by :* $V(G) = V(\overrightarrow{G})$ *and* $\{x, y\}$ *is an edge of* G *if there is a directed* 2*-path or a directed* 3*-path between* x *and* y *in* $\overrightarrow{G}$*. Then, we have:*

i. $\chi_o(\overrightarrow{G}) = \sum_{i=1}^{l} \chi(G[V_i])$.
ii. For all i *in* $\{1, \ldots, l\}$*, induced graph* $G_i = G[V_i]$ *is a cograph.*

Then, `Min Oriented Coloring` *can be solved in polynomial time for complete multipartite oriented graphs.*

Proof (i.) Any optimal oriented coloring of $\overrightarrow{G}$ induces a (usual) coloring of (undirected) graphs $G_i = \overrightarrow{G}[V_i]$, for i in $\{1, ..., l\}$. As no oriented color class contains vertices from both V_i and V_j for $1 \leq i \neq j \leq l$, we have: $\chi_o(\overrightarrow{G}) \geq \sum_{i=1}^{l} \chi(G_i)$.

For $i \in \{1, 2, \ldots, l\}$, we denote by k_i the chromatic number of G_i and by c_i a k_i-coloring of G_i. We claim that, for $1 \leq i < j \leq l$, and for every $a \in \{1, \ldots, k_i\}$ and $b \in \{1, \ldots, k_j\}$, sets $c_i^{-1}(a)$ and $c_j^{-1}(b)$ verifies the unidirection property in the oriented graph $\overrightarrow{G}$. Indeed, let x be a vertex of $c_i^{-1}(a)$ and z, t be vertices of $c_j^{-1}(b)$. Without loss of generalty, we may assume that (x, z) is an arc of $\overrightarrow{G}$.



Since $c_j(z) = c_j(t)$, $\{z, t\}$ is not an edge of G_j. Then, (x, t) is an arc of $\overrightarrow{G}$. For every vertex y in $c_i^{-1}(a)$, since $\{x, y\}$ is not an edge of G_i, we deduce that (y, z) and (y, t) are arcs of $\overrightarrow{G}$. Then, $\{x, y\} \to \{z, t\}$, and by considering every vertex of $c_i^{-1}(a)$ and $c_j^{-1}(b)$, we obtain that $c_i^{-1}(a) \to c_j^{-1}(b)$, and that such pair of monochromatic classes verifies the unidirection property in $\overrightarrow{G}$.
Mapping $c : V(\overrightarrow{G}) \to \{1, 2, ..., k_1 + k_2 + .. + k_l\}$ defined by $c(x) = c_i(x) + \sum_{j=1,...,i-1} k_j$ if $x \in V_i$ is an oriented $(k_1 + k_2 + \cdots + k_l)$-coloring of $\overrightarrow{G}$. We then deduce the formula: $\chi_o(\overrightarrow{G}) = \sum_{i=1}^{l} \chi(G_i)$.

(ii.) Induced graphs $\{G_i\}_{1 \leq i \leq l}$ are cographs because there are P_4-free. Consequently the chromatic number of every G_i can be computed in polynomial time ([4]). We deduce the value of the oriented chromatic number of $\overrightarrow{G}$ by formula i. □

We now show that OCN_k is **NP**-complete even if the input oriented graph is supposed to be bounded degree planar bipartite and circuit-free.

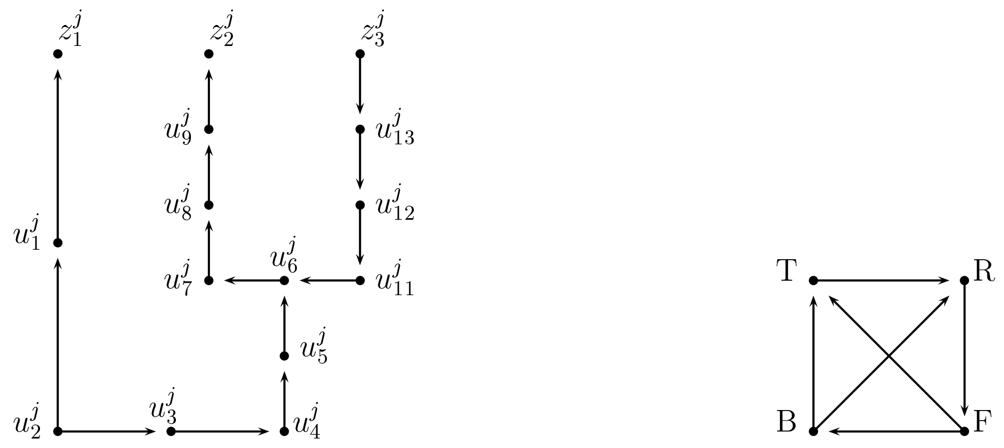


Fig. 1. Oriented graphs $\overrightarrow{L_j}$ (left) and T_4^1 (right).

We consider oriented graph $\overrightarrow{L_j}$ (Fig. 1.) and a mapping c of the following oriented graph $\overrightarrow{L_j} = (V(\overrightarrow{L_j}), A(\overrightarrow{L_j}))$ to the tournament T_4^1 which satisfies $c(z_1^j), c(z_2^j), c(z_3^j) \in \{T, F\}$. We are interested in completing such mapping c into an homomorphism of $\overrightarrow{L_j}$ to T_4^1.

Lemma 1. *Mapping c can be completed into an homomorphism of $\overrightarrow{L_j}$ to T_4^1 if and only if $(c(z_1^j), c(z_2^j), c(z_3^j)) \neq (F, F, F)$.*

The main argument of the proof is: if c satisfies $c(z_3^j) = c(z_2^j) = F$, then $c(u_6^j) = F$. □

Theorem 2. *OCN_4 is **NP**-complete even if the input is restricted to bounded degree circuit-free bipartite planar oriented graph.*

Proof : The proof is done by a reduction from `Planar 3-Sat` to OCN_4.

Let us consider an instance of `Planar 3-Sat` problem: let $X = \{x_1, x_2, .., x_n\}$ be a set of boolean variables, and let $\{C_1, ..., C_m\}$ be m 3-clauses, where each 3-clause is the disjunction of 3 literals over the finite set X. Without loss of generalty, we may assume that no clause contains x_i and $\overline{x_i}$, for $i \in \{1, 2, ..., n\}$, so the three literals of each clause can be sorted by using the index of related variable. Using this order, we denote by $z_1^j \vee z_2^j \vee z_3^j$ the clause C_j.

We associate to this instance a planar graph G define by: Create one vertex for every literal x_i and $\overline{x_i}$ and for every clause c_j. For every variable, create the edge $(x_i, \overline{x_i})$, create an edge (l_i, c_j) if and only if clause c_j contains literal l_i and create the cycle $(x_1, x_2, ..., x_n)$ of edges.

The reduction uses oriented graph $\overrightarrow{G}$ obtained from G by the following operation:

- Delete cycle $(x_1, ..., x_n)$.
- For every variable, substitute the edge $(x_i, \overline{x_i})$ by the circuit-free bipartite planar oriented graph $\overrightarrow{U_i}$.
- For each clause $C_j = z_1^j \vee z_2^j \vee z_3^j$, with $z_1^j, z_2^j, z_3^j \in \{x_i, \overline{x_i}, i \in \{1, 2, ..., n\}\}$, $(j \in \{1, 2, ..., m\})$, replace edges $\{c_j, z_1^j\}, \{c_j, z_2^j\}$ and $\{c_j, z_3^j\}$ of G by a copy of the circuit-free bipartite planar oriented graph $\overrightarrow{L_j}$.

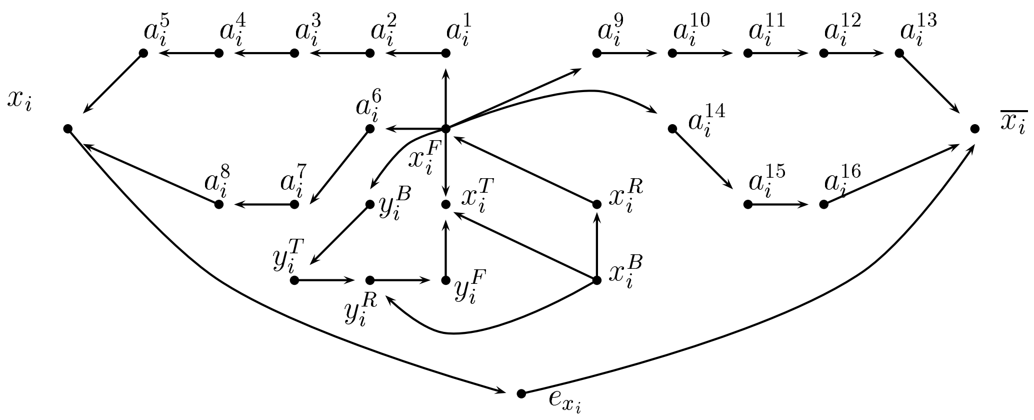


Fig. 2. Oriented graph $\overrightarrow{U_i}$

Starting from G, the construction of $\overrightarrow{G}$ can be performed in polynomial time. It is straigforward to verify that $\overrightarrow{G}$ is planar since G, $\overrightarrow{L_j}$ and $\overrightarrow{U_i}$ are planar. We also verify that oriented graph $\overrightarrow{G}$ is circuit-free bipartite and its degree is bounded by $Max(p + 3; 7)$, where p denotes the maximum number of occurrences of a literal in clauses.

If c is an oriented 4-coloring of $\overrightarrow{G}$, as x_i and $\overline{x_i}$ are linked by a 2-path, $c(x_i) \neq c(\overline{x_i})$. For any fixed i in $\{1, 2, ..., n\}$, the subdigraph $\overrightarrow{U_i}[\{x_i^F, x_i^T, x_i^R, x_i^B, y_i^F, y_i^T, y_i^R, y_i^B\}]$ implies that the color digraph associated to c is T_4^1. Then, x_i^F is necessarily colored by F and the existences of a directed 5-path and a directed 7-path from x_i^F to vertices x_i and $\overline{x_i}$ implies that $\{c(x_i), c(\overline{x_i})\} = \{T, F\}$.
Given a truth distribution $\mathcal{T} : \{x_i, \overline{x_i}/i \in \{1, 2, 3, .., n\}\} \longrightarrow \{True, False\}$,

we associate mapping $c : V(\overrightarrow{G}) \rightarrow \{T, F\}$ defined as

$$\begin{cases} c(x_i) = T & \text{if } \mathcal{T}(x_i) = True, \\ c(x_i) = F & \text{otherwise.} \end{cases}$$

If $\mathcal{T}$ satisfies all clauses $\{C_j\}_{1 \leq j \leq m}$, then applying lemma 1, there exists an homomorphism of $\overrightarrow{G}$ to T_4^1.

Conversely, if such an homomorphism c exists, then we define the truth assignment $\mathcal{T}$ by $\mathcal{T}(x_i) = True$ if $c(x_i) = T$, False otherwise. Since at least one variable of each clause is colored by T (lemma 1), we deduce that $\mathcal{T}$ satisfies all clauses $\{C_j\}_{1 \leq j \leq m}$.

Then, a truth distibution $\mathcal{T} : \{x_i, \overline{x_i}, i \in \{1, 2, 3, .., n\}\} \longrightarrow \{True, False\}$ which satifies all clauses $\{C_j\}_{1 \leq j \leq m}$, exists if and only if $\overrightarrow{G}$ admits an oriented 4-coloring, and the theorem is proved. □

3 Approximation

Since OCN_k is `NP`-complete for $k \geq 4$ and even for very specific classe of oriented graph, we are interested in approximate the oriented chromatic number of an oriented graph. In the first subsection, we obtain negative result by use of a reduction from the well known `Maximum Independent Set` problem. In the second subsection, some positive result are obtained by the analysis of a greedy algorithm.

3.1 Reduction from Maximum Independent Set

Let $G = (V, E)$ be an instance of the Maximum Independent Set problem with $V = \{1, 2, 3, ..., n\}$. We define digraph $\overrightarrow{G'}$ as follows: $V(\overrightarrow{G'}) = X \cup Y \cup Z$ with: $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$ and $Z = \{z_1, z_2, ..., z_n\}$.
$A(\overrightarrow{G'}) = A_{XY} \cup A_{XZ} \cup A_{YZ}$ with: $A_{XZ} = \{(x_i, z_j), i \leq j\} \cup \{(z_i, x_j), i < j\}$, $A_{YZ} = \{(y_i, z_j), i \leq j\} \cup \{(z_i, y_j), i < j\}$ and $A_{XY} = \{(x_i, y_j), i < j\} \cup \{(x_j, y_i), i < j \text{ and } (j, i) \in E\} \cup \{(y_i, x_j), i < j, (i, j) \notin E\}$.

We first remark that digraphs $\overrightarrow{G'}[X \cup Z]$ and $\overrightarrow{G'}[Y \cup Z]$ are isomorphic, while $\overrightarrow{G'}[X \cup Z]$ and $\overrightarrow{G'}[Y \cup Z]$ are complete bipartite digraphs and that $\overrightarrow{G'}[X \cup Y] \cup \{(x_i, y_i)/1 \leq i \leq n\}$ is a complete bipartite digraph. Moreover, $\chi_o(\overrightarrow{G'}[X \cup Z]) =$

$\chi_o(\overrightarrow{G'}[Y \cup Z]) = 2n$: indeed, as there is always an oriented 2-path from x_i to x_j $(i < j)$, and from z_i to z_j, color classes contain only one vertex.

Lemma 2. *Let $n = |G|$, then, $\chi_o(\overrightarrow{G'}) = 3n - \alpha(G)$, where $\alpha(G)$ denotes the independent number of G, and every k-oriented coloring of $\overrightarrow{G'}$ allows us to compute in polynomial time an independent set of G of size $3n - k$.*

Proof Any color class of $\overrightarrow{G'}$ is either a single vertex or the pair $\{x_i, y_i\}$ for $i \in \{1, 2, ..., n\}$. Consequently $\chi_o(\overrightarrow{G'}) = k$ with $2n \leq k \leq 3n$. Any k-oriented coloring of $\overrightarrow{G'}$ is formed by $(3n-k)$ pair of vertices and $(2k-3n)$ single set. Let S be the set $\{i \in V/\{x_i, y_i\}$ *is a color* $\}$. Forall vertices i,j in S with $i < j$, both definition of A_{XY} and the unidirection property imply that $\{(x_i, y_j), (y_i, x_j)\} \subset A(\overrightarrow{G'})$, implying that $\{i, j\} \notin E$. Then, S is an independent set of G.

Conversely, let $S \subset V$ be an independent set of G. By construction of $\overrightarrow{G'}$, it is straightforward to verify that an optimal oriented coloring is given by the following algorithm: color vertices x_i and y_i by the same color if $i \in S$, by different colors if not is an oriented coloring of $\overrightarrow{G'}$.

Consequently, there is a bijection between the oriented colorings of $\overrightarrow{G'}$ and the independent sets of G, which achieves the lemma. □

Theorem 3. *There exists a reduction from* `Maximum Independent Set` *to* `Min Oriented Coloring` *transforming any differential ratio $\rho(n)$ for* `Min Oriented Coloring` *into a $\rho(3n)$ for* `Maximum Independent Set`*.*

Proof: We show how to obtain a polynomial algorithm for `Maximum Independent Set` from a hypothetical polynomial algorithm $\mathcal{A}$ which guarantees a differential ratio of $\rho(n)$ for `Min Oriented Coloring`. Let G be a graph instance of `Maximum Independent Set`. We define the oriented graph $\overrightarrow{G'}$ as previously and denote by $m_{\mathcal{A}}(\overrightarrow{G'})$ be the number of color classes used by algorithm $\mathcal{A}$ for instance $\overrightarrow{G'}$. By lemma 2 we get an independent set of G of size $\alpha'(G) = 3n - m_{\mathcal{A}}(\overrightarrow{G'})$. So we have: $\frac{\alpha'(G)}{\alpha(G)} = \frac{3n - m_{\mathcal{A}}(\overrightarrow{G'})}{3n - \chi_o(\overrightarrow{G'})} \geq \rho(3n)$, which concludes the proof. □

Corollary 1. *If* $\mathbf{P} \neq \mathbf{NP}$*, then* `Min Oriented Coloring` *is not approximable within a constant differential approximation ratio.*
If $\mathbf{P} \neq \mathbf{ZPP}$*, then* `Min Oriented Coloring` *is not approximable within a differential ratio of $O(n^{\epsilon-1})$, $\epsilon > 0$.*

3.2 A greedy algorithm

In this section, we propose a natural generalization of the usual greedy algorithm : we greedily computes independent set algorithm (like in [10]). The main difference between the undirected and the oriented cas arises from the fact that an oriented coloring of a sub-digraph cannot systematically be completed into an oriented coloring of the oriented graph since two vertices of the same color in the sub-digraph can be connected by a 2-path in the whole graph. To overcome this difficulty, the algorithm is devised in the framework of mixed graphs.

First, we introduce a generalization of oriented coloring to mixed graph. A *mixed k-coloring* of a mixed graph $\mathcal{G} = (V, A, E)$ is a mapping $c : V \to \{1, 2, ..., k\}$ such that, for all $1 \leq i \leq k$, the induced mixed-graph $\mathcal{G}[c^{-1}(i)]$ contains no arc nor edge, and for all $1 \leq i \leq k$, color classes $c^{-1}(i)$ and $c^{-1}(j)$ verify the unidirection property in $\mathcal{G}$. Given a vertex v of $\mathcal{G}$, we denote by $B_{\mathcal{G}}(v, 2)$ the set of vertices y such that, for every mixed coloring c of $\mathcal{G}$, $c(v) \neq c(y)$. We then obtain:

$$B_{\mathcal{G}}(v, 2) = \Gamma(v) \cup \Gamma^+(v) \cup \Gamma^-(v) \cup \{y | \exists z \in V, (v, z), (z, y) \in A \text{ or } (y, z), (z, v) \in A\}.$$

We associate to a given mixed graph $\mathcal{G}$, the mixed graph $\mathcal{M}(\mathcal{G})$ that have same vertices and arcs set than $\mathcal{G}$, and set of edges defined by:
$E(\mathcal{M}(\mathcal{G})) = \{(u, v) : u \in B_{\mathcal{G}}(v, 2)\}$.

Obviously, an oriented k-coloring of $\overrightarrow{G}$ is also a mixed k-coloring of $\mathcal{M}(\overrightarrow{G})$ and conversely. Let us also notice that, given a vertex x, $\Gamma^+(x)$ and $\Gamma^-(x)$ are equals in $\overrightarrow{G}$ and in $\mathcal{M}(\overrightarrow{G})$.

It is straightforward to verify that the following proposition holds for mixed coloring of $\mathcal{M}(\overrightarrow{G})$ and does not hold for oriented coloring of $\overrightarrow{G}$. Nevertheless, every mixed k-coloring of $\mathcal{M}(\overrightarrow{G})$ induces an oriented k-coloring of $\overrightarrow{G}$.

Proposition 5. *Let $\overrightarrow{G}$ be a digraph and $z \in V(\overrightarrow{G})$. Every mixed k-coloring c of $\mathcal{M}(\overrightarrow{G})[V(\overrightarrow{G}) \setminus \{z\}]$ can be completed into a mixed $(k+1)$-coloring of $\mathcal{M}(\overrightarrow{G})$.*

We then consider `Greed-monochromatic` (`GMC`) algorithm which can be seen as an adaptation of the usual greedy independent set algorithm:

Proposition 6. *: Let $\overrightarrow{G}$ be a directed graph and $\mathcal{M}(\overrightarrow{G})$ its associated mixed graph. `GMC` computes an independent set S of $\mathcal{M}(\overrightarrow{G})$ (and hence of $\overrightarrow{G}$) such that $|S| \geq log_{\chi_o(\overrightarrow{G})}(|\overrightarrow{G}|)$ and $\forall z \in V(\overrightarrow{G})$, $\{z\}$ and S verify the unidirection property implying:* $\Gamma^+(S) \cap \Gamma^-(S) = \emptyset$

The proof is a simple adaptation of the usual analysis of greedy independent set algorithm [10].

Algorithm GMC
Input: A mixed graph $\mathcal{G} = (V, A, E)$.
Output: GMC$\mathcal{G}$ is an independent set S of $\mathcal{G}$.
(0) $S \leftarrow \emptyset$, $U \leftarrow V$;
(1) While $U \neq \emptyset$ do:
(2) Let v minimizing $|B_{\mathcal{G}[U]}(v, 2)|$ in $\mathcal{G}[U]$;
(3) $S \leftarrow S \cup \{v\}$; $U \leftarrow U \setminus B_{\mathcal{M}[U]}(v, 2)$

Let us now consider algorithm `Greed-Oriented-Coloring` (`GOC`) that iteratively calls `GMC` in order to find a mixed coloring of the input mixed graph $\mathcal{G}$.

Algorithm GOC
Input A mixed graph $\mathcal{G} = (V, A, E)$.
Output `GOC`($\mathcal{G}$) is a mixed coloring of $\mathcal{G}$.
(0) Construct mixed graph $\mathcal{M} = \mathcal{M}(\mathcal{G})$; $U \leftarrow V$, $i \leftarrow 1$.
(a) While $|U| > 0$ do:
(b) Select at most $\log(|U|)$ vertices in `GMC`($\mathcal{M}[U]$) for color i.
(c) Let V_{min} be the subset of minimum order between Γ^+(`GMC`($\mathcal{M}[U]$)) and Γ^-(`GMC`($\mathcal{M}[U]$)).
(d) Every vertex of V_{min} receives a different color in $\{i+1, ..., i+|V_{min}|\}$.
(e) $U \leftarrow U \setminus$ (`GMC`($\mathcal{M}[U]$) $\cup V_{min}$); $i \leftarrow i + |V_{min}| + 1$.

Theorem 4. `Min Oriented Coloring` *admits a differential* $O\left(\frac{\log^2(n)}{n \log(\chi_o(\overrightarrow{G}))}\right)$*-algorithm. In particular, if* $\chi_o(\overrightarrow{G})$ *is bounded, then a differential ratio of* $O\left(\frac{\log^2(n)}{n}\right)$ *is guaranteed.*

Proof: Let $\mathcal{M}_i$ denote the mixed graph $\mathcal{M}[U]$ at the i^{th} iteration of inner loop. Let $n_i = |\mathcal{M}_i|$ and $\lambda_i = Min\{\log(n_i); |\text{GMC}(\mathcal{M}_i)|\}$ and let $k = \chi_m(\mathcal{M})$. Then we have: $\log_k(n_i) \leq |\text{GMC}(\mathcal{M}_i)| \leq \log(n_i)$ and $n_{i+1} \geq \frac{n_i - \lambda_i}{2} \geq \frac{n_i - \log(n_i)}{2} \geq \frac{n_i}{3}$ if $n_i \geq 5$.
Thus, with $p = \lfloor \log_3(n) \rfloor$ calls of algorithm `GMC`, the number of vertices colored by these p colors is at least:

$$\log_k(n) + \log_k(\frac{n}{3}) + \log_k(\frac{n}{3^2}) + ... \log_k(\frac{n}{3^{p-1}}) = O\left(\frac{\log^2(n)}{\log(k)}\right).$$

The number of colors used by the algorithm GOC is at most $\log_3(n) + n - O\left(\frac{\log^2(n)}{\log(k)}\right)$.

We deduce : $\frac{n-\lambda}{n-k} = O\left(\frac{\log^2(n)}{n \log k}\right)$, which achieve the proof. □

3.3 Link between classical and oriented chromatic number

A first link between chromatic number and oriented chromatic number is the following one: if G is a (simple) graph, then $\chi(G) = Min_{\overrightarrow{G}}\ \chi_o(\overrightarrow{G})$, where the minimum is taken over all orientations $\overrightarrow{G}$ of G. In this section, we study some links between the oriented chromatic number of an oriented graph $\overrightarrow{G}$ and the chromatic number of graph associated to $\overrightarrow{G}$. Given an oriented graph $\overrightarrow{G}$, we consider the *undirected graph* $Und(\overrightarrow{G})$ with the same vertices set as $\overrightarrow{G}$ in which $\{x, y\}$ is an edge if and only if (x, y) or (y, x) is an arc of $\overrightarrow{G}$. It is straightforward to see that a k-oriented coloring of digraph $\overrightarrow{G}$ is a k-coloring of $Und(\overrightarrow{G})$.

Given an oriented graph $\overrightarrow{G}$, we introduce set $\mathcal{C}(\overrightarrow{G})$ in order to characterize some digraph $\overrightarrow{G}$ such that every oriented coloring of $\overrightarrow{G}$ is also a (classical) coloring of $Und(\overrightarrow{G})$. For an oriented graph $\overrightarrow{G}$, we denote by $\mathcal{C}(\overrightarrow{G})$ the set

$$\mathcal{C}(\overrightarrow{G}) = \{\{(x, y), (z, t)\} \in A(\overrightarrow{G})^2 : \quad \{x, t\}, \{z, y\} \notin E(Und(M(\overrightarrow{G})))\ \}.$$

Proposition 7. *If $\overrightarrow{G}$ is an oriented graph such as $\mathcal{C}(\overrightarrow{G}) = \emptyset$, then every k-coloring of $Und(M(\overrightarrow{G}))$ is a oriented k-coloring of $\overrightarrow{G}$.*

Proof : Let $c : V(\overrightarrow{G}) \rightarrow \{1, 2, ..., k\}$ be a k-coloring of $Und(\mathcal{M}(\overrightarrow{G}))$. Then, every monochromatic class $\{c^{-1}(i)\}_{1 \leq i \leq k}$ is an independent set of $\overrightarrow{G}$. Let i, j two colors $(1 \leq i < j \leq k)$. If classes $c^{-1}(i)$ and $c^{-1}(j)$ do not verify the unidirection property, then there exists $\{x, x', y, y'\} \in V(\overrightarrow{G})$ such that $c(x) = c(x') = i$; $c(y) = c(y') = j$ and such that (x, y) and (y', x') are arcs of $\overrightarrow{G}$. Since monochromatic classes are independent sets, $\{(x, y), (y', x')\} \in \mathcal{C}(\mathcal{M}(\overrightarrow{G}))$, which contradicts the hypothesis. □

Corollary 2. *Let $\overrightarrow{G}$ be an oriented graph such that $\mathcal{C}(\overrightarrow{G}) = \emptyset$. Then, $\chi_o(\overrightarrow{G}) = \chi(Und(\mathcal{M}(\overrightarrow{G})))$, and the set of such oriented graph $\overrightarrow{G}$ admits an $\frac{289}{360}$-differential ratio algorithm.*

Condition $\mathcal{C}(\overrightarrow{G}) = \emptyset$ is tight because oriented graphs $\overrightarrow{G}$ such that $|\mathcal{C}(\overrightarrow{G})| = 1$ and $\chi_o(\overrightarrow{G}) > \chi(Und(\mathcal{M}(\overrightarrow{G})))$ exist. For example, let $\overrightarrow{K_k}$ be a tournament of order k and let us consider the oriented graph $\overrightarrow{G_k}$ obtained, starting from $\overrightarrow{K_k}$, by adding four vertices $\{x, y, z, t\}$, the arcs (x, y), (z, t), and for any vertex a of $\overrightarrow{K_k}$, (a, x), (a, t), (z, a) and (y, a) are arcs of $\overrightarrow{G_k}$. We then have $\chi_o(\overrightarrow{G}) = k+2$ while $\chi(Und(\mathcal{M}(\overrightarrow{G}))) = k+1$.

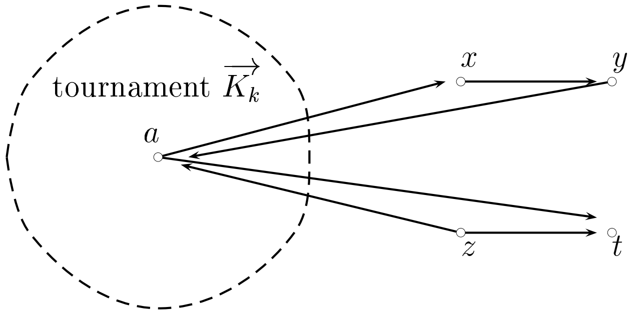


Fig. 3. Oriented graph $\overrightarrow{G_k}$

Any oriented graphs $\overrightarrow{H}$ such that $|\mathcal{C}(\overrightarrow{H})| = 1$ and $\chi_o(\overrightarrow{H}) = k+2 > \chi(Und(\mathcal{M}(\overrightarrow{H})))$ is subdigraph of a $\overrightarrow{G_k}$. More precisely, we have the following proposition:

Proposition 8. *Let $\overrightarrow{H}$ by an oriented graph of order n such that $|\mathcal{C}(\overrightarrow{H})| = 1$ and $\chi_o(\overrightarrow{H}) > \chi(Und(M(\overrightarrow{H})))$. Then, a tournament $\overrightarrow{K_{n-4}}$ of order $n-4$ exists such that $\overrightarrow{H}$ is a subdigraph of $\overrightarrow{G_{n-4}}$ obtained by previous construction.*

4 Discussion:

It will be interesting to consider other particular class of digraph, like interval or permutation graph and to study the chromatic number of there orientations. For example, it seems that adaptation of cograph to the oriented case is **NP**-complete since it is **NP**-complete to determine the oriented chromatic number of the 2-connected composant oriented graph $T_1 \cup T_2$, where T_1 and T_2 denote tournaments.

For bounded degree digraph, the ratio of $\frac{\log^2 n}{n}$ obtained in theorem 4 is the better known approximation ratio obtained by greedy algorithm for `Maximum Independent Set` in both classical and differential theory. Any improvement of our result will also improve theses ratios for `Maximum Independent Set` problem.

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