Symmetric Functions
and
Many Body Integrable Systems

Matthieu Deneufchâtel
Supervisor: Professor J.Y. Thibon
Institut d'électronique et d'informatique Gaspard Monge
Laboratoire d'informatique

January - May 2009

Acknowlegments

First of all I want to thank Professor Jean-Yves Thibon who accepted to supervise this work. I also want to express my gratitude to Alain Lascoux who took time to explain me his algorithms, and to Jean-Gabriel Luque for his classes which helped me to become familiar with the symmetric functions. I thank all the members of the *Algebraic Combinatorics and Symbolic Computation* team that I met. They have been helpful and were always ready to give me the explanations I needed.

Contents

1 General introduction

The subject that I studied during this training is a link between two disciplines: first, it has a big part of physics because it deals with the many body problem; then, it is an interesting problem of combinatorics, because it involves multivariate symmetric functions and polynomials.

The fact that there exist several classical and quantum many-body solvable systems is not very famous. Most of the time, physics students are told that when N, the number of bodies, is greater than 2, we cannot solve the N-body problem. This is not true. Indeed, for some pair interaction potentials, it is possible to show the integrability of the problem and it is even possible to exhibit eigenfunctions of the hamiltonian, as we shall see in this work.

The long-term motivation of this work is the following: by a certain transformation, it is possible to show that the following differentiel operator play a key role in some of the N-body solvable systems:

$$\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \epsilon \sum_{i \neq j} \frac{1}{x_i - x_j} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right) \tag{1}$$

This differential operator is interesting because it also appears in the theory of random matrices. The dynamics of some classes of random matrices is in fact governed by this operator (or by slighty modified operators that are linked to this one). The aim of my work is to give a simple interpretation of these links. A good way to get a deeper understanding of the problem is to use the symmetric functions that are linked to this operator (eigenfunctions, functions that belong to its kernel...). The theory of symmetric functions is rich and is one of the interests of the team I worked with during this training. In particular, Jack polynomials are of great interest and play an important role in some of the many body integrable problems. It would be interesting to explain why.

So far, I only began to learn the basic notions that allow me to understand the way people deal with this problem; I also spent a lot of time to get used to the algorithms that have been developed by Alain Lascoux and Jean-Yves Thibon to manipulate symmetric functions with Maple (the ACE package). I gathered informations about these questions and the different ways that can be followed to solve the problem. In particular, I made a short review of four different ways of proving the integrability of the Calogero problem. This work gave me the opportunity of studying a lot of methods and ideas that play a key role in this problem.

2 Introduction to Many Body Integrable Systems

Let us recall some basic notions of analytical mechanics. In hamiltonian mechanics, a system with N degrees of freedom is called *integrable* if it is possible to find N functions of the generalized coordinates $\{q_1, ..., q_N, p_1, ..., p_N\}$, denoted by $F_i, ..., F_N$ such that:

$$\{F_i, F_j\} = 0, \forall i, j \in 1, ..., N \tag{2}$$

where the *Poisson crochet* is defined by $\{A, B\} = \sum_{i=1}^{N} \left[\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right]$. When the motion is bounded, it is possible to find a canonical transformation that maps $H(q_1, ..., q_N, p_1, ..., p_N)$ onto a new function, $K(F_1, ..., F_N, \omega_1, ..., \omega_N)$, which satisfies Hamilton's equation, such that the ω_i's vary linearly with the time (the time derivative of the F_i's being zero). This new system, which is equivalent to the physical one, has a much simpler dynamics. However it is always hard to give the expressions of the solutions in terms of physical variables.

Calogero showed that integrable many-body systems have a potential such that $V(x) = a^2(x)$ with a satisfying the functional equation:

$$a(x)a'(y) - a(y)a'(x) = a(x+y)(f(x) - f(y)) \tag{3}$$

f being some arbitrary function which can be appropriately chosen for every solution a. The most general form of the potential is $a^2\wp(ax)$, where $\wp$ denotes the *Weierstrass elliptic function*, defined by:

$$\wp(z) = \frac{1}{z^2} + \sum_{n,m\in\mathbb{Z}}{}' \left[\frac{1}{(z-n-m)^2} - \frac{1}{(n+m)^2}\right] \tag{4}$$

(the prime over the sum excludes the case $n = m = 0$). Note that the relation (3) implies

$$(V(x)V'(y) - V(y)V'(x)) + (V(y)V'(z) - V(z)V'(y)) + (V(z)V'(x) - V(x)V'(z)) = 0 \tag{5}$$

for $x + y + z = 0$.

By taking different limits of the $\wp$ function, we find that systems with the following pair interaction potential are integrable: $\frac{1}{x^2}, \frac{a^2}{\sinh^2(ax)}, \frac{a^2}{\sin^2(ax)}$.

We will study systems whose hamiltonian has the following form:

$$H = \sum_{j=1}^{N} U_1(x_j) + \sum_{j<k}^{N} U_2(x_j, x_k) \tag{6}$$

where U_1 represents the kinetic part, and U_2 is one of the potentials listed above, with possibly a harmonic potential ($\omega^2 x_j^2$ or $\omega^2\left(x_i^2 - x_j^2\right)$ up to a constant). We are especially interested in the Calogero problem, with a potential defined by:

$$U_2(x_j, x_k) = \frac{1}{(x_j - x_k)^2} \tag{7}$$

3 Links between Matrix Theory and Integrable Systems

3.1 Links between Random Matrix Theory and Integrable Systems

There are many links between different physical subjects and the random matrix theory. The study of random matrix theory applied to physics began with the interest of Wigner for high dimensional Hamiltonians that describe the energy levels of nuclei. His idea, formulated in the 1950's, was that the properties (eigenvalues, eigenvectors...) of such high dimensional matrices are not very different from the properties of random matrices. In particular, it appeared that, for matrices of a given form, the probability density function of the eigenvalues was the Boltzmann weight

$$\frac{1}{C_\beta} \exp\left[-\beta\left(\sum_{j=1}^{N} U_1(x_j) + \sum_{j<k} U_2(x_j, x_k)\right)\right]$$

for a gaz with a peculiar interaction between the particules in equilibrium at "temperature" β. Indeed, the two-body potential was the potential of charged particules in a two dimensional space that move on a line. But the parameter β can take on a small numbers of values, depending on the form of the matrix. Thus the probability density function is the Boltzmann weight for a gaz at fixed temperature. This is the first link between a many-body problem (but with a statistical point of view and not a hamiltonian one) and random matrix theory.

The link between random matrix theory and Calogero models appears more precisely when we consider the dynamics of the eigenvalues of a random hermitian matrix X, $N \times N$, that obey the equation $\ddot{X}(t) = 0$ (see, for example, in [1]). Indeed, the dynamics of the eigenvalues is described by the following equation:

$$\ddot{q}_i(t) = -2 \sum_{j=1}^{N}{}' \frac{C_{ik}C_{ki}}{(q_i(t) - q_j(t))^3} \tag{8}$$

$\forall\, i \in \{1, ..., N\}$, where the coefficients C_{ij} are functions of the coefficients of the matrix X. This system of differential equations represents the equations of motions for a classical and conservative system of N particules whose energy is $\frac{1}{2}\left[\sum_{i=1}^{N} p_i^2 - \sum_{i \neq j}^{N}{}' \frac{C_{ik}C_{ki}}{(q_i(t) - q_j(t))^2}\right]$. In terms of statistical mechanics, the Boltzmann weight can be written as $\exp(-\beta H)$ for a system in equilibrium at temperature β. However for some random matrices, the probability density for the eigenvalues (with some conditions on the form of the matrix) is (see [2]):

$$\frac{1}{C_\beta} \exp\left[\sum_{j=1}^{N} U_1(x_j) + \sum_{j<k}^{N} U_2(x_j, x_k)\right] \tag{9}$$

Thus the probability density of the eigenvalues is, up to a constant, the Boltzmann weight for a Calogero system.

It is possible to go further by considering one parameter random matrices (*e.g.* time-dependent random matrices). For example, for Wishart matrices, the probability density for the positive root of the eigenvalues obeys a Fokker - Planck equation of the form:

$$\gamma \frac{\partial p}{\partial \tau} = \mathcal{L}p \text{ avec } \mathcal{L} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\frac{\partial W}{\partial x_j} + \frac{1}{\beta}\frac{\partial}{\partial x_j}\right) \tag{10}$$

This equation also describes the dynamics of a gaz of N particules in the potential W with a friction coefficient γ, at temperature β, p denoting the probability density of finding the N particules at the points $x_1, \ldots, x_N$. This equation can be transformed, with a change of variables, into the Schrödinger equation that describes the dynamics of a N-body Calogero - Sutherland model.

3.2 Radial part of Laplace-Beltrami Operator on a matrix space

Let us consider the radial part of the Laplacian on $\mathcal{F}(\mathcal{H}_N)$, the space of functions on $\mathcal{H}_N$, where $\mathcal{H}_N$ denotes the set of hermitian $N \times N$ matrices. On a Riemannian manifold (M, g), the Laplace-Beltrami operator (that generalizes the Laplacien) is given by:

$$\frac{1}{\sqrt{\bar{g}}} \sum_k \frac{\partial}{\partial x_k} \sum_i g^{ik} \sqrt{\bar{g}} \frac{\partial}{\partial x_i} \tag{11}$$

where $\bar{g} = |\det(g_{ij})|$, $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $g^{ij} = [(g_{ij})]^{-1}$.

Let us begin by defining the gradient. For $f \in \mathcal{C}^\infty(M)$, ∇f is defined by:

$$\vec{\nabla} f \cdot X = df(X) \tag{12}$$

Thus $\vec{\nabla} f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$.

In order to define the divergence, we define before a $(n-1)$-form, $X.\omega$, from the volume form $\omega = \sqrt{g} dx_1 \wedge \cdots \wedge dx_n$, by the relation:

$$X.\omega(X_2, \ldots, X_n) = \omega(X, X_2, \ldots, X_n) \tag{13}$$

Its differential is a n-form. But the n-forms are all proportionnal to the volume form. We call divergence the coefficient of proportionality between ω and $d(X.\omega)$.

The computation gives:

$$\vec{\nabla} f = \frac{1}{\sqrt{g}} \sum_i \frac{\partial}{\partial x_i} (\sqrt{g} X^i) \tag{14}$$

Laplace-Beltrami operator is thus generally given by:

$$\Delta f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{1}{\sqrt{g}} \sum_k \frac{\partial}{\partial x_k} \sum_i g^{ik} \sqrt{g} \frac{\partial}{\partial x_i} \tag{15}$$

On $\mathcal{H}_N$, the scalar product of two matrices, defined as $\langle A, B \rangle = \mathrm{Tr}(AB^\dagger)$, becomes

$$\langle A, B \rangle = \sum_{i,j} A_{ij} B^\dagger_{ji} = \sum_{i,j} A_{ij} B_{ji}$$

The metric is constant, in particular its determinant is constant and this simplifies the expression of the Laplacian: $\Delta = g^{ik} \sum_{k,i} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i}$.

Finally,

$$\Delta f(H) = \sum_i \frac{\partial^2 f}{\partial H_{ii}^2} + \sum_{i \neq j} \frac{\partial^2 f}{\partial H_{ij} \partial H_{ji}} \tag{16}$$

But it is possible to prove differential equations linking the derivatives of f with respect to the components of H to the derivatives of f with respect to the eigenvalues of H if f is invariant under conjugation/cy, *i.e.* $f(UHU^{-1}) = f(H)$. Let us consider, in order to prove these equations, the perturbation of a diagonal matrix Λ with a perturbation $\epsilon \delta H$ whose diagonal form is $\delta \Lambda$.

We denote by λ_i the non zero elements of Λ, that we suppose all different. Applying the theory of stationary perturbations in the non-degenerate case, we get (for a more rigourous demonstration, see [3]):

$$\lambda_i(\epsilon) = \lambda_i + \epsilon \delta H_{ii} + \epsilon^2 \sum_{i \neq j} \frac{\delta H_{ij} \delta H_{ji}}{\lambda_i - \lambda_j} + \mathcal{O}(\epsilon^3) \tag{17}$$

$$f(\Lambda + \epsilon \delta H) = f(\Lambda) + \epsilon \sum_{a,b} \frac{\partial f}{\partial H_{ab}} \delta H_{ab} + \frac{\epsilon^2}{2} \sum_{a,b,c,d} \frac{\partial^2 Z}{\partial H_{ab} \partial H_{cd}} \delta H_{ab} \delta_{cd} + \mathcal{O}(\epsilon^3).$$

But since f is $U(N)$ invariant,

$$f(\Lambda + \epsilon \delta H) = f(\Lambda + \epsilon \delta \Lambda) = f(\Lambda) + \sum_i \frac{\partial f}{\partial \lambda_i} \delta \Lambda_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j} \delta \lambda_i \delta \lambda_j + \mathcal{O}(\delta \lambda^3)$$

Replacing $\delta\lambda_i$ by $\epsilon\delta H_{ii} + \epsilon^2 \sum_{i\neq j} \frac{\delta H_{ij}\delta H_{ji}}{\lambda_i - \lambda_j}$ and identifying term to term, we obtain the following relations:

$$\begin{cases} \dfrac{\partial f}{\partial H_{ij}} = \delta_{ij} \dfrac{\partial f}{\partial \lambda_i} \\ \dfrac{\partial^2 f}{\partial H_{ij} \partial H_{ji}} = \dfrac{1}{\lambda_i - \lambda_j} \left(\dfrac{\partial f}{\partial \lambda_i} - \dfrac{\partial f}{\partial \lambda_j} \right) \end{cases} \tag{18}$$

Using the previous relations in (16), we obtain the following expression for the radial part of the Laplacian:

$$\Delta_{\text{rad}} = \sum_i \frac{\partial^2}{\partial \lambda_i^2} + \sum_{i\neq j} \frac{1}{\lambda_i - \lambda_j} \left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j} \right) \tag{19}$$

This operator is a special case of D for $\epsilon = 1$. D seems to be the generalization of the previous Laplace-Beltrami operator on a space with a modified metric. But the geometrical interpretation of this metric is still to be found.

Note that an operator of this type appears (see [2], page 126) in the differential equation that is satisfied by the probability density function $P(H; H^{(0)}, t)$ for the elements of a random matrix H real and orthogonal, hermitian or "self-dual real quaternion" (with, respectively $\beta = 1, 2, 4$) depending on the parameter t:

$$\frac{\partial P}{\partial t} = \sum_\mu \left(\frac{\partial}{\partial H_\mu} (H_\mu P) + \frac{1}{\beta} D_\mu \frac{\partial^2 P}{\partial H_\mu^2} \right) \tag{20}$$

where μ describes the set of the independent elements of the matrix, and with $D_\mu = 1$ for the diagonal elements, $D_\mu = \frac{1}{2}$ for the non-diagonal elements.

4 Introduction to Symmetric Functions

Let us begin with some definitions. When we deal with N particules in one dimension, their coordinates are denoted by $X_1, \ldots, X_N$. The alphabet is the sum of the coordinates: $X = X_1 + \cdots + X_N$. Therefore, when we deal with a function of the N variables $x_1, \ldots, x_N$, we will often write $f(X)$ instead of $f(x_1, \ldots, x_N)$ We denote by S_N the symmetric group on the set $\{1, \ldots, N\}$.

Definition 1 *A function f of the variables $X_1, \ldots, X_N$ is called* symmetric *if*

$$f(X_{\sigma(1)}, \ldots, X_{\sigma(N)}) = f(X_1, \ldots, X_N), \forall \sigma \in S_N$$

Definition 2 *A function f of the variables $X_1, \ldots, X_N$ is called* homogeneous of degree d *if*

$$f(\alpha X_1, \ldots, \alpha X_N) = \alpha^d f(X_1, \ldots, X_N)$$

If f is a polynomial, then it is made up of monomials of the same degree d.

Definition 3 *A* partition λ *of the integer n of length l is a set of l positive integers, $\lambda_1, \ldots, \lambda_l$, such that*

$$\sum_{k=1}^{l} \lambda_k = n$$

There are several bases of the space of symmetric functions. The bases that are easy to define are the elementary symmetric functions, the power sums and the monomial symmetric functions.

Definition 4 *The previous functions are defined as follow:*

- *Elementary symmetric functions:* $e_n(X) = \sum_{\substack{m_1,\ldots,m_n \\ \text{all different}}} X_{m_1} \ldots X_{m_n}$*;*
- *Power sums:* $p_n(X) = \sum_{i=1}^{N} X_i^n$*;*
- *Monomial symmetric functions:* $m_\lambda(X) = \sum_{\sigma \in S_N} X_1^{\sigma(\lambda_1)} \ldots X_N^{\sigma(\lambda_l)}$*, where* λ *is a partition. Each permuation* σ *appears only once.*

The elementary symmetric functions (sometimes denoted by ESF) are the functions that allow us to express the coefficients of a polynomial as functions of its roots. The monomial symmetric functions are simply the functions obtained after symmetrisation of a monomial of degree $\sum_i \lambda_i$. It is the same thing to take the sum over the permutation of the variables or the sum over the permutation of the exponents.

5 Different proofs of the integrability of the Calogero system

5.1 Lax pairs and Conserved quantities

A discussion of the existence of N conserved quantities in involution can be found in *e.g.* [1] (chapter 2). A rigourous proof is given in [4]. However the integrability of these systems can be proved by others arguments as we shall see.

In this section, we will consider the Hamiltonian of the Calogero system in the form

$$H = \frac{1}{2}\sum_{k=1}^{N} p_k + \frac{1}{2}\epsilon \sum_{i \neq j} V(x_i - x_j)$$

Let us come back to a question that we already mentionned (see Section 3.1): the evolution of the eigenvalues of a hermitian $N \times N$ matrix X satisfying the dynamical equation

$$\ddot{X}(t) = 0 \tag{21}$$

We mentionned the fact that the eigenvalues satisfy the following equation:

$$\ddot{q}_i(t) = -2 {\sum_{j=1}^{N}}' \frac{C_{ik}C_{ki}}{(q_i(t) - q_j(t))^3} \tag{22}$$

To derive these equations (see [1]) we need to introduce what is called a *Lax pair*, *i.e.* a pair of time dependant matrices $(L(t), A(t))$ such that the dynamics of X is represented by

$$\frac{dL(t)}{dt} = [L, A] \tag{23}$$

The Lax pair method is a very general method to solve partial differential equations which consists in rewriting the PDE of interest in the form (23). Indeed, denoting by $Q(t)$ the matrix that diagonalize $X(t)$, we have: $X(t) = U(t)Q(t)U^{-1}(t)$. The first derivative of X with respect to t may be written $U(t)L(t)U^{-1}(t)$ with $L(t) = \left[U^{-1}\dot{U}, Q\right] + \dot{Q}$. Equation (21) then gives

$$\dot{L} = [L, M]\,,\; M = U^{-1}\dot{U} \tag{24}$$

and the coeeficient C_{ij} are defined as $[Q(0), L(0)]_{ij}$.

Usually, the set of integrals is the set of the $\mathrm{Tr}(L^k)$, $k \in \{1, \ldots, N\}$ (the $\mathrm{Tr}(L^k)$ for $k \in \mathbb{N}$ are all constant, but only N of them are functionally independant). The involutivity of these integrals is not proved directly. For $t \to \pm\infty$, the system is equivalent to a system of free particules (because of the repulsive interaction potential) and thus

$$\mathrm{Tr}(L^k) \to \sum_{i=1}^{N} p_i^k = \mathfrak{p}_k(\mathbf{p})$$

(where $\mathfrak{p}$ denotes the power sum) in both limits. These quantities are functionally independent since they are polynomials of different degrees in the p_k and are thus in involution. Since they are constant, they are also in involution during the evolution of the system.

Let us now define another set of integrals (following [4]). The first one, denoted by I_N, is defined as:

$$I_N = \exp\left(-\frac{1}{2}\epsilon \sum_{i \neq j} V(x_i - x_j)\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\right)\prod_{k=1}^{N} p_k \tag{25}$$

This quantity is well-defined. Indeed, the number of terms that are involved in the exponential is finite. The argument of the exponential is a differential operator that acts upon a quantity that is linear in each of the p_k's. Thus the non vanishing terms are these that do not have second derivative with respect to one of the p_k's. But if $2n > N$, it is not possible to find a N-uple with each subscribe different from the others, and therefore the components of the development of the exponential for $2n > N$ are equal to zero. I_N is a symmetric function of the N-th order of the p_k's; it is translationally invariant, *i.e.* $\{p_k, I_N\} = 0$. This is easily seen from the expression of I_N that involves only differences of the spatial coordinates $(x_i - x_j)$. The other integrals are defined inductively by taking the successive Poisson brackets of I_N with $\sum_{k=1}^{N} x_k$:

$$I_{n-1} = \{I_n, \sum_{k=1}^{N} x_k\} \tag{26}$$

We will need the Jacobi identity for the next proofs. This relation is true for every arguments of the Poisson bracket, *i.e.* for every function of the x_k's and p_k's. It states that $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$. Therefore,

$$\{H, \{I_n, \sum_{k=1}^{N} x_k\}\} + \{I_n, \{\sum_{k=1}^{N} x_k, H\}\} + \{\sum_{k=1}^{N} x_k, \{H, I_n\}\} = 0$$

and

$$\{\sum_{j=1}^{N} p_j, \{I_n, \sum_{k=1}^{N} x_k\}\} + \{I_n, \{\sum_{k=1}^{N} x_k, \sum_{j=1}^{N} p_j\}\} + \{\sum_{k=1}^{N} x_k, \{\sum_{j=1}^{N} p_j, I_n\}\} = 0$$

The following relation, whose proof is straightforward, will also be useful:

$$\{\sum_{k=1}^{N} x_k, H\} = -\sum_{j=1}^{N} p_j$$

Assuming that $\{H, I_n\} = 0$ and that $\{\{p_j, I_n\} = 0, \forall\, j\}$, and using the previous relation in the two expressions of the Jacobi identity yields:

$$\{H, I_{n-1}\} + \sum_{j=1}^{N}\{I_n, p_j\} + 0 = 0$$

and

$$\sum_{j=1}^{N}\{p_j, I_{n-1}\} + \{I_n, (-N)\} + 0 = 0$$

or $\{H, I_{n-1}\} = 0$ and $\sum_{j=1}^{N}\{p_j, I_{n-1}\} = 0$. Since the p_j are functionally independant, we find that $\{H, I_n\} = 0$ and $\{\{p_j, I_N\} = 0, \forall\, j\}$ implies that $\{H, I_{n-1}\} = 0$ and that $\{\{p_j, I_{n-1}\} = 0, \forall\, j\}$. Since I_N is translationally invariant, we have proved by induction that I_n is translationally invariant $\forall\, n < N$. If we prove that I_N commutes with the Hamiltonian, the induction will assure us that the I_n are constant. This proof is to be found in Appendix (Section 6.5).

It remains to be proved that the constants I_n are in involution. Let us consider the quantity $\{I_m, I_n\}$ which is equal to a function of the spatial coordinates and of the momenta through the I_n's:

$$\{I_m, I_n\} = F(I_1, \ldots I_N, \mathbf{x}) \tag{27}$$

Applying $\partial = \sum_{k=1}^{N} \frac{\partial}{\partial p_k}$ on both sides, we find that

$$\{I_{m-1}, I_n\} + \{I_m, I_{n-1}\} = \sum_{k=1}^{N} \frac{\partial F}{\partial I_k}\partial(I_k) = \sum_{k=1}^{N} \frac{\partial F}{\partial I_k} I_{k-1}$$

Let us assume that all the Poisson bracket of the form $\{I_r, I_s\}$ with $r + s = m + n - 1$ vanish.Then $\{I_{m-1}, I_n\}+\{I_m, I_{n-1}\} = 0$ and the functional independency of the I_k's implies that $\frac{\partial F}{\partial I_k} = 0$. Thus, $F(I_1, \ldots I_N, \mathbf{x}) = F(\mathbf{x})$. But from the independency of p_k and from the form of the Hamiltonian, it immediatly follows that $\frac{\partial F}{\partial x_k} = 0$. It is always possible to chose appropriate coordinates such that $F(\mathbf{x})$ is equal to zero. Moreover, $\{I_1, I_2\} = 0$ because H is translationally invariant and $I_1 = \alpha \sum_{k=1}^{N} p_k,\ I_2 = \beta I_1^2 + \gamma H,\ \alpha, \beta, \gamma \in \mathbb{R}$. Therefore we proved by induction that the I_n's are in involution.

This makes one proof of the integrability of the Calogero problem.

5.2 Rotation of Calogero's Hamiltonian

See for example [5], [6] or [7].

We recall that the Hamiltonian of the Calogero system has the following form:

$$H_C = \frac{1}{2}\sum_{j=1}^{N}\left(-\frac{\partial^2}{\partial x_j^2} + \omega^2 x_j^2\right) + \frac{1}{2}\sum_{j,k;j\neq k}\frac{a(a-1)}{(x_j - x_k)^2} \tag{28}$$

Let us first consider the form of the eigenfunctions of this Hamiltonian and the equation of the dynamics of the polynomial part of the eigenfunctions of the Calogero system. This idea is developped in the books by J. Hoppe ([1]) or Perelomov ([8]). Let us begin by noting that, using the notation $r^2 = \frac{1}{N}\sum_{i>j}(x_i - x_j)^2$, the second term of the Hamiltonian presents a $\frac{1}{2}r^2$. We can thus think about a polynomial in r times a gaussian function: $P(r)\exp\left(-\frac{r^2}{2}\right)$ where $P(r)$ is the polynomial part. Changing the coordinates to work in the center of motion frame, it is possible to show that this problem is equivalent to a problem with an harmonic potential of the form $\sum_i x_i^2$ (see [8]).

Then a physical argument forces us to multiply the previous term by a power of the Vandermonde determinant. Indeed, the singularity of the potentiel for $x_i = x_j$, $i \neq j$ implies that two particules cannot be at the same point at the same time. The solution is equal to zero when $x_i = x_j$, $i \neq j$ and must respect this feature of the system. Thus it must be divisible by $z = \prod_{i>j}(x_i - x_j)$ or by a power of this product. Let us look for the eigenfunctions of H_C with the form:

$$\psi(\mathbf{x}) = z^\epsilon \phi(r) Q(\mathbf{x}) \tag{29}$$

The eigenvalue equation can be written, denoting by E the energy associated to ψ:

$$\begin{aligned}
-\frac{1}{2}&\left[\epsilon(\epsilon-1)\sum_{i\neq k}\frac{1}{(x_i-x_k)^2}\phi + (N(N-1)\epsilon + N - 2)\frac{\phi'}{r} + \phi''\right] z^\epsilon Q \\
&\qquad -\frac{1}{2}\left[2\frac{\phi'}{r}\sum_i (x_i - X)\partial_i Q\right] z^\epsilon \\
&\qquad -\frac{1}{2}(DQ)z^\epsilon\phi + \left[\frac{1}{2}r^2 + \epsilon(\epsilon-1)\sum_{i>j}\frac{1}{(x_i-x_j)^2}\right]\phi z^\epsilon Q = E\phi z^\epsilon Q
\end{aligned} \tag{30}$$

Assuming that $DQ = 0$ (which is the reason for my interest in the kernel of D; I spent quite a long time during my training to write a program that makes easy the study of the kernel of D) and that $\sum_i x_i \partial_i Q = kQ$ (relations whose physical meaning is to determine), the equation for ϕ looks like a one dimension Schrödinger equation:

$$-\frac{1}{2}\phi'' - \frac{1}{2}\frac{\phi'}{r}\left[N(N-1)\epsilon + N - 2 + 2k\right] + \frac{1}{2}r^2\phi = E\phi \tag{31}$$

Therefore, we consider the eigenfunctions of this operator in the form

$$\psi(\mathbf{x}) = \phi(\mathbf{x})\mathcal{P}(\mathbf{x})\exp\left(-\frac{1}{2}\omega r^2\right)$$

(with ϕ a symmetric polynomial, $\mathbf{x} = (x_1, \ldots, x_N)$, $\mathcal{P}(\mathbf{x}) = \prod_{1\leq j<k\leq N} |x_j - x_k|^b$, $b = a$ or $a-1$ and $r^2 = \sum_{l=1}^{N} x_l^2$). The system forbids the situations in which two particules are at the same point, so that we can consider only the cases where $x_j - x_k$ for $j < k$ is positive and take off the absolute value. The energy of the ground state is $E_g = \frac{1}{2}\omega N\left[(N-1)a+1\right]$ and the eigenfunction that is associated is $\psi_g(\mathbf{x}) = \mathcal{P}(\mathbf{x})\exp\left(-\frac{1}{2}\omega r^2\right)$. In order to get the equation that the polynomial part ψ obey, we are going to transform the Hamiltonian to get rid of the ground state. Let us now define the transformation:

$$H = \psi_g^{-1}(H_C - E_g)\psi_g \tag{32}$$

which gives us the two following relations (whose proof can be found in Appendix):

$$\begin{aligned} H &= \sum_{l=1}^{N}\left(-\frac{1}{2}\frac{\partial^2}{\partial x_l^2} + \omega x_l \frac{\partial}{\partial x_l}\right) - \frac{1}{2}a \sum_{l,m;l\neq m} \frac{1}{x_l - x_m}\left(\frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_m}\right) \\ &= -\frac{1}{2}D + \sum_{l=1}^{N} \omega x_l \frac{\partial}{\partial x_l} \end{aligned} \tag{33}$$

Note that we call respectively *Lassale* (denoted by $\mathcal{O}_L$) and *Euler operator* (denoted by $\mathcal{O}_E$) D and $\sum_{i=1}^{N} x_j \frac{\partial}{\partial x_j}$. We can thus write $H_C = \omega\mathcal{O}_E - \frac{1}{2}\mathcal{O}_L$.

5.3 Explicit construction of the eigenfunctions of D

Let us now exhibit some commutation relations between the four following operators: $\mathcal{O}_L$, $\mathcal{O}_E$, $\mathbf{x}^2$ and $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$. We will use the notation $\partial_i = \frac{\partial}{\partial x_i}$.

First

$$[\mathcal{O}_L, \mathcal{O}_E] = 2\mathcal{O}_L \tag{34}$$

Then

$$[\Delta, \mathcal{O}_E] = 2\Delta \tag{35}$$

(because

$$\begin{aligned} [\Delta, \mathcal{O}_E] &= \sum_{i,j} \partial_i \left(\delta_{i,j}\partial_j + x_j \partial_{ij}^2\right) - \sum_{i,j} x_j \partial_{ij}^2 \\ &= \sum_{i} \partial_i^2 + \sum_{i,j} \delta_{i,j}\partial_{i,j}^2 = 2\Delta) \end{aligned}$$

and

$$\left[\mathbf{x}^2, \mathcal{O}_E\right] = -2\mathbf{x}^2 \tag{36}$$

(because

$$\left[\mathbf{x}^2, \mathcal{O}_E\right] = \sum_{i,j} x_i^2 x_j \partial_j - \sum_{i,j} x_j \partial_j (x_i^2) = \sum_{i,j} x_i^2 x_j \partial_j - 2\sum_{i,j} x_j \left(\delta_{i,j} x_i + x_i^2 x_j \partial_j\right))$$

Finally,

$$\left[\Delta, \mathbf{x}^2\right] = 2(2\mathcal{O}_E + N) \tag{37}$$

Using these relations and relation (38) which stems from the Baker-Campbell-Hausdorff's formula, **Eq**. (39), it is possible to prove the relations (40), (41) and (42).

$$\begin{aligned}\exp(a)\, b\, \exp(-a) = \exp(-a)\, b = \sum_{n \leq 1} \frac{\overbrace{[a, [a, \ldots\ [a, b]]}^{n-1\,\text{times}}}{(n-1)!} \\ = b + [a, b] + \frac{1}{2}\,[a, [a, b]] + \ldots\end{aligned} \tag{38}$$

$$\exp(X) \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \ldots) \tag{39}$$

(the dots indicate that there is a series of terms that can be written as commutators of the commutator of X and Y).

$$\exp(\frac{1}{4\omega}\mathcal{O}_L) H_C \exp(-\frac{1}{4\omega}\mathcal{O}_L) = \omega\mathcal{O}_E \tag{40}$$

$$\exp(-\frac{1}{4\omega}\Delta)\, \omega\mathcal{O}_E \exp(-\frac{1}{4\omega}\Delta) = \omega\mathcal{O}_E - \frac{1}{2}\Delta \tag{41}$$

$$\exp(-\frac{1}{2}\omega\mathbf{x}^2) \exp(-\frac{1}{4\omega}\Delta)\, \omega\mathcal{O}_E \exp(\frac{1}{4\omega}\Delta) \exp(\frac{1}{2}\omega\mathbf{x}^2) = \frac{1}{2}\sum_{j=1}^{N}(p_j^2 + \omega^2 x_j^2) - \frac{1}{2}N\omega \tag{42}$$

These relations show that, after some transformations, the system is equivalent to a system of N independent quantum harmonic oscillators. Indeed, the Hamiltonian of this system of N oscillators is (when we take off the ground state energy) $\displaystyle\frac{1}{2}\sum_{j=1}^{N}(p_j^2 + \omega^2 x_j^2) - \frac{1}{2}N\omega$, which is precisely the expression that we find if we conjugate H_C with T^{-1}, *i.e.* if we consider the operator $T^{-1}H_C T$, with $T = \exp(-\frac{1}{4\omega}\mathcal{O}_L) \exp(\frac{1}{4\omega}\Delta) \exp(\frac{1}{2}\omega\mathbf{x}^2)$, using the relations proved above.

Let us now introduce creation and annihilation operators, respectively $a_j^\dagger$ and a_j, defined by:

$$a_j^\dagger = \frac{1}{2\mathrm{i}\omega}(p_j + \mathrm{i}\omega x_j)\,;\; a_j = \mathrm{i}(p_j - \mathrm{i}\omega x_j)$$

Denoting by $n_j = a_j^\dagger a_j = \dfrac{1}{2\omega}(p_j^2 + \omega^2 x_j^2 - \dfrac{1}{2})$ the number operator associated with the jth oscillator, we can write that:

$$T^{-1} H_C T = \omega \sum_{j=1}^{N} n_j \tag{43}$$

The eigenfunctions of the total number operator $\omega \sum_{j=1}^{N} a_j^\dagger a_j$ are indexed with N integers and their form is:

$$|n_1, \ldots, n_N\rangle = \prod_{j=1}^{N} (a_j^\dagger)^{n_j} |0\rangle \tag{44}$$

where $\langle \mathbf{x}|0\rangle = \exp(-\dfrac{1}{2}\omega \mathbf{x}^2)$. The search for the eigenfunctions of the Hamiltonian is thus easier after these transformations. But how can we find the expression of the eigenfunctions in terms of the original variables?

It is possible to show that the action of T on $|n_1, \ldots, n_N\rangle$ allows us to get back to expression in terms of the x_i:

$$T|n_1, \ldots, n_N\rangle = \exp(-\frac{1}{4\omega}\mathcal{O}_L) \prod_{j=1}^{N} x_j^{n_j} \tag{45}$$

First let us note that the transformation T can be factorized into a product of N transformations of the same form acting on each of the variables (the x_i being independent). We will note $T_j = \exp(-\partial_j^2 - a \sum_{k \neq j} \dfrac{\partial_j - \partial_k}{x_j - x_k}) \exp(\dfrac{1}{4\omega}\partial_j^2) \exp(\dfrac{1}{2}\omega x_j^2)$. Let us take $\omega = 1$ (it is always possible with a change of units). With these notations, we obtain that:

$$T|n_1, \ldots, n_N\rangle = \left(\prod_{j=1}^{N} T_j (a_j^\dagger)^{n_j} \right) |0\rangle \tag{46}$$

The eigenstates of the harmonic oscillators obey the following relations (see *e.g.* [9] page 397, with attention to the normalization of the operators):

$$\frac{(a_j^\dagger)^{n_j}}{\sqrt{n_j!}}|0\rangle = \frac{1}{\pi^{1/4}\sqrt{n_j!}} \frac{1}{2^{n_j}} H_{n_j}(x_j) \exp(-\frac{x_j^2}{2}) \tag{47}$$

or

$$|n_j\rangle = (a_j^\dagger)^{n_j}|0\rangle = \frac{1}{\pi^{1/4}\, 2^{n_j}} H_{n_j}(x_j) \exp(-\frac{x_j^2}{2}) \tag{48}$$

with (generating series of Hermite polynomials)

$$\sum_{n_j \geq 0} \frac{H_{n_j}(x_j) t^{n_j}}{n_j!} = \exp(2x_j t - t^2) \tag{49}$$

and thus

$$\begin{aligned} \sum_{n_j \geq 0} \frac{(a_j^\dagger)^{n_j}}{n_j!} t^{n_j} |0\rangle &= \frac{\exp(-x_j^2/2)}{\pi^{1/4}} \sum_{n_j \geq 0} \frac{H_{n_j}(x_j)}{n_j!} \left(\frac{t}{2}\right)^{n_j} \\ &= \frac{\exp(-x_j^2/2)}{\pi^{1/4}} \exp(x_j t - \frac{t^2}{4}) \end{aligned} \tag{50}$$

Therefore, (the t_j are parameters),

$$T_j \sum_{n_j \geq 0} \frac{|n_j\rangle}{n_j!} t_j^{n_j} = \exp(-\partial_j^2 - a \sum_{k \neq j} \frac{\partial_j - \partial_k}{x_j - x_k}) \exp(\frac{1}{4}\partial_j^2) \frac{1}{\pi^{1/4}} \exp(x_j t_j - \frac{t_j^2}{4}) \tag{51}$$

But

$$\frac{1}{4}\partial_j^2 \left(\exp(t_j x_j)\right) = \frac{1}{4} t_j^2 \exp(t_j x_j)$$

therefore

$$\sum_{k\geq 0}\frac{\left(\frac{1}{4}\partial_j^2\right)^k}{k!}\exp(t_jx_j)=\exp(\frac{1}{4}\partial_j^2)\exp(x_j)=\sum_{k\geq 0}\frac{\left(\frac{1}{4}t_j^2\right)^k}{n_j!}\exp(t_jx_j)=\exp(\frac{t_j^2}{4})\exp(t_jx_j)$$

Thus we have:

$$T_j\sum_{n_j\geq 0}\frac{t_j^{n_j}}{n_j!}|n_j\rangle=\frac{1}{\pi^{1/4}}\exp(-\partial_j^2-a\sum_{k\neq j}\frac{\partial_j-\partial_k}{x_j-x_k})\exp(x_jt_j)$$

and

$$T\sum_{n_1,\ldots,n_N\geq 0}\frac{t_1^{n_1}\ldots t_N^{n_N}}{n_1!\ldots n_N!}|n_1,\ldots,n_N\rangle=\frac{1}{\pi^{N/4}}\exp(-\frac{1}{4}\mathcal{O}_L)\exp(\mathbf{x}\cdot\mathbf{t}) \tag{52}$$

Developping the factor $\exp(\mathbf{x}\cdot\mathbf{t})$ we get:

$$\begin{aligned}\exp(\mathbf{x}\cdot\mathbf{t})&=\prod_{j=1}^{N}\sum_{n_j\geq 0}\frac{(x_jt_j)^{n_j}}{n_j!}\\&=\sum_{k_1,\ldots,k_N\geq 0}\prod_{j=1}^{N}\frac{(x_jt_j)^{n_j}}{n_1!\ldots n_N!}\end{aligned}$$

Identifying the left-hand side of (52) and the right-hand side of the previous relation we obtain, by linearity of T, the relation that we wanted to prove for $\omega=1$, up to a constant.

The following relations give the expressions of the operators in the physical variables in terms of annihilation and creation operators:

$$x_j=\exp(\frac{1}{4\omega})\exp(\frac{1}{2}\omega\mathbf{x}^2)a_j^{\dagger}\exp(-\frac{1}{2}\omega\mathbf{x}^2)\exp(-\frac{1}{4\omega}) \tag{53}$$

$$\frac{\partial}{\partial x_j}=\exp(\frac{1}{4\omega})\exp(\frac{1}{2}\omega\mathbf{x}^2)a_j\exp(-\frac{1}{2}\omega\mathbf{x}^2)\exp(-\frac{1}{4\omega}) \tag{54}$$

$$x_j\frac{\partial}{\partial x_j}=\exp(\frac{1}{4\omega})\exp(\frac{1}{2}\omega\mathbf{x}^2)n_j\exp(-\frac{1}{2}\omega\mathbf{x}^2)\exp(-\frac{1}{4\omega}) \tag{55}$$

But we cannot apply this inverse transformation to the $|n_1,\ldots,n_N\rangle$ functions, because such a transformation creates essential singularities. Indeed, this transformation is created by the exponentiation of the Lassalle operator; this operator has a $\dfrac{1}{x_i-x_j}$ term. Applied to $|n_1,\ldots,n_N\rangle$, this term gives rise by exponentiation to essential singularities. In order to solve this problem, it is necessary to symmetrize the functions that we used so far. This symmetrization is the same that defines the bosonic wave functions in quantum mechanics and is thus called *bosonization*. Let us consider a function such as:

$$\sum_{\sigma\in S_N}\prod_{j=1}^{N}x_j^{\lambda_{\sigma(j)}} \tag{56}$$

where the symmetrization is done so that each permutation appears only once. Acting with the $\dfrac{\partial_i-\partial_j}{x_i-x_j}$ term of the Lassale operator, we find that:

$$\sum_{i\neq j}\sum_{\sigma\in S_N}\frac{\lambda_{\sigma(i)}x_i^{\lambda_{\sigma(i)}-1}x_j^{\lambda_{\sigma(j)}}-\lambda_{\sigma(j)}x_j^{\lambda_{\sigma(j)}-1}x_i^{\lambda_{\sigma(i)}}}{x_i-x_j}\prod_{k\neq i,\,k\neq j}x_k^{\lambda_{\sigma(k)}}$$

The sum over the (different) elements of S_N can be written as a sum over the (different) elements of $\tau_{i,j}S_N$ where $\tau_{i,j}$ is the transposition that switches i and j. Let us do this transformation in the term $\displaystyle\sum_{i\neq j}\sum_{\sigma\in S_N}\frac{\lambda_{\sigma(j)}x_j^{\lambda_{\sigma(j)}-1}x_i^{\lambda_{\sigma(i)}}}{x_i-x_j}$. We obtain the following expression: $\displaystyle\sum_{i\neq j}\sum_{\sigma\in S_N}\frac{\lambda_{\sigma(i)}x_j^{\lambda_{\sigma(i)}-1}x_i^{\lambda_{\sigma(j)}}}{x_i-x_j}$. This transformation allows us to factorize $\lambda_{\sigma(i)}(x_i-x_j)$, which can be simplified with the denominator and this takes off the problem of the singularities.

This symmetrization acting on a fixed monomial gives rise to the monomial function of degree $\sum_{j=1}^N \lambda_j$. Thus,

$$m_\lambda(\mathbf{x}) = \sum_{\sigma\in S_N} x_1^{\lambda_{\sigma(1)}}\dots x_N^{\lambda_{\sigma(N)}} \tag{57}$$

We will note $|\lambda\rangle = \displaystyle\sum_{\sigma\in S_N}|\lambda_{\sigma(1)},\dots,\lambda_{\sigma(N)}\rangle = m_\lambda(\mathbf{a}^\dagger)|0\rangle$. The $|\lambda\rangle$ are eigenfunctions of every operator that is itself a function of the number operators n_j (because these operators commute two by two). In particular, they are eigenfunctions of the *power sums* (see above) of the number operators. Therefore, we have:

$$\boldsymbol{p}_l(\mathbf{n})|\lambda\rangle = p_l(\lambda)|\lambda\rangle \tag{58}$$

The $\boldsymbol{p}_l(\mathbf{n})$ form a set of conserved operators commuting with each other. The eigenvalue of the $\boldsymbol{p}_l(\mathbf{n})$, $\boldsymbol{p}_l(\lambda)$ are thus numbers that can be used to index the states: the symmetrized states $|\lambda\rangle$ are fully described by the set of the $\boldsymbol{p}_l(\lambda)$ pour $l\in\{1,\dots,N\}$.

The $|\lambda\rangle$ functions form a set of eigenfunctions for the Hamiltonian we consider and we can act on them with the inverse transformation to get expression in terms of the physical variables. Let us denote by $M_\lambda(\mathbf{x};\frac{1}{a},\omega)$ the function obtained by applying T to $|\lambda\rangle$: $T|\lambda\rangle = M_\lambda(\mathbf{x};\frac{1}{a},\omega)$. In order to express the solutions in terms of the original variables, we recall the M_λ are only the polynomial part of the eigenfunctions of the Calogero Hamiltonian.

To get compact formulae, we act on creation and annihilation operators directly with the inverse transformation T. Therefore we define:

$$b_j^+ = Ta_j^\dagger T^{-1} \qquad = \exp(-\frac{1}{4\omega}\mathcal{O}_L)\,x_j\,\exp(\frac{1}{4\omega}\mathcal{O}_L) \tag{59a}$$

$$b_j = Ta_jT^{-1} \qquad = \exp(-\frac{1}{4\omega}\mathcal{O}_L)\,\frac{\partial}{\partial x_j}\,\exp(\frac{1}{4\omega}\mathcal{O}_L) \tag{59b}$$

$$\nu_j = b_j^\dagger b_j \tag{59c}$$

then, taking into consideration the non-polynomial part:

$$\hat{b}_j^+ = \psi_g b_j^+ \psi_g^{-1} \tag{60a}$$

$$\hat{b}_j = \psi_g b_j \psi_g^{-1} \tag{60b}$$

$$\hat{\nu}_j = \hat{b}_j^\dagger \hat{b}_j \tag{60c}$$

The eigenfunctions of H_C can then be expressed as:

$$|\lambda\rangle = m_\lambda\left(\hat{\mathbf{b}}^+\right)|0\rangle \tag{61}$$

Let us note n_j the degeneracy of λ_j in λ (*i.e.* the number of times that λ_j appears in λ, considering that the λ_j are all different; $\lambda = \{\underbrace{\lambda_1 \dots \lambda_1}_{n_1}, \dots, \underbrace{\lambda_N \dots \lambda_N}_{n_N}\})$. Then this basis is orthogonal and:

$$\langle \mu | \lambda \rangle = \frac{N!}{n_1! \dots n_N!} \prod_{j=1}^{N} \lambda_j! \langle 0|0 \rangle \delta_{\mu,\lambda} \tag{62}$$

This section presents and explains the content of an article that shows how to map the Calogero problem (N bodies in interaction through a potential of the form $1/r^2$) onto a system of N independent quantum harmonic oscillators. This transformation simplifies greatly the problem and shows its integrability. The eigenfunctions of this problem can also be expressed in a simple manner, provided we make use of the good symmetrization in order to avoid the singularities due to the potential. The symmetrization is the same that is used when one deals with the wave functions of a system of bosons. We showed that the eigenfunctions of the Hamiltonian are simply the result of the action of monomial symmetric functions (with creation operators as variables) on a gaussian function:

$$|\lambda\rangle = m_\lambda \left(\hat{\mathbf{b}}^+\right) |0\rangle$$

where λ is a partition of size N. It is possible to express the monomial symmetric functions as superpositions of Jack polynomials, and this gives the first link between our problem and the Jack polynomials. Indeed, we have:

$$J_\lambda(\mathbf{x}, 1/a) = \sum_{\mu \leq \lambda} v_{\lambda\mu}(a) m_\mu(\mathbf{x}),\ v_{\lambda\lambda} \tag{63}$$

where $v_{\lambda\mu}$ is the matrix of the change of basis, and $\leq$ denotes the dominance order. Thus, we can write:

$$\begin{aligned} m_\lambda(\mathbf{b}^+) &= \exp(-\frac{1}{4\omega}\mathcal{O}_L) m_\lambda(\mathbf{x}) \\ &= \exp(-\frac{1}{4\omega}\mathcal{O}_L) \sum_{\mu \leq \lambda} v^{-1}_{\lambda\mu}(a) J_\mu(\mathbf{x}, 1/a) \end{aligned}$$

5.4 Expression of the Calogero Hamiltonian in terms of the Elementary Symmetric Functions

We have already presented the elementary symmetric functions and used the monomial functions, two types of symmetric functions. We also showed that the interesting functions in this problem are symmetric. It is useful, if we want to go further in this problem, to express the differential operators that are used by the Calogero Hamiltonian in terms of the elementary symmetric functions (see [6], paper in which the new coordinates are invariant under translation - the new frame is the frame of the center of mass of the system) whose definition is recalled below:

$$\sigma_k(\mathbf{x}) = \sum_{1 < i_1 < \dots < i_N \leq N} \prod_{j=1}^{k} x_{i_j} \tag{64}$$

Indeed, this transformation accelerates the computations, especially when we deal with eigenfunctions that have simple expressions in one or another basis of the space of symmetric functions.

The Calogero Hamiltonian is a differential operator of degree 2. It is thus entirely defined by its action on the elementary symmetric functions and on products of two of these functions. Indeed, we can write that:

$$\sum_{i=1}^{N}\left(x_i\partial_i-\partial_i^2\right)-2a\sum_{i<j}\frac{\partial_i-\partial_j}{x_i-x_j}=$$

$$\sum_{i=1}^{N}\left(\sum_{k=1}^{N}x_i\frac{\partial e_k}{\partial x_i}\frac{\partial}{\partial e_k}-\sum_{k,l}\left[\partial_i(e_l)\partial_i(e_k)\frac{\partial^2}{\partial e_l\partial e_k}+\partial_i(e_l)\frac{\partial^2 e_k}{\partial e_l\partial_i}\frac{\partial}{\partial e_k}\right]\right)$$

$$-2a\sum_{i<j}\frac{1}{x_i-x_j}\sum_{k=1}^{N}\left(\frac{\partial e_k}{\partial x_i}\frac{\partial}{\partial_{e_k}}-\frac{\partial e_k}{\partial x_j}\frac{\partial}{\partial_{e_k}}\right)$$

The term $\dfrac{\partial^2 e_k}{\partial e_l\partial_i}$ is trivially equal to zero: of course when $l\neq k$; if $l=k$, $\dfrac{\partial e_k}{\partial e_l}=1$ and the second derivative makes it zero.

Let us consider the action of H_C on $e_i(X)$ (where $X=x_1+\cdots+x_N$), and first of all, let us consider the action of the laplacian. The e_i are all linear in each of the variables so their second derivative with respect to each of the variable is equal to zero. Therefore:

$$\sum_{i=1}^{N}\frac{\partial^2 e_k}{\partial x_i^2}=0$$

Moreover,

$$\partial_i(e_k(X))=e_{k-1}(X-x_i)$$

where $e_{k-1}(X-x_i)=\displaystyle\sum_{\substack{1<i_1<\cdots<i_N\\ i_1,\ldots i_N,\neq i}}\prod_{j=1}^{k-1}x_{i_j}$. Thus,

$$\sum_{i<j}\frac{\partial_i-\partial_j}{x_i-x_j}(e_k(X))=\sum_{i<j}\frac{1}{x_i-x_j}\left(\sum_{\substack{1<l_1<\cdots<l_N\\ l_1,\ldots l_N,\neq i}}\prod_{n=1}^{k-1}x_{l_n}-\sum_{\substack{1<l_1<\cdots<l_N\\ l_1,\ldots l_N,\neq j}}\prod_{n=1}^{k-1}x_{l_n}\right)$$

The sums over l_n are the same if all indices are different from i and j. These sums simplify with each other and in the remainder, we can factorize x_j-x_i. Thus, the term that remains is:

$$2\sum_{i<j}\frac{\partial_i-\partial_j}{x_i-x_j}(e_k(X))=2\sum_{i<j}e_{k-2}(X-x_i-x_j)$$

It is possible to show that:

$$\sum_{i<j}e_{k-2}(X-x_i-x_j)=\frac{(N-k+2)(N-k+1)}{2}e_{k-2}(X)$$

Finally,

$$\sum_{i=1}^{N}x_i\partial_i(e_k(X))=\sum_{i=1}^{N}x_ie_{k-1}(X-x_i)=ke_k(X)$$

The Laplacian applied on the product of two e_i only uses the double product of the first derivatives because the second derivatives are equal to zero. We just have to express the following term in terms of the $e_k(X)$:

$$\sum_i \partial_i(e_n(X))\partial_i(e_m(X)) = \sum_i e_{n-1}(X-x_i)e_{m-1}(X-x_i)$$

It is equal to

$$A_{nm} = (2-\delta_{n,m})\left[(N-m+1)e_{n-1}e_{m-1} - \sum_{k=1}^{\min(n-1,N-m+1)} (m-n+2k)e_{n-1-k}e_{m-1-k}\right]$$

where e_k means $e_k(X)$.
Putting together the different elements, we get the following result:

$$H_C = \sum_{l\le m} A_{lm}\frac{\partial^2}{\partial e_l \partial e_m} - a\sum_{k=2}^{N}\frac{(N-k+2)(N-k+1)}{2}e_{k-2}\frac{\partial}{\partial e_k} + \sum_{k=1}^{N} k e_k \frac{\partial}{\partial e_k} \tag{65}$$

If we consider translation invariant coordinates, $y_i = x_i - \frac{1}{N}\sum_{j=1}^{N} x_j$, [6] gives the following result: (with our conventions):

$$H_C = -\frac{1}{2}\sum_{j,k=2}^{N} B_{jk}\frac{\partial^2}{\partial\tau_j\partial\tau_k} + \sum_{i=2}^{N} i\tau_i\frac{\partial}{\partial\tau_i} + \frac{1}{2}\left(\frac{1}{N}+a\right)\sum_{i=2}^{N}(N-i+2)(N-i+1)\tau_{i-2}\frac{\partial}{\partial\tau_i} \tag{66}$$

where

$$B_{jk} = \frac{(N-j+1)(k-1)}{N}\tau_{j-1}\tau_{k-1} + \sum_{l\ge\max(1,k-j)}(k-j-2l)\tau_{j+l-1}\tau_{k-l-1}$$

and where $\tau_i = \sigma_i(y(x))$ (the sum begins with $j,k = 2$ because the coordinates transformation separes the center of mass dynamics, that is, for the relative motion, takes off one degree of freedom).

Note that the expression of H_C as a differential operator on the elementary symmetric functions is a way to prove the integrability of the underlying problem (see [6]). Indeed, this expression allows us to write H_C as an element of the Lee algebra of de $GL_N(\mathbb{R})$. It is possible to represent this Lee algebra in terms of differential operators, using the notations:

$$J_{i,j} = t_i\frac{\partial}{\partial t_j},\ i,j = 2,\ldots,N$$

The Lee algebra of $GL(N,\mathbb{R})$ is formed by the set of all $N\times N$ with real valued coefficients, with the commutator ($[A,B] = AB - BA$). We denote by t_i the canonical basis of $\mathbb{R}^N$. A basis of $M_N(\mathbb{R})$ (the set of all real $N\times N$ matrices) is given by the E_{ij}, matrices with a one at the intersection of the i^{th} row and the j^{th} column. With these notations, $E_{ij}t_k = t_i\delta_{j,k}$. We would have the same result if we considered the differential operator $J_{i,j} = t_i\frac{\partial}{\partial t_j}$. It is easy to verify that the $J_{i,j}$ operators obey the same commutation relation E_{ij} obey, *i.e.* $\left([E_{ij},E_{kl}]\,t_m\right)_n = \delta_{n,i}\delta_{j,k}\delta_{m,l} - \delta_{n,k}\delta_{j,m}\delta_{l,i}$. The operators $J_{i,j}$ are thus a representation of the Lee algebra of $GL(N,\mathbb{R})$.

The following expression of H_C in terms of the $J_{i,j}$ is available, replacing t_i by τ_i:

$$
\begin{aligned}
H_C = & \sum_{j=2}^{N} \left\{ \frac{(N-j+1)(j-1)}{N} \left(J_{j-1,j}\right)^2 - 2\sum_{l=1}^{j-1} l J_{j+l-1,j} J_{j-l-1,j} \right\} \\
& + 2 \sum_{2\leq k\leq j\leq N} \left\{ \frac{(N-j+1)(j-1)}{N} J_{j-1,j} J_{k-1,k} - \sum_{l=1}^{k-1} (j-k+2l) J_{j+l-1,j} J_{k-l-1,k} \right\} \\
& - 2\sum_{k=2}^{N} k J_{k,k} - \left(\frac{1}{N} + a\right) \sum_{k=2}^{N} (N-k+2)(N-k+1) J_{k-2,k}
\end{aligned}
\tag{67}
$$

The expression of H_C in terms of the $J_{i,j}$ shows us that the set of the

$$
V_n(\mathbf{t}) = \mathrm{Vec}\Big(\prod_{j=2}^{N} t_j^{n_j}, 0 \leq \sum_j n_j \leq n\Big) \tag{68}
$$

is invariant under the action of this operator (indeed, the $J_{i,j}$ preserve the degree of every homogeneous polynomial of n variables; their products conserve also the degree). But the integrability of a system in hamiltonian mechanics is equivalent to the conservation by the Hamiltonian of the system of a set such as $V_n(\mathbf{t})$ (see [6]).

5.5 Equivalence of Calogero's problem to a system of free particules

See [10].

It is possible to construct explicitly a transformation that maps Calogero's Hamiltonian onto the Hamiltonian of a system of N free particules. Let us prove this by introducing the following operators:

$$
T_+ = \frac{1}{\omega} \left(\sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{a}{2} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{1}{(q_i - q_j)^2} \right) \tag{69a}
$$

$$
T_- = \omega \sum_{i=1}^{N} \frac{q_i^2}{2} \tag{69b}
$$

$$
T_0 = \frac{1}{2} \sum_{i=1}^{N} q_i p_i \tag{69c}
$$

with the canonical notation $q_i, p_i = \dot{q}_i$. From now on, we take $\omega = 1$.

The following "commutation" relations between these operators hold:

$$
\{T_0, T_\pm\} = \pm T_\pm \tag{70a}
$$

$$
\{T_+, T_-\} = -2T_0 \tag{70b}
$$

These commutation relations do not involve a. They also hold for $a = 0$, in which case we denote T_0, $T_\pm$ by $\tilde{T}_0$ and $\tilde{T}_\pm$ respectively. These relations define the *Poisson algebra*, the algebra of $SL(2, \mathbb{R})$.

Consider now the following transformation:

$$A \to \exp(i\lambda T_1) * A = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \underbrace{\{T_1, \dots \{T_1, A\}\}}_{n\,\text{times}} \tag{71}$$

where $T_1 = \frac{i}{2}\,(T_+ + T_-)$ where A is any operators functions of the q_k's and the p_k's. This transformation maps T_+ onto T_- for $\lambda = \pi$:

$$\exp(i\pi T_1) * T_+ = T_- = \tilde{T}_- \tag{72}$$

Indeed, it is possible to show by induction the following relations $p \in \mathbb{N}^*$ and $\mathbb{N}$ respectively:

$$\begin{cases} \underbrace{\{i\pi T_1, \dots, \{i\pi T_1, T_+\}\}}_{2p\,\text{times}} = (-1)^p\, \dfrac{\pi^{2p}}{2}(T_+ - T_-) \\ \underbrace{\{i\pi T_1, \dots, \{i\pi T_1, T_+\}\}}_{2p+1\,\text{times}} = (-1)^{p+1}\, \pi^{2p+1} T_0 \end{cases} \tag{73}$$

Therefore,

$$\begin{aligned}
\exp(i\pi T_1) * T_+ &= \sum_{n=0}^{\infty} \frac{1}{n!} \{i\pi T_1, \dots, \{i\pi T_1, T_+\}\} \\
&= T_+ + \sum_{p=1}^{\infty} \frac{1}{(2p)!} \underbrace{\{i\pi T_1, \dots, \{i\pi T_1, T_+\}\}}_{2p\,\text{fois}} + \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \underbrace{\{i\pi T_1, \dots, \{i\pi T_1, T_+\}\}}_{2p+1\,\text{fois}} \\
&= T_+ + (T_+ - T_-)\frac{1}{2} \sum_{p=1}^{\infty} (-1)^p \frac{\pi^{2p}}{(2p)!} - T_0 \sum_{p=0}^{\infty} (-1)^p \frac{\pi^{2p+1}}{(2p+1)!} \\
&= T_+ + (T_+ - T_-) \left[\cos(\pi) - \frac{1}{2}(T_+ - T_-)\right] - \sin(\pi) T_0 \\
&= \frac{1}{2}(T_+ + T_-) - \frac{1}{2}(T_+ - T_-) \\
&= T_-
\end{aligned}$$

Note that the transformation $\exp(i\lambda T_1) * A$ also defines the time evolution (with $t = \dfrac{\lambda}{2\omega}$) of a system whose Hamiltonian is the Hamiltonian of the Calogero model. Let us now consider another transformation that we apply to $\tilde{T}_-$:

$$\exp(-i\pi \tilde{T}_1) * \tilde{T}_- = \sum_{i=1}^{N} \frac{p_i^2}{2}$$

As for the previous computation, we have:

$$
\begin{aligned}
\exp(-i\pi\tilde{T}_1) * \tilde{T}_- &= \tilde{T}_- + \frac{1}{2}\sum_{p=1}^{\infty}(-1)^{p-1}\frac{\pi^{2p}}{(2p)!}\left(\tilde{T}_+ - \tilde{T}_-\right) + \sum_{p=0}^{\infty}(-1)^p\frac{\pi^{2p+1}}{(2p+1)!}\tilde{T}_0 \\
&= \tilde{T}_- - \frac{1}{2}\left[\sum_{p=0}^{\infty}(-1)^p\frac{\pi^{2p}}{(2p)!}\left(\tilde{T}_+ - \tilde{T}_-\right) - \left(\tilde{T}_+ - \tilde{T}_-\right)\right] \\
&= \tilde{T}_- - \frac{1}{2}\left[-\left(\tilde{T}_+ - \tilde{T}_-\right) - \left(\tilde{T}_+ - \tilde{T}_-\right)\right] \\
&= \tilde{T}_+
\end{aligned}
$$

We have thus established that the following transformation maps the Hamiltonian of the Calogero model onto the Hamiltonian of a free particule system:

$$
\exp(-i\pi\tilde{T}_1) * (\exp(i\pi T_1) * T_+) = \sum_{i=1}^{N}\frac{p_i^2}{2} \tag{74}
$$

6 Appendix

6.1 Proof of the relation 33

To make the proof, we consider the action of H as a differential operator on a function f of $\mathbf{x}$. We have:

$$\begin{aligned} Hf &= \psi_g^{-1}(H_C - E_g)(\psi_g f) \\ &= \frac{1}{2}\psi_g^{-1}\left[(\sum_{j=1}^{N} \omega^2 x_j^2 + \sum_{j,k;j\neq k} \frac{a(a-1)}{(x_j-x_k)^2})\psi_g f - \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}(\psi_g f) - E_g \psi_g f \right] \end{aligned}$$

We recall that $\partial_j^2(\psi_g f) = \partial_j^2(f)\psi_g + 2(\partial_j f)(\partial_j \psi_g) + f\partial_j^2\psi_g$. Let us start with $\partial_j \psi_g$ and $\partial_j^2 \psi_g$.

$$\partial_j \psi_g = \mathcal{P}(\mathbf{x}) \exp(-\frac{1}{2}\omega r^2) \left[-\omega x_j + a \sum_{k,k\neq j} \frac{1}{x_j - x_k} \right]$$

$$\partial_j^2 \psi_g = \mathcal{P}(\mathbf{x}) \exp(-\frac{1}{2}\omega r^2) \left[-\omega - a \sum_{k,k\neq j} \frac{1}{(x_j-x_k)^2} + \left(-\omega x_j + \sum_{k,k\neq j} \frac{a}{x_j - x_k} \right)^2 \right]$$

Indeed, $\partial_j \left(\exp\left(-\frac{1}{2}\omega r^2\right)\right) = -\omega x_j \exp\left(-\frac{1}{2}\omega r^2\right)$ and $\partial_j \mathcal{P} = a\mathcal{P} \sum_{k,k\neq j} \frac{1}{x_j - x_k}$.

The ground state part disappears because we can factorize by it in the differential term, so that:

$$\begin{aligned} Hf = \frac{1}{2}\left[\sum_{j,k;j\neq k} \frac{a(a-1)}{(x_j-x_k)^2} + \sum_{j=1}^{N}\omega^2 x_j^2 - 2E_g - \sum_{j=1}^{N} \left(\partial_j^2 + 2\left\{ \sum_{k,k\neq j} \frac{a}{x_j-x_k} - \omega x_j \right\} \partial_j \right) \right] f \\ -\frac{1}{2}\left[\left(\sum_{k,k\neq j} \frac{a}{x_j - x_k} - \omega x_j \right)^2 - \omega - \sum_{k,k\neq j} \frac{a}{(x_j-x_k)^2} \right] f \end{aligned}$$

However $2 \sum_{j,k,j\neq k} \frac{\partial_j f}{x_j - x_k} = \sum_{j,k,j\neq k} \frac{\partial_j f}{x_j - x_k} - \sum_{j,k,j\neq k} \frac{\partial_k f}{x_j - x_k}$ if we change the subscripts j and k in the second term.

Thus,

$$\begin{aligned} Hf = \omega \sum_{j=1}^{N} x_j \partial_j f - \frac{1}{2}Df + \frac{1}{2}\left[\sum_{j,k,j\neq k} \frac{a(a-1)}{(x_j-x_k)^2} + -a\omega N(N-1) \right] \\ +\frac{1}{2}\left[\sum_{j,k,j\neq k} \frac{a}{(x_j-x_k)^2} + 2a\omega \sum_{j,k,j\neq k} \frac{x_j}{x_j - x_k} - \sum_{j,k,k';k,k'\neq j} \frac{a^2}{(x_j-x_k)(x_j-x_{k'})} \right] \end{aligned}$$

Moreover, $2a\omega \sum_{j,k,j\neq k} \frac{x_j}{x_j-x_k} = a\omega \left[\sum_{j,k,j\neq k} \frac{x_j}{x_j - x_k} - \sum_{j,k,j\neq k} \frac{x_k}{x_j-x_k} \right] = a\omega \sum_{j,k,j\neq k} \frac{x_j - x_k}{x_j - x_k}$.

We thus have $2a\omega \sum_{j,k,j\neq k} \frac{x_j}{x_j - x_k} = a\omega N(N-1)$

A factorization into simple elements then yields, for j, k, k' all different:

$$\frac{1}{(x_j - x_k)(x_j - x_{k'})} = \frac{1}{(x_j - x_k)(x_k - x_{k'})} + \frac{1}{(x_j - x_{k'})(x_{k'} - x_k)}$$

The part with $k = k'$ in the sum of the $\dfrac{1}{(x_j - x_k)(x_j - x_{k'})}$ can be simplified, and the sum of the simple elements should be zero.

After simplification, the following term remains:

$$Hf = \omega \sum_{j=1}^{N} x_j \frac{\partial f}{\partial x_j} - \frac{1}{2} D f \tag{75}$$

6.2 Proofs of several relations of Section 5.3

6.2.1 Proof of relation (34)

$$\begin{aligned}
[\mathcal{O}_L, \mathcal{O}_E] &= \sum_{i,j} \partial_i^2 (x_j \partial_j) + a \sum_{i \neq j, k} \frac{\partial_i - \partial_j}{x_i - x_j} (x_k \partial_k) - \sum_{i,j} x_j \partial_j \partial_i^2 - a \sum_{i \neq j, k} x_k \partial_k \left(\frac{\partial_i - \partial_j}{x_i - x_j} \right) \\
&= \sum_{i,j} \left(2\partial_i^2 + x_j \partial_j \partial_i^2 \right) - \sum_{i,j} x_j \partial_j \partial_i^2 + \dots
\end{aligned}$$

The last part yields:

$$a \sum_{i \neq j, k} \left[\frac{1}{x_i - x_j} \left(\delta_{k,i} \partial_k - \delta_{k,j} \partial_k + x_k (\partial_{ik}^2 - \partial_{jk}^2) \right) - \frac{x_k}{x_i - x_j} (\partial_{ik}^2 - \partial_{kk}^2) + x_k \frac{\delta_{k,i} - \delta_{k,j}}{(x_i - x_j)^2} (\partial_l - \partial_m) \right]$$

or $2a \sum_{i \neq j} \dfrac{\partial_i - \partial_j}{x_i - x_j}$. As the third derivatives in the first term are equal to zero, we get the relation that we wanted to prove.

6.2.2 Proof of relation (37)

If we develop the commutator, we get:

$$\begin{aligned}
\left[\sum_i \partial_i^2, \sum_j x_j^2 \right] f &= \sum_{i,j} \left(\partial_i^2 \left(x_j^2 f \right) - x_j^2 \partial_i^2 (f) \right) \\
&= \sum_{i,j} \left(\partial_i^2 \left(x_j^2 \right) f + 2 \partial_i \left(x_j^2 \right) \partial_i (f) \right) \\
&= 4 \sum_i x_i \partial_i (f) + 2 \sum_i \partial_i (x_i) f
\end{aligned}$$

and the relation written before.

6.2.3 Proof of relations (40), (41) and (42)

Relation (40) is proved as follows: using formula (38), we find that

$$\exp(\frac{1}{4\omega} \mathcal{O}_L) H_C \exp(-\frac{1}{4\omega} \mathcal{O}_L) = \omega \mathcal{O}_E - \frac{\mathcal{O}_L}{2} + \left[\frac{\mathcal{O}_L}{4\omega}, \omega \mathcal{O}_E - \frac{\mathcal{O}_L}{2} \right] + \left[\frac{\mathcal{O}_L}{4\omega}, \left[\frac{\mathcal{O}_L}{4\omega}, \omega \mathcal{O}_E - \frac{\mathcal{O}_L}{2} \right] \right] + \dots$$

$\frac{1}{4\omega}\mathcal{O}_L$ commutes with itself, so the commutators that are in the right-hand side of the previous equation only contain commutators of commutators of $\frac{1}{4\omega}\mathcal{O}_L$ with $\mathcal{O}_E$. However, since $[\mathcal{O}_L, \mathcal{O}_E] = 2\mathcal{O}_L$, $[\mathcal{O}_L, [\mathcal{O}_L, \mathcal{O}_E]]$ and the following terms (forgetting the constants for the moment) are zero. Only remains:

$$\exp(\frac{1}{4\omega}\mathcal{O}_L)H_C \exp(-\frac{1}{4\omega}\mathcal{O}_L) = \omega\mathcal{O}_E - \frac{1}{2}\mathcal{O}_L + \frac{\omega}{4\omega}[\mathcal{O}_L, \mathcal{O}_E] \tag{76}$$

and the relation we wanted to prove. In the same way, as $[\Delta, \mathcal{O}_E] \propto \Delta$

$$\exp(-\frac{1}{4\omega}\Delta)\omega\mathcal{O}_E \exp(\frac{1}{4\omega}\Delta) = \omega\mathcal{O}_E + \left[\frac{1}{4\omega}\Delta, \omega\mathcal{O}_E\right] = -\frac{1}{4}2\Delta + \omega\mathcal{O}_E$$

we prove the relation (41). Let us apply the same method to

$$\exp(-\frac{1}{2}\omega\mathbf{x}^2)\left(\omega\mathcal{O}_E - \frac{1}{2}\Delta\right)\exp(\frac{1}{2}\omega\mathbf{x}^2) = \omega\mathcal{O}_E - \frac{1}{2}\Delta + \left[-\frac{1}{2}\omega\mathbf{x}^2, \omega\mathcal{O}_E - \frac{1}{\Delta}\right]$$
$$+\left[-\frac{1}{2}\omega\mathbf{x}^2, \left[-\frac{1}{2}\omega\mathbf{x}^2, \omega\mathcal{O}_E - \frac{1}{\Delta}\right]\right] + \left[-\frac{1}{2}\omega\mathbf{x}^2, \left[-\frac{1}{2}\omega\mathbf{x}^2, \left[-\frac{1}{2}\omega\mathbf{x}^2, \omega\mathcal{O}_E - \frac{1}{\Delta}\right]\right]\right] + \dots$$

We have:

$$A = \left[-\frac{1}{2}\omega\mathbf{x}^2, \omega\mathcal{O}_E - \frac{1}{\Delta}\right] = \omega^2\mathbf{x}^2 - \omega\mathcal{O}_E - \frac{1}{2}\omega N$$

$$B = \left[-\frac{1}{2}\omega, A\right] = -\omega^2\mathbf{x}^2$$

$$C = \left[-\frac{1}{2}\omega, B\right] \propto \left[\mathbf{x}^2, \mathbf{x}^2\right] = 0$$

Since C is equal to 0, the next terms are also equal to zero. The different parts of the equation yield:

$$\exp(-\frac{1}{2}\omega\mathbf{x}^2)\left(\omega\mathcal{O}_E - \frac{1}{2}\Delta\right)\exp(\frac{1}{2}\omega\mathbf{x}^2) = \omega\mathcal{O}_E - \frac{1}{2}\Delta + \omega^2\mathbf{x}^2 - \omega\mathcal{O}_E - \frac{1}{2}\omega N - \omega^2\mathbf{x}^2$$

We recall that $p_i = -i\dfrac{\partial}{\partial x_i}$. Thus, using relation (41), we prove relation (42).

6.2.4 Proof of relations (53), (54) and (55)

We can, for example, prove the relation (55) using the relation (38). We have:

$$C = \exp(\frac{1}{2}\omega\mathbf{x}^2)\left[\frac{1}{2\omega}\left(-\partial_j^2 + \omega^2 x_j^2\right) - \frac{1}{2}\right]\exp(-\frac{1}{2}\omega\mathbf{x}^2) =$$
$$\frac{1}{2\omega}\left[-\partial_j^2 + \omega^2 x_j^2\right] - \frac{1}{2} - \frac{1}{4}\left[x_j^2, \partial_j^2\right] + \frac{1}{2}\frac{1}{2\omega}\left[x_j^2, \left[x_j^2, \partial_j^2\right]\right] + \dots$$

thus

$$C = \frac{1}{2\omega}\left[-\partial_j^2 + \omega^2 x_j^2\right] - \frac{1}{2} + \frac{1}{4}(2 + 4x_j\partial_j) + \frac{1}{4}\omega\left[x_j^2, x_j\partial_j\right] + \dots$$

As the x_j variables are independent, the commutation relations proved for the sums also hold for each term. As $\left[x_j^2, x_j\partial_j\right] = -2x_j^2$, the commutators that are involved in the rest of the developpement are all equal to zero because they involve x_j^2 on both side. Therefore we have $C = -\frac{1}{2\omega}\partial_j^2 + x_j\partial_j$. Moreover,

$$\exp(\frac{1}{4\omega})\, C\, \exp(-\frac{1}{4\omega}) = -\frac{1}{2\omega}\partial_j^2 + x_j\partial_j + \frac{1}{4\omega}\left[\partial_j^2, x_j\partial_j\right] + \frac{1}{2}\frac{1}{4\omega}\left[\partial_j^2, \left[\partial_j^2, x_j\partial_j\right]\right]$$

But $\left[\partial_j^2, x_j\partial_j\right] \propto \partial_j^2$ so all terms but the first commutator are equal to zero and we get the relation (55).

6.3 Proof of relation (66)

The following expression of H_C in terms of the $J_{i,j}$ is available, replacing t_i by τ_i:

$$\begin{aligned} H_C = & \sum_{j=2}^{N}\left\{\frac{(N-j+1)(j-1)}{N}\left(J_{j-1,j}\right)^2 - 2\sum_{l=1}^{j-1} lJ_{j+l-1,j}J_{j-l-1,j}\right\} \\ & + 2\sum_{2\le k\le j\le N}\left\{\frac{(N-j+1)(j-1)}{N}J_{j-1,j}J_{k-1,k} - \sum_{l=1}^{k-1}(j-k+2l)J_{j+l-1,j}J_{k-l-1,k}\right\} \\ & - 2\sum_{k=2}^{N} kJ_{k,k} - \left(\frac{1}{N}+a\right)\sum_{k=2}^{N}(N-k+2)(N-k+1)J_{k-2,k} \end{aligned} \tag{77}$$

Indeed, we have, for every function f:

$$\begin{aligned} \left[\sum_{j=2}^{N}\frac{(N-j+1)(j-1)}{N}\left(J_{j-1,j}\right)^2\right] f &= \sum_{j=2}^{N}\frac{(N-j+1)(j-1)}{N}t_{j-1}\frac{\partial}{\partial t_j}\left(t_{j-1}\frac{\partial f}{\partial t_j}\right) \\ &= \sum_{j=2}^{N}\frac{(N-j+1)(j-1)}{N}\left\{t_{j-1}\delta_{j,j-1}\frac{\partial f}{\partial t_j} + t_{j-1}^2\frac{\partial^2 f}{\partial t_j^2}\right\} \\ &= \sum_{j=2}^{N}\frac{(N-j+1)(j-1)}{N}t_{j-1}^2\frac{\partial^2 f}{\partial t_j^2} \end{aligned} \tag{78}$$

$$\begin{aligned} \left[-2\sum_{j=2}^{N}\sum_{l=1}^{j-1} lJ_{j+l-1,j}J_{j-l-1,j}\right] f &= -2\sum_{j=2}^{N}\sum_{l=1}^{j-1} lt_{j+l-1}\frac{\partial}{\partial t_j}\left(t_{j-l-1}\frac{\partial f}{\partial t_j}\right) \\ &= -2\sum_{j=2}^{N}\sum_{l=1}^{j-1} l\left\{t_{j+l-1}\delta_{j,j-l-1}\frac{\partial f}{\partial t_j} + t_{j+l-1}t_{j-l-1}\frac{\partial^2 f}{\partial t_j^2}\right\} \\ &= -2\sum_{j=2}^{N}\sum_{l=1}^{j-1} lt_{j+l-1}t_{j-l-1}\frac{\partial^2 f}{\partial t_j^2} \end{aligned} \tag{79}$$

$$\sum_{2\le k<j\le N}\frac{2(N-j+1)(k-1)}{N}J_{j-1,j}J_{k-1,k}f=\sum_{2\le k<j\le N}\frac{2(N-j+1)(k-1)}{N}t_{j-1}\frac{\partial}{\partial t_j}\left(t_{k-1}\frac{\partial f}{\partial t_k}\right)$$
$$=\sum_{2\le k<j\le N}\frac{2(N-j+1)(k-1)}{N}t_{j-1}t_{k-1}\frac{\partial^2 f}{\partial t_j\partial t_k} \tag{80}$$

Moreover

$$-2\sum_{2\le j<k\le N}\sum_{l=1}^{k-1}(j-k+2l)J_{j+l-1,j}J_{k-l-1,k}f=-2\sum_{2\le j<k\le N}\sum_{l=1}^{k-1}(j-k+2l)t_{j+l-1}\partial_j\left(t_{k-l-1}\partial_k f\right)$$
$$=-2\sum_{2\le j<k\le N}\sum_{l=1}^{k-1}(j-k+2l)t_{j+l-1}t_{k-l-1}\frac{\partial^2 f}{\partial t_j\partial t_k}$$

(with $\partial_k=\dfrac{\partial}{\partial t_k}$). The previous term for $j=k$ is precisely equal to (79). In the same way, (78) is equal to the sum (80) for $k=j$. In each sum with $j<k$, we make use of the factor 2 to consider the the sum over $j\neq k$, and we add the term for $k=j$. The last two terms of (67) are obtained directly. Thus we obtain the terms of the expression (66).

6.4 Proof of the relations (70)

Indeed,

$$\{T_0,T_+\}=\frac{1}{4}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial(q_jp_j)}{\partial q_i}\frac{\partial(p_k^2)}{\partial p_i}+a\sum_{j=1}^{N}\sum_{k\neq l}^{N}\frac{\partial(q_jp_j)}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{1}{(q_k-q_l)^2}\right)\right)$$
$$-\frac{1}{4}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial(q_jp_j)}{\partial p_i}\frac{\partial(p_k^2)}{\partial q_i}+a\sum_{j=1}^{N}\sum_{k\neq l}^{N}\frac{\partial(q_jp_j)}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{1}{(q_k-q_l)^2}\right)\right)$$

thus

$$\{T_0,T_+\}=\frac{1}{4}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}2\delta_{ij}\delta_{ik}p_jp_k+2a\sum_{j=1}^{N}\sum_{k\neq l}^{N}\delta_{ij}q_j\frac{\delta_{ik}-\delta_{il}}{(q_k-q_l)^3}\right)$$
$$=\frac{1}{2}\sum_{i=1}^{N}p_i^2+\frac{2a}{4}\sum_{k\neq l}\frac{q_k-q_l}{(q_k-q_l)^3};$$

$$\{T_0,T_-\}=\frac{1}{4}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial(q_jp_j)}{\partial q_i}\frac{\partial(q_k^2)}{\partial p_i}-\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial(q_jp_j)}{\partial p_i}\frac{\partial(q_k^2)}{\partial q_i}\right)$$
$$=-\frac{2}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\delta_{ij}\delta_{ik}q_jq_k=-\frac{1}{2}\sum_{i=1}^{N}q_i^2,$$

and

$$\{T_+,T_-\} = \frac{1}{4}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial(p_j^2)}{\partial q_i}\frac{\partial(q_k^2)}{\partial p_i} + a\sum_{j=1}^{N}\sum_{k\neq l}^{N}\frac{\partial(q_j^2)}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{1}{q_k-q_l}\right)\right)$$

$$-\frac{1}{4}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{\partial(p_j^2)}{\partial p_i}\frac{\partial(q_k^2)}{\partial q_i} + a\sum_{j=1}^{N}\sum_{k\neq l}^{N}\frac{\partial(q_j^2)}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{1}{q_k-q_l}\right)\right)$$

Therefore

$$\{T_+,T_-\} = -\frac{4}{4}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}\sum_{k=1}^{N}\delta_{ij}\delta_{ik}p_jq_k\right) = -2T_0$$

6.5 Proof of the commutation of H et I_N

$\{H,I_N\} = \sum_{k=1}^{N}\left[\frac{\partial H}{\partial p_k}\frac{\partial I_N}{x_k} - \frac{\partial I_N}{\partial p_k}\frac{\partial H}{\partial x_k}\right] = \sum_{k=1}^{N}\left[p_k\frac{\partial I_N}{\partial x_k} - \frac{\partial I_N}{\partial p_k}\frac{\partial H}{\partial x_k}\right]$. Denoting by M the differential operator $\sum_{i\neq j}V(x_i-x_j)\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}$, and using the fact that since V is an even function of its argument, its derivative, denoted by V', is an odd function, we have

$$\begin{aligned}
\sum_{k=1}^{N}p_k\left(\frac{\partial M}{\partial x_k}\right) &= \sum_{k=1}^{N}\sum_{i\neq j}p_kV'(x_i-x_j)\left(\delta_{ik}-\delta_{jk}\right)\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}\\
&= \sum_{k}\sum_{l,l\neq k}p_k\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_k}\left[V'(x_k-x_l) - V'(x_l-x_k)\right]\\
&= 2\sum_{k\neq l}V'(x_k-x_l)\frac{\partial}{\partial p_l}p_k\frac{\partial}{\partial p_k}\\
&= -2\sum_{k\neq l}V'(x_k-x_l)\frac{\partial}{\partial p_k}p_l\frac{\partial}{\partial p_l}
\end{aligned}$$

The term in ϵ^n in the development of the exponential of the first term in the previous relation is equal to

$$\left(\sum_{k=1}^{N}p_k\frac{\partial}{\partial x_k}\right)\left[\frac{1}{n!}\left(-\frac{1}{2}\epsilon\right)^n M^n\right]\prod_{m=1}^{N}p_m = \sum_{k=1}^{N}p_k\left(-\frac{1}{2}\epsilon\right)^n\frac{\partial M}{\partial x_k}\frac{n}{n!}M^{n-1}\prod_{m=1}^{N}p_m$$

and thus

$$\epsilon\sum_{k\neq l}V'(x_k-x_l)\frac{\partial}{\partial p_k}p_l\frac{\partial}{\partial p_l}\left[\frac{1}{(n-1)!}\left(-\frac{1}{2}\epsilon\right)^{n-1}M^{n-1}\right]\prod_{m=1}^{N}p_m$$

The term in ϵ^n in the second term is equal to

$$\epsilon\sum_{k\neq l}V'(x_k-x_l)\frac{\partial}{\partial p_k}\left[\frac{1}{(n-1)!}\left(-\frac{1}{2}\epsilon\right)^{n-1}M^{n-1}\right]\prod_{m=1}^{N}p_m$$

Gathering the term of the same power of ϵ, we find

$$-\epsilon\frac{1}{(n-1)!}\left(-\frac{1}{2}\epsilon\right)^{n-1}\sum_{k\neq l}V'(x_k-x_l)\frac{\partial}{\partial p_k}\left(1-p_l\frac{\partial}{\partial p_l}\right)M^{n-1}\prod_{m=1}^{N}p_m \tag{81}$$

The terms of the previous expression that are non vanishing are those that are in $M^{n-1}\prod_{m=1}^{N}p_m$ where p_k is present and p_l absent. Indeed, if p_k is absent, then $\frac{\partial}{\partial p_k}$ acting on the term will be equal to zero, and if p_l is present, then $p_l\frac{\partial}{\partial p_l}$ does not change the expression and is thus equal to 1 (since the product of the p_i's is linear in each of the p_i's); but since it has to be subtracted to 1, it gives 0. Therefore, for the non vanishing terms, $1-p_l\frac{\partial}{\partial p_l}$ is equal to the identity.

Therefore, exhibiting one of the factor M, the last expression without the constant gives

$$\sum_{k\neq l}V'(x_k-x_l)\frac{\partial}{\partial p_k}p_l\frac{\partial}{\partial p_l}\sum_{i\neq j}V(x_i-x_j)\frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j}M^{n-2}\prod_{m=1}^{N}p_m$$

Assuming (without loss of generality) that differentiation with respect to p_l appears in the left factor of M, this factor can be rewritten as

$$\sum_{k\neq l}\left[V(x_l-x_m)+V(x_m-x_l)\right]\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}+\text{non essential terms}$$

and, within to a factor, (81) yields

$$\sum_{k\neq l\neq m}V'(x_k-x_l)\left[V(x_l-x_m)+V(x_m-x_l)\right]\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}M^{n-2}\prod_{t=1}^{N}p_t$$

$$\sum_{k\neq l\neq m}V'(x_k-x_l)\left[V(x_l-x_m)+V(x_m-x_l)\right]\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}M^{n-2}\prod_{t=1}^{N}p_t$$

$$=\sum_{k\neq l\neq m}V'(x_k-x_l)\left[V(x_l-x_m)+V(x_l-x_m)\right]\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}M^{n-2}\prod_{t=1}^{N}p_t$$

$$=\sum_{k\neq l\neq m}V'(x_k-x_l)\left[V(x_l-x_m)-V(x_m-x_l)\right]\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}M^{n-2}\prod_{t=1}^{N}p_t$$

by permuting k,l in the second sum. For fixed (k,l,m), $\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_l}\frac{\partial}{\partial p_m}$ is multiplied by

$$V'(x_k-x_l)\left[V(x_l-x_m)-V(x_m-x_k)\right]+V'(x_l-x_m)\left[V(x_m-x_k)-V(x_k-x_l)\right]$$
$$+V'(x_m-x_k)\left[V(x_k-x_l)-V(x_l-x_m)\right]$$

But this last quantity is equal to zero by virtue of (5) and the identity $(x_k-x_l)+(x_l-x_m)+(x_m-x_k)=0$.

References

[1] J. Hoppe. *Lectures on Integrables Systems*. Springer-Verlag, 1992.

[2] P. J. Forrester. Random Matrices, Log-gases and the Calogero-Sutherland Model. In *Quantum Many-body Problems and Representation Theory*, volume 1, pages 97–181. Memoirs Math. Soc. Japan, 1998.

[3] N. Sugiura. Derivatives of the characteristic root of a symmetric or a hermitian matrix with two applications in multivariate analysis. *Communications in Statistics*, 1(5):393–417, 1973.

[4] S. Wojciechowski. Involutive set of integrals for completely integrable many-body problems with pair interaction. *Lett. N. Cim.*, 18:103–107, 1977.

[5] Akinori Nishino, Hideaki Ujino, and Miki Wadati. Symmetric Fock space and orthogonal symmetric polynomials associated with the Calogero model, 1998.

[6] Werner Ruhl and Alexander Turbiner. Exact Solvability of the Calogero and Sutherland Models. *MOD.PHYS.LETT.A*, 10:2213, 1995.

[7] T. H. Baker and P. J. Forrester. The Calogero-Sutherland Model and Generalized Classical Polynomials. *Communications in Mathematical Physics*, 188:175, 1997.

[8] A. Perelomov. *Generalized Coherent States and their Applications*. Springer-Verlag, 1986.

[9] M. Le Bellac. *Physique quantique*. Savoirs actuels. Edp Sciences, 2003.

[10] Tomasz Brzezinski, Cezary Gonera, and Pawel Maslanka. On the equivalence of the rational Calogero-Moser system to free particles. *Physics Letters A*, 254:185, 1999.