Shifts and characterization of the elements of $\mathcal{A}^\circ$

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We take the following notations : if $\mathcal{A}$ be an algebra and $f \in \mathcal{A}^*$,
- $f_x : y \mapsto f(xy)$ (left shift) ;
- ${}_xf : y \mapsto f(yx)$ (right shift) ;
- ${}_xf_y : z \mapsto f(yzx)$;
- A° denotes the finite dual (also called Sweedler's dual).

Theorem 0.1. ***Theorem by Abe (extended by Schützenberger's condition (property*** (vi) ***below))*** - Characterization of the elements of $\mathcal{A}^\circ$:
Let $\mathcal{A}$ be an algebra and $f \in \mathcal{A}^*$. The following properties are equivalent :
- (i) ${}^t\mu(f) \in \mathcal{A}^* \otimes \mathcal{A}^*$;
- (ii) *The family* $(f_x)_{x\in\mathcal{A}}$ *is of finite rank ;*
- (iii) *The family* $({}_xf)_{x\in\mathcal{A}}$ *is of finite rank ;*
- (iv) *The family* $({}_xf_y)_{x,y\in\mathcal{A}}$ *is of finite rank ;*
- (v) $f(xy) = \sum_{i=1}^{n} f_i(x)g_i(y)$;
- (vi) $\exists\ \alpha : \mathcal{A} \to k^{n\times n}$ *and* $(\lambda, \gamma) \in k^{1\times n} \times k^{n\times 1}$ *such that*

$$\forall x \in \mathcal{A}, \quad f(x) = \lambda\, \alpha(x)\, \gamma;$$

- (vii) $\mathrm{Ker}(f)$ *contains an ideal of finite codimension (i.e. there exists an ideal I such that*

$$\dim(\mathrm{Ker}(f)/I) < \infty). \tag{1}$$

Proof : We begin by two lemmas :

Lemma 0.2. *Let t be an element of $V \otimes W$. If we decompose t as $\sum_{i=1}^{n} f_i \otimes g_i$ with n minimal, then the f_i's and the g_i's form free families.*

Proof : Indeed, if that was not the case, one could write that $y_n = \sum_{i=1}^{n-1} \alpha_i y_i$ and

$$t = x_n \otimes y_n + \sum_{i=1}^{n} x_i \otimes y_i = \sum_{i=1}^{n-1} (\alpha_i x_n + x_i) \otimes y_i.$$

But this equation shows that n is not minimal and it is impossible by hypothesis. □

Lemma 0.3. *Let U, V and W be vector spaces and $\phi : U \times V \to W$ a bilinear map. Then if $z \in \mathrm{Im}(\phi)$ and if $z = \sum_{i=1}^{n} \phi(x_i, y_i)$ with n minimal among all decompositions of this form, the families $(x_i)_{1\le i\le n}$ and $(y_i)_{1\le i\le n}$ are free in their respective spaces.*

Note that the first lemma is a consequence of the second one : take $\phi = \otimes$.

Proof : Assume, without loss of generality, that $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$. Then

$$\begin{aligned} z &= \sum_{i=1}^{n-1} \phi(x_i, y_i) + \phi(x_n, y_n) \\ &= \sum_{i=1}^{n-1} \phi(x_i, y_i) + \phi(\sum_{j=1}^{n-1} \alpha_j x_j, y_n) \\ &= \sum_{i=1}^{n-1} \phi(x_i, y_i) + \sum_{j=1}^{n-1} \alpha_j \phi(x_j, y_n) \end{aligned}$$

since ϕ is bilinear. The same argument allows us to "factor" the x_i's in the two sums :

$$z = \sum_{i=1}^{n-1} \phi(x_i, y_i + \alpha_i y_n)$$

Therefore, there would be another decomposition of z with $n' = n - 1 < n$ terms. This is impossible since n is minimal. Therefore, $(x_i)_i$ is a free family. The same argument also applies for $(y_i)_i$. □

The first lemma is in fact a consequence of the second (take $\mu = \otimes$).

1. $(i) \Rightarrow (v)$ and $(v) \Rightarrow (ii), (iii)$: We assume that ${}^t\mu(f) \in \mathcal{A}^* \otimes \mathcal{A}^*$. Therefore ${}^t\mu(f) = \sum_{i=1}^{n} f_i \otimes g_i$. Now
$$\langle {}^t\mu(f) | x \otimes y \rangle = f(xy) = \sum_{i=1}^{n} \langle f_i \otimes g_i | x_i \otimes y_i \rangle = \sum_{i=1}^{n} f_i(x) g_i(y) \tag{2}$$
which is the condition (v).
Hence we can write that
$$({}_y f) = \sum_{i=1}^{n} g_i(y) f_i;\ {}_{\mathcal{A}} f \subset \mathrm{span}(f_i).$$
$$(f_x) = \sum_{i=1}^{n} f_i(x) g_i;\ f_{\mathcal{A}} \subset \mathrm{span}(g_i).$$
The orbits and ${}_{\mathcal{A}} f$ and $f_{\mathcal{A}}$ are of finite rank.
2. $(v) \Rightarrow (iv)$: If the g_i's are in $f_{\mathcal{A}}$, their shifts are also in this orbit (which is finite dimensional). Thus, they are of finite rank and $g_i(yz) = \displaystyle\sum_{j=1}^{m} g_{ij}^1(y) g_{ij}^2(z)$.
$$f(yzx) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(y) g_{ij}^1(y) g_{ij}^2(z),\ \forall\, z \in \mathcal{A}.$$
This is equivalent to the following equation :
$${}_x f_y = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(y) g_{ij}^1(y) g_{ij}^2 \tag{4}$$
Let us show now that $(g_i)_i \subset f_{\mathcal{A}}$. We assume now that n is minimal (it is always possible). Since ${}^t\mu$ is bilinear, the lemma 0.3 applied to eq. (2) implies that the f_i's and g_i's form free families.
$(g_i)_i$ is a free family of minimal rank. Hence, $(g_i)_i$ is a basis of $\mathrm{span}_{x \in \mathcal{A}}(f_x)$ and the g_i's are in $f_{\mathcal{A}}$.
Therefore Eq. (4) implies that
$$({}_x f_y)_{x,y \in \mathcal{A}}$$
is of finite rank.

3. $(iii) \Rightarrow (vi)$: Let ${}_{x_1}f, {}_{x_2}f, \ldots, {}_{x_n}f$ be a basis of the orbit ${}_{\mathcal{A}}f$ of f under the action of $\mathcal{A}$. ${}_1f = f \in {}_{\mathcal{A}}f$. Thus,

$$ {}_1f = (\lambda_1 \ldots \lambda_n) \begin{pmatrix} {}_{x_1}f \\ \vdots \\ {}_{x_n}f \end{pmatrix} $$

and

$$ {}_1f(y) = (\lambda_1 \ldots \lambda_n) \begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix}. $$

Now, if we do another shift on the left, we stay in the same space generated by ${}_{x_1}f, {}_{x_2}f, \ldots, {}_{x_n}f$ and we have another decomposition which involves a (uniquely defined) matrix $\alpha(y)$:

$$ {}_y\begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix} = \begin{pmatrix} {}_{yx_1}f(y) \\ \vdots \\ {}_{yx_n}f(y) \end{pmatrix} $$

with ${}_{yx_i}f(y) = \sum_{j=1}^{n} \alpha_{ij}(y)\, {}_{x_j}f$. Therefore,

$$ {}_y\begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix} = \begin{pmatrix} \alpha_{11}(y) & \ldots & \alpha_{1n}(y) \\ \vdots & & \vdots \\ \alpha_{n1}(y) & \ldots & \alpha_{nn}(y) \end{pmatrix} \begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix} $$

$f(z) = {}_zf(1)$ thus

$$ \begin{aligned} {}_z\left[(\lambda_1 \ldots \lambda_n) \begin{pmatrix} {}_{x_1}f \\ \vdots \\ {}_{x_n}f \end{pmatrix} \right]\Bigg|_{y=1} &= (\lambda_1 \ldots \lambda_n) \; {}_z\begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix}\Bigg|_{y=1} \\ &= \left[(\lambda_1 \ldots \lambda_n)\alpha(z) \begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix} \right]\Bigg|_{y=1} \end{aligned} $$

Finally,

$$ f(z) = (\lambda_1 \ldots \lambda_n)\alpha(z) \begin{pmatrix} {}_{x_1}f(1) \\ \vdots \\ {}_{x_n}f(1) \end{pmatrix}. $$

4. $(iii) \Rightarrow (i)$: If $({}_xf)_{x\in\mathcal{A}}$ is of finite rank, there exists a linear representation $(\lambda, \alpha, \gamma)$ of f :

$$ f(z) = (\lambda_1 \ldots \lambda_n)\alpha(z) \begin{pmatrix} {}_{x_1}f(1) \\ \vdots \\ {}_{x_n}f(1) \end{pmatrix} $$

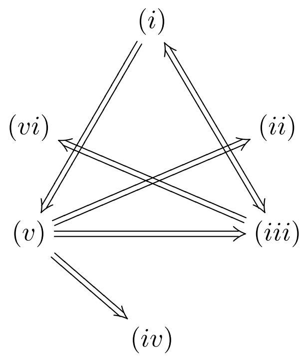
with :

$$ \alpha(x+y) = \alpha(x) + \alpha(y), $$
$$ \alpha(\delta x) = \delta\alpha(x), $$
$$ \alpha(xy) = \alpha(x)\alpha(y). $$

Therefore,

$$\begin{aligned}
f(xy) &= \sum_{k,\ell=1}^{n} \lambda_k \alpha_{k\ell}(xy)\gamma_\ell \\
&= \sum_{k,\ell=1}^{n} \sum_{j=1}^{m} \lambda_k \alpha_{kj}(x)\alpha_{j\ell}(y)\gamma_\ell \\
&= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \lambda_k \alpha_{kj}(x) \right) \left(\sum_{\ell=1}^{n} \alpha_{j\ell}(y)\gamma_\ell \right)
\end{aligned}$$

This equation tells us that $f(xy)$ and therefore ${}^t\mu(f)$ belongs to $\mathcal{A}^* \otimes \mathcal{A}^*$.
We proved the following implications :



The proofs of the following implications are straightforward now : $(ii) \Rightarrow (i)$, $(iv) \Rightarrow (i)$ and $(vi) \Rightarrow (i)$.