

Séminaire CIP, LIPN, Université Paris 13  
Algèbres de Koszul et Combinatoire des mots

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## 1 $N$ -Koszul algebras

Koszul algebras were introduced by Koszul in the 1950's. They are based on a complex in a cohomological algebra sense.

Let  $V$  be a  $n$ -dimensional  $\mathbb{C}$ -vector space with basis  $x_1, \dots, x_n$  and  $A = S(V)$  the polynomial algebra in the variables  $x_1, \dots, x_n$ . We denote by  $A^l = \Lambda(V^*)$  the exterior algebra in  $x_1^*, \dots, x_n^*$  where  $x_i^*$  is the dual of  $x_i$ .

The Koszul (complex) algebra of  $A$  is defined as :

$$S(V) \otimes \Lambda^l(V) \xrightarrow{d_l} S(V) \otimes \Lambda^{l-1}(V) \xrightarrow{d_{l-1}} \dots \quad (1)$$

where

$$d_l(u_1 \dots u_k \otimes V_1 \wedge \dots \wedge V_l) = \sum_{1 \leq j \leq l} (-1)^{j+1} (u_1 \dots u_k \otimes v_j) \otimes v_1 \wedge \dots \wedge \check{v}_j \wedge \dots \wedge v_l \quad (2)$$

Koszul showed that this complex is exact in degree  $l > 0$ .

*Consequence* : for any  $m > 0$ , the sub-complex defined by  $k + l = m$  ( $S^k(V) \otimes \Lambda^l(V) \rightarrow S^{k+1}(V) \otimes \Lambda^{l-1}(V) \rightarrow \dots$ ) is exact and has a finite length (because  $\Lambda^l$  vanishes for  $l < \deg(V)$ ?).

The Euler-Poincaré identity provides the following relation :

$$\sum_{k+l=m} (-1)^k \dim(S^k(V)) \cdot \dim(\Lambda^l(V)) = 0 \quad (3)$$

Equivalently,

$$\sum_{k+l=m} (-1)^k \binom{n+k-1}{k} \cdot \binom{n}{l} = 0 \quad (4)$$

We recall that  $\binom{n+k-1}{k}$  counts the increasing words  $(x_{j_1} \dots x_{j_k}, j_1 \leq \dots \leq j_k)$  and  $\binom{n}{l}$  counts the increasing words without repetition  $(x_{i_1} \dots x_{i_k}, i_1 < \dots < i_k)$ . This is the first link between Koszul algebra and combinatorics on words.

Let us now define the Hilbert series of  $A$  and  $A^l$  :

$$H_A(t) = \sum_{k \geq 0} \dim(S^k(V)) \cdot t^k = \sum_{k \geq 0} \dim(A_k) \cdot t^k \quad (5a)$$

$$H_{A^l}(t) = \sum_{l \geq 0} \dim(\Lambda^l(V)) \cdot t^l \quad (5b)$$

Note that relation (3) is equivalent to

$$H_A(t) = H_{A^l}(t) \quad (6)$$

*Remark* :  $A = S(V) = T(V)/(R)$  where

- $T(V)$  is the free associative algebra of  $V$  ;
- $R$  is the subspace of  $V \otimes V$  generated by

$$x_i \otimes x_j - x_j \otimes x_i, 1 \leq i < j \leq n; \quad (7)$$

- $(R)$  is the ideal generated by  $R$ .

We have  $A^! = \Lambda(V^*) = T(V^*)/(R^\perp)$  where  $R^\perp$  is the orthogonal of  $R$  in  $(V \otimes V^*) = V^* \otimes V$  for the usual duality between  $V$  and  $V^*$ .

We want to generalize these idea. The first step is to consider a general quadratic relation instead of relation (7). This was done by Priddy in the 1970's. The next step is to consider a relation involving  $N > 2$   $x_i$ 's.

Let  $V$  be a  $\mathbb{C}$  vector space and  $N \geq 2$ .  $R$  is a subspace of  $V^{\otimes N}$ ,  $A = T(V)/(R)$ .  $A$  is a graded algebra (the elements of  $V$  are of degree 1) called a  $N$ -homogeneous algebra.  $A^! = T(V^*)/(R^\perp)$  is a  $N$ -homogeneous algebra. The Koszul complex  $K(A)$  makes sense :

$$\begin{array}{ccccccc} \dots & \rightarrow & A \otimes (R \otimes V \cap V \otimes R) & \xrightarrow{d_3} & A \otimes R & \xrightarrow{d_2} & A \otimes V & \xrightarrow{d_1} & A & \rightarrow & 0 \\ & & N+1 & & N & & 1 & & & & \\ & & & & & & & & & & \end{array}$$

We define now  $\nu_N : \mathbb{N} \rightarrow \mathbb{N} : \nu_N(2i) = Ni$  and  $\nu_N(2i+1) = Ni+1$ .  $\nu_N$  gives the degree of ?

**Definition 1** *The  $N$ -homogeneous algebra is said to be  $N$ -Koszul if  $K(A)$  is exact in any positive degree.*

(the case  $N = 2$  was treated by Priddy ; for  $N > 2$  see [Berger(2001)]).

**Theorem 1** *(Dubois, Violette, Popov) For any  $N$ -Koszul algebra  $A$  so that  $\dim_{\mathbb{C}} < \infty$ , we have the duality formula :*

$$H_A(t) \left[ \sum_{l \geq 0} (-1)^l \dim \left( A^!_{\nu_N(l)} \right) t^{\nu_N(l)} \right] = 1 \quad (8)$$

## Références

[Berger(2001)] R. Berger. Koszulity for nonquadratic algebras. *Journal of Algebra*, 239(7) :705–734, 2001.