R. P. Stanley, Enumerative combinatorics. — Solution de l'exercice 5.37

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Contents

1 Preamble **1**

2 Solution of a). **2**

3 Solution of b). **5**
- 3.1 Notations 5
- 3.2 Solution of the first part 5
- 3.3 Solution of the second part 8

4 Solution of c). **10**

5 Solution of d). **10**

6 Solution of e). **11**

1 Preamble

As this exercise uses mainly exponentials, the following result has to be stated.

Proposition 1.1 *Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra and, for $a \in \mathcal{A}$ and $k \in \mathbb{N}$ define*

$$\binom{a}{k} := \frac{a(a-1)\cdots(a-k+1)}{k!} \tag{1}$$

Then, in $\mathcal{A}[[X]]$ one has

$$e^{a\ \ln(1+X)} = \sum_{k\geq 0} \binom{a}{k} X^k\ . \tag{2}$$

This last quantity will be denoted $(1+X)^a$. Of course, due to (2), one has for commuting $a, b \in \mathcal{A}$, $(1+X)^{a+b} = (1+X)^a(1+X)^b$.

2 Solution of a).

a) (iii) $\Longrightarrow$ (i) If (iii) is true, then

$$\begin{aligned}\sum_{n\geq 0} p_n(x+y)\frac{u^n}{n!} &= \Big(\sum_{n\geq 0} p_n(1)\frac{u^n}{n!}\Big)^{x+y} = \Big(\sum_{n\geq 0} p_n(1)\frac{u^n}{n!}\Big)^{x}\Big(\sum_{n\geq 0} p_n(1)\frac{u^n}{n!}\Big)^{y}\\ &= \Big(\sum_{n\geq 0} p_n(x)\frac{u^n}{n!}\Big)\Big(\sum_{n\geq 0} p_n(y)\frac{u^n}{n!}\Big)\end{aligned}$$

then as

$$\begin{aligned}p_n(x+y)\frac{u^n}{n!} &= n!\,[u^n]\Big(\sum_{r\geq 0} p_r(x+y)\frac{u^r}{r!}\Big) = n!\,[u^n]\Big(\sum_{m\geq 0} p_m(x)\frac{u^m}{m!}\Big)\Big(\sum_{n\geq 0} p_m(y)\frac{u^m}{m!}\Big)\\ &= n!\sum_{0\leq k\leq n}[u^k]\Big(\sum_{n\geq 0} p_n(x)\frac{u^n}{n!}\Big)[u^{n-k}]\Big(\sum_{r\geq 0} p_r(y)\frac{u^r}{r!}\Big)\\ &= \sum_{0\leq k\leq n}\frac{n!}{k!(n-k)!}p_k(x)p_{n-k}(y) = \sum_{k\geq 0}\binom{n}{k}p_k(x)p_{n-k}(y)\end{aligned}$$

b) (i) $\Longrightarrow$ (ii) Set

$$T(x,u) = \sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!}. \tag{3}$$

From (i), one has $T(x+y,u) = T(x,u)T(y,u)$ and then (one supposes that x and y are independant)[1]

$$\begin{aligned}T(y,u)\frac{\partial T}{\partial x}(x,u) &= T(y,u)\frac{\partial}{\partial x}T(x,u) = \frac{\partial}{\partial x}T(x,u)T(y,u)\\ &= \frac{\partial}{\partial x}T(x+y,u) = \frac{\partial T}{\partial x}(x+y,u)\end{aligned} \tag{4}$$

specializing x to 0 in (4), we get the evolution equation

$$\left\{\begin{array}{c} T(y,u)\frac{\partial T}{\partial x}(0,u) = \frac{\partial T}{\partial x}(y,u)\\ T(0,u) = 1\end{array}\right. , \tag{5}$$

Lemma 2.1 *Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra and $T \in \mathcal{A}[[y]]$ an element satisfying the following equation:*

$$T'(y) = T(y)a$$

with $T(0) = 1$. Then $T(y) = \exp\left(y\ln\left(T(1)\right)\right)$.

Indeed, for any evolution equation of the form $Y'(t) = aY(t)$, the space of solutions is one-dimensional. Let Y_0 be an invertible solution and Y be another solution and write $Z(t) = Y_0^{-1}(t)Y(t)$. One has

$$Z'(t) = -Y_0^{-1}Y_0'Y_0^{-1}Y + Y_0^{-1}Y' = -aY_0^{-1}Y_0Y_0^{-1}Y + aY_0^{-1}Y = 0$$

---

[1] As in thermodynamics, for a function $T(x,u)$, $\frac{\partial T}{\partial x}(f(x,u),g(x,u))$ is the derivative with respect to the first place of T computed at the point $(f(x,u),g(x,u))$ whereas $\frac{\partial}{\partial x}$ stands for the derivative with respect to x. For example $\frac{\partial}{\partial x}T(f(x),u) = \frac{\partial T}{\partial x}(f(x),u).f'(x)$.

Therefore, z is constant and $Y = Y_0 C,\ C \in \mathcal{A}^2$.
In order to apply this lemma we first look for a solution of the type $Y(t) = \exp(tc)$ and find that the solutions are of the following form: $T(y) = \lambda \exp(yc)$. But $\lambda = 1$ (evaluation at $y = 0$), and $T(1) = \exp(c)$. Hence $c = \ln(T(1))$.
Therefore, our equation integrates in $T(y,u) = e^{y\ln(T(1,u))}$. As $T(1,u) = 1 + up_1(1) + \sum_{n\geq 2} p_n(1)\frac{u^n}{n!}$, the function $f(u) = \ln(T(1,u))$ has the required form.

c) (ii) $\Longrightarrow$ (iii) obvious as $\exp(f(u)) = 1 + uH(u)$.
d). Let us finally show that (i) $\Leftrightarrow$ (iv).

- First, (i) $\Longrightarrow$ (iv).
  - We show by induction that $p_n(0) = \delta_{n,0},\ \forall k \leq n$. This is true by definition for $n = 0$. Let us investigate also[3] the case $n = 1$.

    $$p_1(x+y) = p_1(x)p_0(y) + p_0(x)p_1(y) = p_1(x) + p_1(y)$$

    since $p_0(x) = 1$. But $\deg(p_1) = 1$ and $p_1(0) = \delta_{1,0} = 0$. Hence, $p_1 = \alpha x$ with $\alpha \neq 0$. $p_0(0) = 1$ since $p_0 = 1$.
    Let us now suppose that $p_n(0) = \delta_{n,0},\ \forall k \leq n$. Then

    $$p_{n+1}(0) = \sum_{k=0}^{n+1} \binom{n+1}{k} p_k(0)p_{n+1-k}(0) = 2p_{n+1}(0).$$

    (the terms that do not vanish are obtained with $k = 0$ and $k = n+1$ because of the hypothesis of the induction). Therefore, for $n > 0$, $p_n(0) = 0,\ \forall k \leq n+1$ and $p_n(0) = \delta_{n,0},\ \forall n$.
  - We now define a linear operator Q on $\mathbb{P}$ by its action on the p_n's that form a triangular basis of $\mathbb{P}$:

    $$Q\left[x \rightarrow p_n(x)\right] = np_{n-1}(x) \text{ for } n \leq 1 \text{ and } Q\left[x \rightarrow 1\right] = 0$$

    $Q\left[x \rightarrow x\right] \neq 0$. Indeed,

    $$Q\left[x \rightarrow \alpha x\right] = Q\left[x \rightarrow p_1(x)\right] = p_0 = 1 \tag{6}$$

    and $Q\left[x \rightarrow x\right] = \dfrac{1}{\alpha}$.
  - Let us now show that Q commutes with $E^\alpha,\ \forall \alpha$. Let P be a polynomial of degree n. We have:

    $$P = \sum_{k=0}^{n} \beta_k p_k.$$

    $$Q\left[E^a\left[x \rightarrow P(x)\right]\right] = Q\left[x \rightarrow P(x+a)\right] = \sum_{k=0}^{n} \beta_k Q\left[p_k(x+a)\right]$$

---

[2] One can prove that C is invertible, realizing then the Galois differential group.
[3] This is not logically needed here, but will be useful in the following.

since Q is linear. We now use (i):

$$\begin{aligned}
Q\left[E^a\left[x \rightarrow P(x)\right]\right] &= \sum_{k=0}^{n} \beta_k \sum_{j=0}^{k} \binom{k}{j} Q\left[x \rightarrow p_j(x)\right] p_{k-j}(a) \\
&= \sum_{k=0}^{n} \beta_k \sum_{j=1}^{k} \binom{k}{j} j p_{j-1}(x) p_{k-j}(a) \\
&= \sum_{k=0}^{n} \beta_k \sum_{j=1}^{k} \binom{k-1}{j-1} k p_{j-1}(x) p_{k-1-(j-1)}(a)
\end{aligned}$$

Indeed,

$$j\binom{k}{j} = \frac{k!}{(j-1)!(k-j)!} = k\binom{k-1}{j-1}. \tag{7}$$

Thus,

$$\begin{aligned}
Q\left[E^a\left[x \rightarrow P(x)\right]\right] &= \sum_{k=0}^{n} k\beta_k \sum_{j=0}^{k-1} \binom{k-1}{j} k p_j(x) p_{k-1-j}(a) \\
&= \sum_{k=0}^{n} k\beta_k p_{k-1}(x+a) \\
&= E^a\left[Q\left[x \rightarrow P(x)\right]\right].
\end{aligned}$$

- (iv) $\Longrightarrow$ (i): Let us show by induction that the following property, that we denote by $\mathcal{R}_n$, is true $\forall n \geq 1$:

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y)$$

This property is true for $n = 0$ and $n = 1$. Suppose that $\mathcal{R}_n$ is true for $n \in \mathbb{N}$. Then

$$\begin{aligned}
Q\left[x \rightarrow p_{n+1}(x+y)\right] &= Q\left[E^y\left[x \rightarrow p_{n+1}(x)\right]\right] \\
&= E^y\left[Q\left[x \rightarrow p_{n+1}(x)\right]\right] \\
&= E^y\left[(n+1)p_n(x)\right] \\
&= (n+1)p_n(x+y).
\end{aligned}$$

By the hypothesis of the induction,

$$\begin{aligned}
(n+1)p_n(x+y) &= (n+1)\sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \\
&= \sum_{k=0}^{n} \binom{n+1}{k+1} (k+1) p_k(x) p_{n+1-(k+1)}(y)
\end{aligned}$$

(see 7). Hence,

$$\begin{aligned}
Q\left[x \rightarrow p_{n+1}(x+y)\right] &= Q\left[x \rightarrow \sum_{k=0}^{n} \binom{n+1}{k+1} p_{k+1}(x) p_{n+1-(k+1)}(y)\right] \\
&= Q\left[x \rightarrow \sum_{k=1}^{n+1} \binom{n+1}{l} p_l(x) p_{n+1-l}(y)\right]
\end{aligned}$$

But $Q\left[x \to 1\right] = 0$. Indeed $E^1\left[p_1\right] = \alpha x + \alpha = p_1 + \alpha p_0$. But $E^1\left[Q\left[p_1\right]\right] = Q\left[E^1\left[p_1\right]\right] = Q\left[p_1 + \alpha p_0\right] = p_0 + \alpha Q\left[p_0\right] = p_0$. Thus

$$Q\left[x \to p_{n+1}(x+y)\right] = Q\left[x \to \sum_{l=0}^{n+1} \binom{n+1}{l} p_l(x) p_{n+1-l}(y)\right]$$

Now $\dim(\mathrm{Ker}(Q)) = 1$:

$$Q\left[\sum_{k=0}^{n} \gamma_k p_k\right] = \sum_{k=1}^{n} \gamma_k k p_{k-1}$$

equals 0 if $\gamma_k = 0$ for $k \neq 0$ and $\gamma_0 = 0$.

Therefore,

$$p_{n+1}(x+y) = \sum_{l=0}^{n+1} \binom{n+1}{l} p_l(x) p_{n+1-l}(y) + C$$

The evaluation of these quantities at $x = y = 0$ shows that $C = 0$.

Remark: A sequence of polynomials such that $T(x,u) = \sum_{n\geq 0} p_n(x) \frac{u^n}{n} = \exp\left(x f(u)\right)$ is called *a one parameter group* as it satifies the following equation:

$$T(x_1 + x_2, u) = T(x_1, u) T(x_1, u). \tag{8}$$

3 Solution of b).

3.1 Notations

- $$\binom{x}{n} = \frac{x(x-1)\dots(x-n+1)}{n!} \text{ (see the preamble)}$$
- $$(x)_n = x(x-1)\dots(x-n+1) = x^{\underline{n}} = \binom{x}{n} n!$$
- $$(x)^{(n)} = x^{\overline{n}} = x(x+1)\dots(x+n-1) = (-1)^n(-x)^{\underline{n}} = (-1)^n(-x)_n$$

3.2 Solution of the first part

1. For $p_n(x) = x^n$, one has $\sum_{n=0}^{\infty} x^n \frac{u^n}{n!} = \exp(xu)$.
2. For $p_n(x) = (x)_n$, one has

$$\begin{aligned}\sum_{n=0}^{\infty} (x)_n \frac{u^n}{n!} &= \sum_{n=0}^{\infty} \binom{x}{n} n! \frac{u^n}{n!} \\ &= (1+u)^x \\ &= \exp\left[x \ln(1+u)\right]\end{aligned}$$

3. One has $(-x)^{(n)} = (-x)(-x-1)\dots(-x-n+1) = (-1)^n x(x+1)\dots(x+n-1)$. Thus, $x^{\overline{n}} = (x)^{(n)} = (-1)^n(-x)_n$ and

$$\begin{aligned}\sum_{n=0}^{\infty}(x)^{(n)}\frac{u^n}{n!} &= \sum_{n=0}^{\infty}(-1)^n(-x)_n\frac{u^n}{n!}\\ &= \exp\left[-x\ln(1-u)\right]\\ &= \exp\left[x\ln\left(\frac{1}{1-u}\right)\right]\end{aligned}$$

4. Abel polynomials: $p_n(x) = x(x-an)^{n-1}$, $a \in \mathbb{K}$. On the one hand,

$$\begin{aligned}DE^a\left[x(x-an)^{n-1}\right] &= D\left[(x+a)(x-a(n-1))^{n-1}\right]\\ &= (x-a(n-1))^{n1} + (x+a)(n-1)(x-a(n-1))^{n-2}\\ &= (x-a(n-1))^{n-2}\left[x-a(n-1)+(n-1)(x+a)\right]\\ &= (x-a(n-1))^{n-2}\left[nx\right]\\ &= np_{n-1}(x)\end{aligned}$$

On the other hand, $E^a = \sum_{n=0}^{\infty}\frac{D^n}{n!}a^n$. It suffices to verify this equality on a basis of the vetor space of polynomials, for example the basis of the x^n's. We have

$$E^a\left[x \mapsto x^n\right] = (x+a)^n$$

and (cf. Taylor expansion)

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{D^m}{m!}a^m\left[x \mapsto x^n\right] &= \sum_{m=0}^{\infty}\frac{a^m}{m!}n(n-1)\dots(n-m+1)x^{n-m}\\ &= \sum_{m=0}^{\infty}\binom{n}{m}a^m x^{n-m}\\ &= (x+a)^n\end{aligned}$$

Therefore, there exists a power series $q(u) = b_1 u + \dots$ with $b_1 \neq 0$ such that $q(D) = DE^a$, namely this expansion of $u\exp(au)$ since $DE^a = D\exp(aD)$.

Assuming that the question d). of the same exercise is solved, the power series $f(u)$ that we are looking for is given by the compositional invert of $u\exp(au)$. Let us introduce Lambert$W$ function: it is the compositional inverse of $x \mapsto x\exp(x)$. We have:

$$f(u) = \frac{1}{a}\ \text{Lambert}W(au)$$

for $a \neq 0$ (for $a = 0$, we recover x^n and we have already dealt with this case).

5. $p_n(x) = \sum_{k=1}^{n} S(n,k)x^k$.

We consider a class of finite labelled graphs closed by

- taking connected components
- relabelling.

We denote by $M(n,k)$ the number of graphs labelled with $\{1 \dots n\}$ and with k connected components.
Then, the exponential formula states that

$$\sum_{n,k\geq 0} M(n,k)\frac{u^n}{n!}x^k = \exp\left[x\left(\sum_{n=1}^{\infty} M(n,1)\frac{u^n}{n!}\right)\right] \tag{9}$$

Therefore, calling the LHS of (9) *mixed generating series* (MGS), one can solve three identities with these combinatorial interpretations

Graphs	Numbers	Generating series
Permutations	$S_1(n,k)$	$\exp\left[-x\ln(1-u)\right]$
Set partitions	$S_2(n,k)$	$\exp\left[x\left(\exp(u)-1\right)\right]$
Graphs of Indempotent endofunctions	$\binom{n}{k}k^{n-k}$	$\exp\left[x\left(u\exp(u)\right)\right]$

6. For $p_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n+(a-1)k-1}{n-k}x^k$, let us consider the generating series of the $p_n(x);\ p_0 = 1$. One has

$$\sum_{n=0}^{\infty} p_n(x)\frac{u^n}{n!} = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{n!}{k!}\binom{n+(a-1)k-1}{n-k}x^k\frac{u^n}{n!}\,. \tag{10}$$

$$\begin{aligned} 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{n!}{k!}\binom{n+(a-1)k-1}{n-k}x^k\frac{u^n}{n!} &= 1 + \sum_{k=1}^{\infty}\frac{x^k}{k!}\sum_{n=k}^{\infty}\binom{n+(a-1)k-1}{n-k}u^n \\ &= 1 + \sum_{k=1}^{\infty}\frac{x^k u^k}{k!}\sum_{m=0}^{\infty}\binom{m+ak-1}{m}u^m \end{aligned} \tag{11}$$

But $\binom{-h}{m} = (-1)^m\binom{h+m-1}{m}$. Indeed,

$$\begin{aligned} \binom{-h}{m} &= \frac{-h(-h-1)\dots(-h-m+1)}{m!} \\ &= (-1)^m\frac{h(h+1)\dots(h+m-1)}{m!} \\ &= (-1)^m\binom{h+m-1}{m} \end{aligned}$$

We use the previous equality for $h = ak$ in the right hand side of equation (11) and then relation (2):

$$\begin{aligned}
1+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{n!}{k!}\binom{n+(a-1)k-1}{n-k}x^k\frac{u^n}{n!} &= 1+\sum_{k=1}^{\infty}\frac{x^k u^k}{k!}\sum_{m=0}^{\infty}(-1)^m\binom{-ak}{m}u^m \\
&= 1+\sum_{k=1}^{\infty}\frac{x^k u^k}{k!}\sum_{m=0}^{\infty}\binom{-ak}{m}(-u)^m \\
&= 1+\sum_{k=1}^{\infty}\frac{x^k u^k}{k!}(1-u)^{-ak} \\
&= 1+\sum_{k=1}^{\infty}\frac{x^k}{k!}\left(\frac{u}{(1-u)^a}\right)^k
\end{aligned} \tag{12}$$

Therefore, the series $f(u)$ that we are looking for is given by $f(u) = \dfrac{u}{(1-u)^a}$ since

$$1+\sum_{k=1}^{\infty}\frac{x^k}{k!}\left(\frac{u}{(1-u)^a}\right)^k = \exp\left(\frac{xu}{(1-u)^a}\right). \tag{13}$$

3.3 Solution of the second part

The last part of this question consists in finding the delta operator corresponding to each of these binomial sequences.

1. For $p_n(x) = x^n$, it is obvious that $Q = \dfrac{d}{dx}$.
2. $p_n(x) = (x)_n$ is binomial with $f(u) = \ln(1+u)$ whose compositional inverse is $\exp(u)-1$. Therefore, using the property of question d)., one has $Q = \exp(D) - 1 = \sum_{n=1}^{\infty}\dfrac{D^n}{n!}$.
3. For $p_n(x) = (x)^{(n)}$, $f(u) = \ln\dfrac{1}{1-u}$ whose compositional inverse is $1-\exp(-u)$. Thus $Q = 1-\exp(-D) = -\sum_{n=1}^{\infty}\dfrac{(-D)^n}{n!}$.
4. We have already shown that, for Abel polynomials, $Q = D\exp(aD)$.
5. For Stirling numbers of the first class, the function $f(u)$ is the same as for the rising factorial.
6. For Stirling numbers of the second class, $f(u) = \exp(u) - 1$ whose compositional inverse is $\ln(1+u)$ (see the falling factorial case). Therefore, $Q = \ln(1+D)$.
7. For $\dbinom{n}{k}k^{n-k}$, $f(u) = u\exp(u)$ whose compositional inverse is $\mathrm{LambertW}(u)$. Hence $Q = \mathrm{LambertW}(D)$.

8. For Laguerre polynomials, the fact that $f(u)$ is invertible can easily be shown in the following cases: $a = 0, 1, 2$, but it seems to be more difficult in the general case. Indeed, one has

   - $$(u)^{-1}(v) = \frac{1}{v} \tag{14}$$

   - $$\left(\frac{u}{(1-u)}\right)^{-1}(v) = \frac{v}{1+v} \tag{15}$$

   - In the case $a = 2$, solving $\frac{u}{(1-u)^2} = y$ leads to the following equation:

     $$u = \frac{2y + 1 \pm \sqrt{4y+1}}{2y} \tag{16}$$

     Since we want the series to begin with y, we take the “negative“ solution. Using equation (2), one finds that

     $$\left(\frac{u}{(1-u)^2}\right)^{-1}(y) = -\sum_{k=2}^{\infty} \binom{\frac{1}{2}}{k} 2^{2k-1} y^{k-1} \tag{17}$$

Remark 1 *Note that we can distinguish among these sequences two pairs where the power series f of the first sequence is the series (in the variable D) that gives the delta operator Q of the second sequence:*

Sequence	$f(u)$	Q
Falling factorial	$\ln(1+u)$	$\exp(D) - 1$
$S_2(n,k)$	$\exp(u) - 1$	$\ln(1+D)$

Sequence	$f(u)$	Q
Abel polynomials	$\frac{1}{a}\mathrm{LambertW(au)}$	$D\exp(aD)$
$\binom{n}{k}k^{n-k}$	$u\exp(au)$	LambertW(D)

Remark 2 *Note that Lambert W function is related to rooted labelled trees. Indeed, rooted labelled trees can constructed as a node to which a forest is attached. Using the* `set` *functor (that takes a data structure whose elements are labelled with distinct and consecutive integers and gives the sets labelled labelled with distinct and consecutive integers), this construction gives the following equation:*

$$T(z) = z \cdot \exp(T(z)) \tag{18}$$

whose solution is $T(z) = -\mathrm{LambertW}(-z)$.

4 Solution of c).

As $(p_n)_{n\in\mathbb{N}}$ is a basis, one just has to check the coincidence RHS=LHS on the p_n.

$$\begin{aligned}\Big(\sum_{k\geq 0} a_k\frac{Q^k}{k!}\Big)[p_n(y)] &= \sum_{0\leq k\leq n} a_k\frac{Q^k}{k!}[p_n(y)] = \sum_{0\leq k\leq n} a_k\frac{1}{k!}n(n-1)\cdots(n-k+1)p_{n-k}(y)\\ &= \sum_{0\leq k\leq n} a_k\binom{n}{k}p_{n-k}(y)\\ &= \sum_{0\leq k\leq n} T(p_k(x))|_{x=0}\binom{n}{k}p_{n-k}(y)\\ &= T\Big(\sum_{0\leq k\leq n}\binom{n}{k}p_k(x)p_{n-k}(y)\Big)\Big|_{x=0}\\ &= T(p_n(x+y))\Big|_{x=0} = T(E^y p_n(x))\Big|_{x=0}\\ &= E^yT[p_n](x)\Big|_{x=0} = T[p_n](x+y)\Big|_{x=0} = T[p_n](y)\end{aligned}$$

the equality $T\Big(p_n(x+y)\Big) = T[p_n](x+y)$ being due to the fact that T is shift-invariant.

5 Solution of d).

As Q is shift-invariant, one can apply the preceding result with Q in the role of T and $D = \frac{d}{dx}$ in the role of Q so that, in $\mathrm{End}\,(K[x])$[4]. One has

$$Q = \sum_{n=0}^{\infty} a_n\frac{D^n}{n!} = q(D) \tag{19}$$

with $a_n = Q[x^n]\Big|_{x=0}$, in particular $a_0 = 0$. So q has the required form. Now, set

$$M = M(x,u) = \sum_{n,k\geq 0} p_n(x)\frac{u^n}{n!}\ .$$

One defines the diagonal extension of Q as a continuous endomorphism of $K[x][[u]]$ (formal power series with coefficients in $K[x]$ endowed with the topology of stationary convergence) by

$$Q\Big(\sum_{n=0}^{\infty} a_n(x)u^n\Big) := \Big(\sum_{n=0}^{\infty} Q(a_n(x))u^n\Big)\ . \tag{20}$$

Then, as all the summands are summable:

$$Q(M) = Q\Big(\sum_{n\geq 0} p_n(x)\frac{u^n}{n!}\Big) = \sum_{n\geq 0} Q(p_n(x))\frac{u^n}{n!} = \sum_{n\geq 1} np_{n-1}(x))\frac{u^n}{n!} = uM(x,u) \tag{21}$$

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[4] Contrary to a common belief a sum $\sum_{k\geq 0} a_k\frac{Q^k}{k!}$ is by no means formal because Q is locally nilpotent, see [1].

On the other hand

$$Q(M) = Q(e^{xf(u)}) = \sum_{n\geq 0} Q[x^n]\frac{f(u)^n}{n!}\ . \tag{22}$$

As the linear form $\cdot\Big|_{x=0} : K[x][[u]] \to K[u]$ is continuous, one gets $u = q(f(u))$ which proves the claim.

6 Solution of e).

Let us first prove the following lemma:

Lemma 6.1 *Let $(r_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$ be two basic sequences with delta operators Q_r and Q_s. Let ϕ be the automorphism of $K[x]$ which sends the first to the second sequence (i. e. $\phi(r_n) = s_n$ for all $n \in \mathbb{N}$). Then, for each basic sequence $(p_n)_{n\in\mathbb{N}}$, the sequence $t_n = \phi(p_n)$ is basic with delta operator $\phi Q_p \phi^{-1}$.*

Proof – It is straightforward that ϕ is degree-preserving so that $\deg(t_n) = \deg(p_n) = n$. Moreover

$$t_0 = \phi(p_0) = \phi(1) = \phi(r_0) = s_0 = 1$$

and, with $\phi Q_p \phi^{-1}$ and $n \geq 1$,

$$Q[t_n] = \phi Q_p \phi^{-1}(t_n) = \phi Q_p \phi^{-1}\phi(p_n) = \phi Q_p\, p_n = \phi n p_{n-1} = n\phi(p_{n-1}) = n\, t_{n-1}\ .$$

Remains to prove that Q is shift-invariant. This is the consequence of the fact that $Q_s = f_s(D)$ for some $f_s(u) = b_1 u + b_2 u^2 + \cdots$ and that $[Q, Q_s] = \phi[Q_p, Q_r]\phi^{-1} = 0$, then Q commutes with $g_s(Q_s) = D$ (g_s is the compositional inverse of f_s) and thus, Q is shift-invariant as

$$E^a = \sum_{n=0}^{\infty} D^n a^n/n!\ .$$

□

We now apply the lemma with $r_n = x(x - \alpha n)^{n-1}$ (Abel polynomials) and $s_n = x^n$. It is clear that the (linear) transformation ϕ defined by $\phi(1)$ and on polynomials of degree $n \geq 1$ without constant term by

$$\phi(P) := \frac{x}{x + \alpha n} P(x + \alpha n)$$

is the automorphism which sends r_n to s_n. Hence the claim.

References

[1] V.G. KAC, *Infinite dimensional Lie algebras*, Cambridge, 1990.