Algebraic type systems

<work-in-progress>

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L I G

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Lineal [Arrighi, Dowek 2008]

Higher-order computation

$\mathbf{t}, \mathbf{r}, \mathbf{s} := \quad x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \quad \mid$

- $(\lambda x.\mathbf{t})\ \mathbf{b} \to \mathbf{t}[\mathbf{b}/x]$

b an abstraction or a variable.

Linear algebra

$\mathbf{t} + \mathbf{r} \mid \alpha.\mathbf{t} \mid \mathbf{0}$ $\quad \underset{\text{commutative ring}}{\alpha \in (\mathcal{S}, +, \times)}$

- Elementary rules such as $\mathbf{u} + \mathbf{0} \to \mathbf{u}$ and $\alpha.(\mathbf{u} + \mathbf{v}) \to \alpha.\mathbf{u} + \alpha.\mathbf{v}$.
- Factorisation rules such as $\alpha.\mathbf{u} + \beta.\mathbf{u} \to (\alpha + \beta).\mathbf{u}$.
- Application rules such as $\mathbf{u}\ (\mathbf{v} + \mathbf{w}) \to (\mathbf{u}\ \mathbf{v}) + (\mathbf{u}\ \mathbf{w})$.

Lineal [Arrighi, Dowek 2008]

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- $(\lambda x.\mathbf{t})\ \mathbf{b} \to \mathbf{t}[\mathbf{b}/x]$

b an abstraction or a variable.

$$(\lambda x.(x)\ x)\ (y+z) \to^* (\lambda x.(x)\ x)\ y + (\lambda x.(x)\ x)\ z \to^* (y)\ y + (z)\ z$$

$$(\lambda x.(x)\ x)\ (y+z) \to^* (y+z)\ (y+z) \to^* (y)\ y + (y)\ z + (z)\ y + (z)\ z$$

Linear algebra

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- Application rules such as $\mathbf{u}\ (\mathbf{v}+\mathbf{w}) \to (\mathbf{u}\ \mathbf{v}) + (\mathbf{u}\ \mathbf{w})$.

Challenge: Type it!

A straight extension of System F ($\lambda 2^{la}$) [Arrighi, Díaz-Caro 2009]

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : U \to T \qquad \Gamma \vdash \mathbf{r} : U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : T} \to_E \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \to_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \forall X.T}{\Gamma \vdash \mathbf{t} : T[U/X]} \forall_E \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \forall X.T} \forall_I$$

$$\frac{}{\Gamma \vdash \mathbf{0} : T}\ ax_0 \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{r} : T}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T} +_I \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : T} \alpha_I$$

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Strengths

- Simple
- Strong normalization: straight

A straight extension of System F ($\lambda 2^{la}$) [Arrighi, Díaz-Caro 2009]

$$\frac{}{\Gamma, x:U \vdash x:U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t}:U \to T \qquad \Gamma \vdash \mathbf{r}:U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r}:T}\ \to_E \qquad \frac{\Gamma, x:U \vdash \mathbf{t}:T}{\Gamma \vdash \lambda x.\mathbf{t}:U \to T}\ \to_I$$

$$\frac{\Gamma \vdash \mathbf{t}:\forall X.T}{\Gamma \vdash \mathbf{t}:T[U/X]}\ \forall_E \qquad \frac{\Gamma \vdash \mathbf{t}:T \qquad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t}:\forall X.T}\ \forall_I$$

$$\frac{}{\Gamma \vdash \mathbf{0}:T}\ ax_0 \qquad \frac{\Gamma \vdash \mathbf{t}:T \qquad \Gamma \vdash \mathbf{r}:T}{\Gamma \vdash \mathbf{t}+\mathbf{r}:T}\ +_I \qquad \frac{\Gamma \vdash \mathbf{t}:T}{\Gamma \vdash \alpha.\mathbf{t}:T}\ \alpha_I$$

Strengths

- Simple
- Strong normalization: straight

Weaknesses

- Too restrictive
- Algebraic features not reflected in types

The Scalar Type System [Arrighi, Díaz-Caro 2009]

$$T, R, S := U \mid \forall X.T \mid \alpha.T \mid \bar{0}$$
$$U, V, W := X \mid U \to T \mid \forall X.U$$

The Scalar Type System [Arrighi, Díaz-Caro 2009]

$$T, R, S := U \mid \forall X.T \mid \alpha.T \mid \bar{0} \qquad \alpha.\bar{0} \equiv \bar{0} \qquad \alpha.(\beta.T) \equiv (\alpha \times \beta).T$$

$$U, V, W := X \mid U \to T \mid \forall X.U \qquad 0.T \equiv \bar{0} \qquad \forall X.\alpha.T \equiv \alpha.\forall X.T$$

$$1.T \equiv T$$

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$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : \alpha.(U \to T) \quad \Gamma \vdash \mathbf{r} : \beta.U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : (\alpha \times \beta).T} \to_E \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \to_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \forall X.T}{\Gamma \vdash \mathbf{t} : T[U/X]}\ \forall_E \qquad \frac{\Gamma \vdash \mathbf{t} : T \quad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \forall X.T}\ \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \quad T \equiv R}{\Gamma \vdash \mathbf{t} : R}\ \equiv$$

$$\frac{}{\Gamma \vdash \mathbf{0} : \bar{0}}\ ax_0 \qquad \frac{\Gamma \vdash \mathbf{t} : \alpha.T \quad \Gamma \vdash \mathbf{r} : \beta.T}{\Gamma \vdash \mathbf{t} + \mathbf{r} : (\alpha + \beta).T} +_I \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \alpha.T}\ \alpha_I$$

The Scalar Type System [Arrighi, Díaz-Caro 2009]

Properties

- Strong normalization (re-using proof of $\lambda 2^{la}$)
- Subject reduction
- Accounting for scalars

  Possible application: A barycentric λ-calculus ($\mathfrak{B}$):

  - Take scalars in $\mathbb{R}$
  - Contexts with only $\mathcal{C}$ (classical types)
  - $\forall_E$: only for $\mathcal{C}$

Theorem

If $\Gamma \vdash \boldsymbol{t}:\mathcal{C}$, then $\boldsymbol{t}$ reduces to a barycentric term.

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Properties

- Strong normalization (re-using proof of $\lambda 2^{la}$)
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  Possible application: A barycentric λ-calculus ($\mathfrak{B}$):
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  - Contexts with only $\mathcal{C}$ (classical types)
  - $\forall_E$: only for $\mathcal{C}$

Theorem

If $\Gamma \vdash \mathbf{t} : \mathcal{C}$, then $\mathbf{t}$ reduces to a barycentric term.

Weakness: Still too restrictive!

- $\Gamma \vdash \mathbf{t} + \mathbf{r} : T$ implies $\mathbf{t}$ and $\mathbf{r}$ have the same type (up to scalars)

The Additive Type System [Díaz-Caro, Petit 2010]

Simplifying the language: Removing scalars

Higher-order computation

$\mathbf{t},\mathbf{r},\mathbf{s} := \quad x \mid \lambda x.\mathbf{t} \mid (\mathbf{t})\ \mathbf{r} \qquad \mid$

- $(\lambda x.\mathbf{t})\ \mathbf{b} \to \mathbf{t}[\mathbf{b}/x]$

b an abstraction or a variable.

Linear algebra

$\mathbf{t} + \mathbf{r} \mid$ ~~$\alpha.\mathbf{t}$~~ $\mid \mathbf{0}$

- Elementary rule:
  $\mathbf{t} + \mathbf{0} \to \mathbf{t}$.
- Application rules:
  $(\mathbf{t} + \mathbf{r})\ \mathbf{s} \to (\mathbf{t})\ \mathbf{s} + (\mathbf{r})\ \mathbf{s}$
  $(\mathbf{t})\ (\mathbf{r} + \mathbf{s}) \to (\mathbf{t})\ \mathbf{r} + (\mathbf{t})\ \mathbf{s}$
  $(\mathbf{0})\ \mathbf{t} \to \mathbf{0}$
  $(\mathbf{t})\ \mathbf{0} \to \mathbf{0}$
- AC equivalences
  $\mathbf{t} + \mathbf{r} \equiv \mathbf{r} + \mathbf{t}$
  $\mathbf{t} + (\mathbf{r} + \mathbf{s}) \equiv (\mathbf{t} + \mathbf{r}) + \mathbf{s}$

The Additive Type System [Díaz-Caro, Petit 2010]

$$T, R, S := U \mid T + R \mid \bar{0}$$
$$U, V, W := X \mid U \to T \mid \forall X.U$$

The Additive Type System [Díaz-Caro, Petit 2010]

$$T, R, S := U \mid T + R \mid \overline{0} \qquad T + \overline{0} \equiv T \quad T + R \equiv R + T$$

$$U, V, W := X \mid U \to T \mid \forall X.U \qquad (T + R) + S \equiv T + (R + S)$$

The Additive Type System [Díaz-Caro, Petit 2010]

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$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{\alpha} (U \to T_i) \quad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{\beta} U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} T_i} \to_E \qquad \frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \to_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \forall X.T}{\Gamma \vdash \mathbf{t} : T[U/X]} \forall_E \qquad \frac{\Gamma \vdash \mathbf{t} : T \quad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \forall X.T} \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \quad T \equiv R}{\Gamma \vdash \mathbf{t} : R} \equiv$$

$$\frac{}{\Gamma \vdash \mathbf{0} : \bar{0}} ax_0 \qquad \frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R} +_I \qquad \frac{}{\Gamma, x : U \vdash x : U} ax$$

- Strong normalisation ✓
- Subject reduction ✓

The Additive Type System [Díaz-Caro, Petit 2010]

System F with pairs

$$t, u := x \mid \lambda x.t \mid tu \mid \star \mid \langle t, u\rangle \mid \pi_1(t) \mid \pi_2(t)$$
$$A, B := X \mid A \Rightarrow B \mid \forall X.A \mid \mathbf{1} \mid A \times B$$

$$(\lambda x.t)\ u \to t[u/x] \qquad \pi_i(\langle t_1, t_2\rangle) \to t_i$$

$$\frac{}{\Delta, x : A \vdash_F x : A}\ Ax \qquad \frac{}{\Delta \vdash_F \star : \mathbf{1}}\ \mathbf{1} \qquad \frac{\Delta \vdash_F t : A \quad \Delta \vdash_F u : B}{\Delta \vdash_F \langle t, u\rangle : A \times B}\ \times_I$$

$$\frac{\Delta \vdash_F t : A \times B}{\Delta \vdash_F \pi_1(t) : A}\ \times_{E_l} \qquad \frac{\Delta \vdash_F t : A \times B}{\Delta \vdash_F \pi_2(t) : B}\ \times_{E_r} \qquad \frac{\Delta \vdash_F t : A \Rightarrow B \quad \Delta \vdash_F u : A}{\Delta \vdash_F tu : B}\ \Rightarrow_E$$

$$\frac{\Gamma, x : A}{\Delta \vdash_F \lambda x.t : A \Rightarrow B}\ \Rightarrow_I \qquad \frac{\Delta \vdash_F t : A \quad X \notin FV(\Gamma)}{\Delta \vdash_F t : \forall X.A}\ \forall_I \qquad \frac{\Delta \vdash_F t : \forall X.A}{\Delta \vdash_F t : A[B/X]}\ \forall_E$$

The Additive Type System [Díaz-Caro, Petit 2010]

Translation of types

Additive		**System F with pairs**
X	$\rightsquigarrow$	X
$U \to T$	$\rightsquigarrow$	$\lvert U\rvert \Rightarrow \lvert T\rvert$
$\forall X.U$	$\rightsquigarrow$	$\forall X.\lvert U\rvert$
$\bar{0}$	$\rightsquigarrow$	$\mathbf{1}$
$T + S$	$\rightsquigarrow$	$\lvert T\rvert \times \lvert S\rvert$

The Additive Type System [Díaz-Caro, Petit 2010]

Sums as Pairs

$$+\ ,\ \overline{0} \qquad \rightsquigarrow \qquad \times\ ,\ \mathbf{1}$$

$$\begin{array}{cc}
T + S \equiv S + T & A \times B \neq B \times A \\
T + (S + R) \equiv (T + S) + R & A \times (B \times C) \neq (A \times B) \times C \\
T + \overline{0} \equiv T & A \times \mathbf{1} \neq A
\end{array}$$

The Additive Type System [Díaz-Caro, Petit 2010]

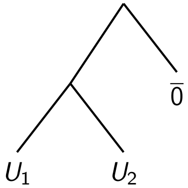
Sums as Pairs

$$+\ ,\ \overline{0} \qquad \rightsquigarrow \qquad \times\ ,\ \mathbf{1}$$

$$T + S \equiv S + T \qquad A \times B \neq B \times A$$
$$T + (S + R) \equiv (T + S) + R \qquad A \times (B \times C) \neq (A \times B) \times C$$
$$T + \overline{0} \equiv T \qquad A \times \mathbf{1} \neq A$$

$$T, R, S := U \mid T + R \mid \overline{0}$$
$$U, V, W := X \mid U \to T \mid \forall X.U$$

Type $\rightsquigarrow$ Binary tree (leaf: U or $\overline{0}$)
Example: $T = (U_1 + U_2)\ +\ \overline{0}$



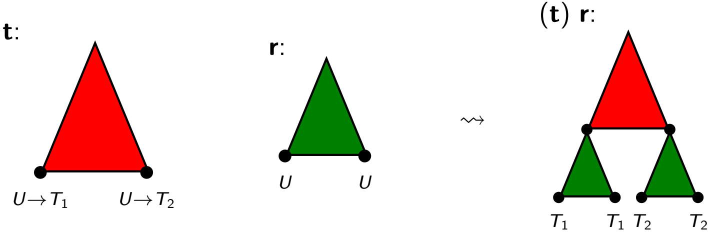
We first keep structured sum types

The Additive Type System [Díaz-Caro, Petit 2010]

Structured Arrow-elimination

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{\alpha}(U \to T_i) \qquad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{\beta} U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{\alpha}\sum_{j=1}^{\beta} T_i}$$

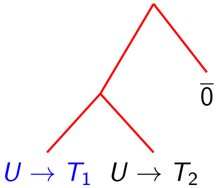
$$\frac{\Gamma \vdash \mathbf{t} : \mathcal{A}[\ell \mapsto (U \to T_\ell)] \qquad \Gamma \vdash \mathbf{r} : \mathcal{A}'[\ell' \mapsto U]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \mathcal{A} \circ \mathcal{A}'[\ell\ell' \mapsto T_\ell]}$$



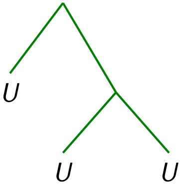
The Additive Type System [Díaz-Caro, Petit 2010]

Structured Arrow-elimination: An example

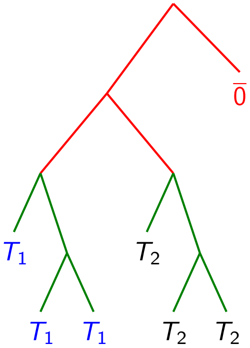
$\mathbf{t} = (\mathbf{t}_1 + \mathbf{t}_2) + 0$



$\mathbf{r} = \mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3)$



$$
\begin{aligned}
(\mathbf{t})\ \mathbf{r} \to^* \ & (\mathbf{t}_1)\ \big(\mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3)\big) + \\
& (\mathbf{t}_2)\ \big(\mathbf{r}_1 + (\mathbf{r}_2 + \mathbf{r}_3)\big) + 0 \\
\to^* \ & (\mathbf{t}_1)\ \mathbf{r}_1 + ((\mathbf{t}_1)\ \mathbf{r}_2 + (\mathbf{t}_1)\ \mathbf{r}_3) + \\
& (\mathbf{t}_2)\ \mathbf{r}_1 + ((\mathbf{t}_2)\ \mathbf{r}_2 + (\mathbf{t}_2)\ \mathbf{r}_3) + 0
\end{aligned}
$$



The Additive Type System [Díaz-Caro, Petit 2010]

Equivalence in System F

What about associativity, commutativity and neutral element?

Theorem

$$T \equiv T' \qquad \textit{implies} \qquad |T| \Leftrightarrow |T'|$$

Where $A \Leftrightarrow B$ means

$$\vdash_F \varepsilon_{A,B} : A \Rightarrow B \qquad \text{and} \qquad \vdash_F \varepsilon_{B,A} : B \Rightarrow A$$

for some terms $\varepsilon_{A,B}, \varepsilon_{B,A}$ s.t.

$$\varepsilon_{A,B} \circ \varepsilon_{B,A} \approx id_A \qquad \text{and} \qquad \varepsilon_{B,A} \circ \varepsilon_{A,B} \approx id_B$$

The Additive Type System [Díaz-Caro, Petit 2010]

What happens with linearity?

$$\mathbf{t} + \mathbf{r} \qquad \rightsquigarrow \qquad \langle [\mathbf{t}], [\mathbf{r}] \rangle$$

$$(\mathbf{t}_1 + \mathbf{t}_2)\ (\mathbf{r}_1 + \mathbf{r}_2) \to^* \qquad \text{but} \quad \langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \not\to$$
$$\mathbf{t}_1\ \mathbf{r}_1 + \mathbf{t}_1\ \mathbf{r}_2 + \mathbf{t}_2\ \mathbf{r}_1 + \mathbf{t}_2\ \mathbf{r}_2 \qquad \langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$$

No linearity in System F with pairs

The Additive Type System [Díaz-Caro, Petit 2010]

What happens with linearity?

$$\mathbf{t} + \mathbf{r} \quad \rightsquigarrow \quad \langle [\mathbf{t}], [\mathbf{r}] \rangle$$

$(\mathbf{t}_1 + \mathbf{t}_2)\ (\mathbf{r}_1 + \mathbf{r}_2) \to^*$ $\mathbf{t}_1\ \mathbf{r}_1 + \mathbf{t}_1\ \mathbf{r}_2 + \mathbf{t}_2\ \mathbf{r}_1 + \mathbf{t}_2\ \mathbf{r}_2$ but $\langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \not\to$ $\langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$

No linearity in System F with pairs

Main ideas:

- The sum is distributed during the translation of application

$$[(\mathbf{t})\ \mathbf{r}] = \langle\ \langle [\mathbf{t}_1][\mathbf{r}_1], [\mathbf{t}_1][\mathbf{r}_2] \rangle\ ,\ \langle [\mathbf{t}_2][\mathbf{r}_1], [\mathbf{t}_2][\mathbf{r}_2] \rangle\ \rangle$$

if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$

The Additive Type System [Díaz-Caro, Petit 2010]

What happens with linearity?

$$\mathbf{t} + \mathbf{r} \quad \rightsquigarrow \quad \langle [\mathbf{t}], [\mathbf{r}] \rangle$$

$(\mathbf{t}_1 + \mathbf{t}_2)\ (\mathbf{r}_1 + \mathbf{r}_2) \to^*$
$\mathbf{t}_1\ \mathbf{r}_1 + \mathbf{t}_1\ \mathbf{r}_2 + \mathbf{t}_2\ \mathbf{r}_1 + \mathbf{t}_2\ \mathbf{r}_2$

but $\langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \not\to$
$\langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle$

No linearity in System F with pairs

Main ideas:

- The sum is distributed during the translation of application
  $$[(\mathbf{t})\ \mathbf{r}] = \langle\ \langle [\mathbf{t}_1][\mathbf{r}_1], [\mathbf{t}_1][\mathbf{r}_2] \rangle\ ,\ \langle [\mathbf{t}_2][\mathbf{r}_1], [\mathbf{t}_2][\mathbf{r}_2] \rangle\ \rangle$$
  if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$
- The "sum structure" of a term is known thanks to its type
  $\Gamma \vdash \mathbf{t} : (T_1 + T_2) + T_3 \quad \rightsquigarrow \quad \mathbf{t}$ "$\simeq$" $(\mathbf{t}_1 + \mathbf{t}_2) + \mathbf{t}_3$
  with $\Gamma \vdash \mathbf{t}_i : T_i$

The Additive Type System [Díaz-Caro, Petit 2010]

Translation of terms

$$\Gamma \ \vdash \ \mathbf{t} : T \qquad \rightsquigarrow \qquad |\Gamma| \ \vdash_F \ [\mathbf{t}]_{\mathcal{D}} : |T|$$

$$\Gamma, x : T \vdash x : T \qquad \rightsquigarrow \qquad [x]_{\mathcal{D}} = x$$

$$\Gamma \vdash \mathbf{0} : \overline{0} \qquad \rightsquigarrow \qquad [\mathbf{0}]_{\mathcal{D}} = \star$$

The Additive Type System [Díaz-Caro, Petit 2010]

Translation of terms

$$\Gamma \ \vdash \ \mathbf{t} : T \qquad \rightsquigarrow \qquad |\Gamma| \ \vdash_F \ [\mathbf{t}]_{\mathcal{D}} : |T|$$

$$\Gamma, x : T \vdash x : T \qquad \rightsquigarrow \qquad [x]_{\mathcal{D}} = x$$

$$\Gamma \vdash \mathbf{0} : \overline{0} \qquad \rightsquigarrow \qquad [\mathbf{0}]_{\mathcal{D}} = \star$$

$$\frac{\Gamma \vdash \mathbf{t} : T \quad T \equiv T'}{\Gamma \vdash \mathbf{t} : T'} \qquad \rightsquigarrow \qquad [\mathbf{t}]_{\mathcal{D}} = \varepsilon_{|T|,|T'|}[\mathbf{t}]_{\mathcal{D}'}$$

The Additive Type System [Díaz-Caro, Petit 2010]

Translation of terms

$$\Gamma \vdash \mathbf{t} : T \quad \rightsquigarrow \quad |\Gamma| \vdash_F [\mathbf{t}]_{\mathcal{D}} : |T|$$

$$\Gamma, x : T \vdash x : T \quad \rightsquigarrow \quad [x]_{\mathcal{D}} = x$$

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$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : S}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + S} \quad \rightsquigarrow \quad [\mathbf{t}+\mathbf{r}]_{\mathcal{D}} = \langle [\mathbf{t}]_{\mathcal{D}_1}, [\mathbf{r}]_{\mathcal{D}_2} \rangle$$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T} \quad \rightsquigarrow \quad [\lambda x.\mathbf{t}]_{\mathcal{D}} = \lambda x.[\mathbf{t}]_{\mathcal{D}'}$$

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$$\Gamma \vdash \mathbf{t} : T \quad \rightsquigarrow \quad |\Gamma| \vdash_F [\mathbf{t}]_{\mathcal{D}} : |T|$$

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$$\frac{\Gamma \vdash \mathbf{t} : \mathcal{A}[_{w \mapsto (U \to T_w)}] \quad \Gamma \vdash \mathbf{r} : \mathcal{A}'[_{v \mapsto U}]}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \mathcal{A} \circ \mathcal{A}'[_{wv \mapsto T_w}]}$$

$$\rightsquigarrow \quad [(\mathbf{t})\ \mathbf{r}]_{\mathcal{D}} = \mathcal{A} \circ \mathcal{A}'[_{wv \mapsto \pi_{\overline{w}}([\mathbf{t}]_{\mathcal{D}_1})\pi_{\overline{v}}([\mathbf{u}]_{\mathcal{D}_2})}]$$

The Additive Type System [Díaz-Caro, Petit 2010]

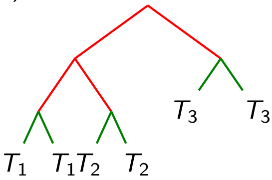
Translation of terms: An example

$$\frac{\Gamma \vdash \mathbf{t} : \Big((U \to T_1) + (U \to T_2)\Big) \; + \; (U \to T_3) \qquad \Gamma \vdash \mathbf{r} : U + U}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \Big((T_1 + T_1) + (T_2 + T_2)\Big) + (T_3 + T_3)}$$

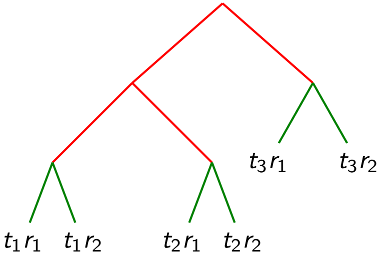
$t_1 = \pi_{11}([\mathbf{t}]); t_2 = \pi_{12}([\mathbf{t}]); t_3 = \pi_2([\mathbf{t}]);$

$r_1 = \pi_1([\mathbf{r}]); r_2 = \pi_2([\mathbf{r}]);$

$(\mathbf{t})\ \mathbf{r} :$



$[(\mathbf{t})\ \mathbf{r}]_{\mathcal{D}} =$



The Additive Type System [Díaz-Caro, Petit 2010]

Translation of terms: Correctness

Correctness of typing:

Theorem

$\Gamma \vdash \boldsymbol{t} : T \quad \Longrightarrow \quad |\Gamma| \vdash_F [\boldsymbol{t}]_{\mathcal{D}} : |T|$

Correctness of reduction:

Theorem

$\Gamma \vdash \boldsymbol{t} : T$ *and* $\boldsymbol{t} \to \boldsymbol{t}'$ *implies* $[\boldsymbol{t}]_{\mathcal{D}} \to^* [\boldsymbol{t}']_{\mathcal{D}'}$ *for some* $\mathcal{D}'$

The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

Grammar

Types grammar:

$$\begin{aligned} T, R, S &:= U \mid T + R \mid \alpha.T \qquad \text{(no type } \bar{0}\text{!)} \\ U, V, W &:= X \mid U \to T \mid \forall X.U \end{aligned}$$

Equivalences:

$$\begin{aligned} 1.T &\equiv T \\ \alpha.(\beta.T) &\equiv (\alpha \times \beta).T \\ \alpha.T + \alpha.R &\equiv \alpha.(T + R) \\ \alpha.T + \beta.T &\equiv (\alpha + \beta).T \\ T + R &\equiv R + T \\ T + (R + S) &\equiv (T + R) + S \end{aligned}$$

The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

Not one zero... a lot of them

$$\dfrac{\dfrac{\Gamma \vdash \lambda x.x : U \to U \qquad \dfrac{\Gamma \vdash \mathbf{r} : R \qquad \Gamma \vdash -\mathbf{r} : -R}{\Gamma \vdash \mathbf{r} - \mathbf{r} : \overline{0}}}{\Gamma \vdash \lambda x.x + \mathbf{r} - \mathbf{r} : U \to U} \qquad \dfrac{}{\Gamma \vdash \mathbf{t} : U}}{\Gamma \vdash (\lambda x.x + \mathbf{r} - \mathbf{r})\ \mathbf{t} : U}$$

The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

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$$\dfrac{\Gamma \vdash \lambda x.x : U \to U \qquad \dfrac{\Gamma \vdash \mathbf{r} : R \qquad \Gamma \vdash -\mathbf{r} : -R}{\Gamma \vdash \mathbf{r} - \mathbf{r} : \overline{0}}}{\dfrac{\Gamma \vdash \lambda x.x + \mathbf{r} - \mathbf{r} : U \to U \qquad \overline{\Gamma \vdash \mathbf{t} : U}}{\Gamma \vdash (\lambda x.x + \mathbf{r} - \mathbf{r})\ \mathbf{t} : U}}$$

However, $(\lambda x.x + \mathbf{r} - \mathbf{r})\ \mathbf{t} \to (\lambda x.x)\ \mathbf{t} + (\mathbf{r})\ \mathbf{t} - (\mathbf{r})\ \mathbf{t}$.

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We need to keep track of the zeros... where they came from!

The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

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However, $(\lambda x.x + \mathbf{r} - \mathbf{r})\ \mathbf{t} \to (\lambda x.x)\ \mathbf{t} + (\mathbf{r})\ \mathbf{t} - (\mathbf{r})\ \mathbf{t}$.
We need to keep track of the zeros... where they came from!

Instead of $\overline{0}$, we have $0.R$ and $T + 0.R \not\equiv T$

The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

The factorisation rule problem

$$\frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{t} : T'}{\Gamma \vdash \alpha.\mathbf{t} + \beta.\mathbf{t} : \alpha.T + \beta.T'}$$

- However, $\alpha.\mathbf{t} + \beta.\mathbf{t} \to (\alpha + \beta).\mathbf{t}$
- In general $\alpha.T + \beta.T' \neq (\alpha + \beta).T \neq (\alpha + \beta).T'$

The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

The factorisation rule problem

Several possible solutions:

- Remove factorisation rule (Done. SR and SN both work)
  - $+$ in scalars not used anymore. Scalars $\Rightarrow$ Monoid
  - It works!... but it is no so expressive, no so useful

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  - It works!... but it is no so expressive, no so useful
- Add several typing rules to allow typing $(\alpha + \beta).\mathbf{u}$ with $\alpha.A + \beta.B$
  - As soon as we add one, we have to add many to make it work
  - Too complex and inelegant

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  - Seems to be the natural solution
  - Big complexness with polymorphism and distributivity

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  - Too complex and inelegant
- Church style (work-in-progress)
  - Seems to be the natural solution
  - Big complexness with polymorphism and distributivity
- Weaker subject reduction (this presentation)

The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

Typing rules

$$\frac{}{\Gamma, x : U \vdash x : U}\ ax \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \mathbf{0} : 0.T}\ 0_I \qquad \frac{\Gamma \vdash \mathbf{t} : T}{\Gamma \vdash \alpha.\mathbf{t} : \alpha.T}\ \alpha_I$$

$$\frac{\Gamma \vdash \mathbf{t} : \sum_{i=1}^{n} \alpha_i.\forall\vec{X}.(U \to T_i) \qquad \Gamma \vdash \mathbf{r} : \sum_{j=1}^{m} \beta_j.V_j \qquad \forall V_j, \exists \vec{W}_j / U[\vec{W}_j/\vec{X}] = V_j}{\Gamma \vdash (\mathbf{t})\ \mathbf{r} : \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \times \beta_j.T_i[\vec{W}_j/\vec{X}]}\ \to_E$$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x.\mathbf{t} : U \to T}\ \to_I \qquad \frac{\Gamma \vdash \mathbf{t} : T \qquad \Gamma \vdash \mathbf{r} : R}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + R}\ +_I$$

$$\frac{\Gamma \vdash \mathbf{t} : U \qquad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t} : \forall X.U}\ \forall_I \qquad \frac{\Gamma \vdash \mathbf{t} : \forall X.U}{\Gamma \vdash \mathbf{t} : U[V/X]}\ \forall_E$$

Vectorial type system

Example

$\Gamma \vdash \mathbf{b}_1 : V_1$

$\Gamma \vdash \mathbf{b}_2 : V_2$

Vectorial type system

Example

$$\begin{array}{l} \Gamma \vdash \mathbf{b}_1 : V_1 \\ \Gamma \vdash \mathbf{b}_2 : V_2 \end{array} \Rightarrow \Gamma \vdash \beta_1.\mathbf{b}_1 + \beta_2.\mathbf{b}_2 : \beta_1.V_1 + \beta_2.V_2$$

Vectorial type system

Example

$$\begin{array}{l}\Gamma \vdash \mathbf{b}_1 : V_1 \\ \Gamma \vdash \mathbf{b}_2 : V_2\end{array} \Rightarrow \Gamma \vdash \beta_1.\mathbf{b}_1 + \beta_2.\mathbf{b}_2 : \beta_1.V_1 + \beta_2.V_2$$

$$\begin{array}{l}\Gamma \vdash \lambda x_1.\mathbf{t}_1 : \forall X.(U \to T_1) \\ \Gamma \vdash \lambda x_2.\mathbf{t}_2 : \forall X.(U \to T_2)\end{array}$$

$$\Rightarrow \Gamma \vdash (\alpha_1.\lambda x_1.\mathbf{t}_1) + (\alpha_2.\lambda x_2.\mathbf{t}_2) : (\alpha_1.\forall X.(U \to T_1)) + (\alpha_2.\forall X.(U \to T_2))$$

Vectorial type system

Example

$$\begin{array}{l}\Gamma \vdash \mathbf{b}_1 : V_1 \\ \Gamma \vdash \mathbf{b}_2 : V_2\end{array} \Rightarrow \Gamma \vdash \beta_1.\mathbf{b}_1 + \beta_2.\mathbf{b}_2 : \beta_1.V_1 + \beta_2.V_2$$

$$\begin{array}{l} U[W_1/X] = V_1 \\ U[W_2/X] = V_2 \end{array}$$

$$\begin{array}{l}\Gamma \vdash \lambda x_1.\mathbf{t}_1 : \forall X.(U \to T_1) \\ \Gamma \vdash \lambda x_2.\mathbf{t}_2 : \forall X.(U \to T_2)\end{array}$$

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Example

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$$(\blacktriangle + \blacktriangledown)\ (\triangle + \triangledown) \to^* (\blacktriangle)\ \triangle + (\blacktriangle)\ \triangledown + (\blacktriangledown)\ \triangle + (\blacktriangledown)\ \triangledown$$

Vectorial type system

Example

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<u>e.g.</u>

$$(\blacktriangledown)\ \vartriangle = (\alpha_2.\lambda x_2.\mathbf{t}_2)\ \beta_1.\mathbf{b}_1 \to^* (\alpha_2 \times \beta_1).(\lambda x_2.\mathbf{t}_2)\ \mathbf{b}_1$$

Vectorial type system

Example

$$\begin{array}{l}\Gamma \vdash \mathbf{b}_1 : V_1 \\ \Gamma \vdash \mathbf{b}_2 : V_2\end{array} \Rightarrow \Gamma \vdash \beta_1.\mathbf{b}_1 + \beta_2.\mathbf{b}_2 : \beta_1.V_1 + \beta_2.V_2$$

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The Vectorial Type System [Arrighi, Díaz-Caro, Valiron 2010]

A weaker subject reduction

$$(\alpha+\beta).T \sqsubseteq \alpha.T + \beta.T' \qquad \text{if } \Gamma \vdash \alpha.\mathbf{t} : \alpha.T \text{ and } \Gamma \vdash \beta.\mathbf{t} : \beta.T'$$

(and its contextual closure)

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(and its contextual closure)

Theorem (Weak subject reduction)

$$\left.\begin{array}{r} \boldsymbol{t} \to_R \boldsymbol{s} \\ \Gamma \vdash \boldsymbol{t} : T \end{array}\right\} \exists S \sqsubseteq T \;/\; \Gamma \vdash \boldsymbol{s} : S$$

Moreover, if R is not a factorisation, then $\Rightarrow S \equiv T$.

▸ Extra: orthogonality

▸ Conclusion

Orthogonality

Why

Now, say we would like normalised terms

$$\alpha.\mathbf{t} + \beta.\mathbf{s}$$

with $|\alpha|^2 + |\beta|^2 = 1$.

Orthogonality

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So, we want to allow $\frac{1}{\sqrt{2}}.\mathbf{t} + \frac{1}{\sqrt{2}}.\mathbf{s}$ and disallow $\frac{1}{2}.\mathbf{t} + \frac{1}{2}.\mathbf{s}$

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Or do we? What happens if $\mathbf{t} \to^* \mathbf{s}$?

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Or do we? What happens if $\mathbf{t} \to^* \mathbf{s}$?
We need a notion of orthogonality between terms (and also a way to check it).

Definition (Terms inner product)

Let $\mathbf{t}, \mathbf{s}, \mathbf{r}$ be any term, and let $\mathbf{u}, \mathbf{v}$ be terms in normal form which are not sum of terms nor a scalar times a term. We define the function 'delta' as follows:

$$\delta_{\mathbf{t},\mathbf{s}} = \begin{cases} 0 & \text{if } \mathbf{t} \neq \mathbf{s} \vee \mathbf{t} = \mathbf{0} \vee \mathbf{s} = \mathbf{0} \\ 1 & \text{in any other case} \end{cases}$$

Then, we define the inner product between terms recursively by

$\langle \mathbf{u} | \mathbf{v} \rangle = \langle \mathbf{v} | \mathbf{u} \rangle = \delta_{\mathbf{u},\mathbf{v}}$

$\langle \mathbf{t} | \mathbf{s} \rangle = \overline{\langle \mathbf{s} | \mathbf{t} \rangle} = \langle \mathbf{t}\downarrow | \mathbf{s}\downarrow \rangle$

$\langle \alpha.\mathbf{t} + \beta.\mathbf{s} | \mathbf{r} \rangle = \alpha \times \langle \mathbf{t} | \mathbf{r} \rangle + \beta \times \langle \mathbf{s} | \mathbf{r} \rangle$

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$$\langle \alpha.\mathbf{t} + \beta.\mathbf{s} | \mathbf{r} \rangle = \alpha \times \langle \mathbf{t} | \mathbf{r} \rangle + \beta \times \langle \mathbf{s} | \mathbf{r} \rangle$$

Examples

$$\langle (\lambda x\ x)\ \mathbf{v} | \mathbf{v} \rangle = 1$$

Definition (Terms inner product)

Let $\mathbf{t}, \mathbf{s}, \mathbf{r}$ be any term, and let $\mathbf{u}, \mathbf{v}$ be terms in normal form which are not sum of terms nor a scalar times a term. We define the function 'delta' as follows:

$$\delta_{\mathbf{t},\mathbf{s}} = \begin{cases} 0 & \text{if } \mathbf{t} \neq \mathbf{s} \vee \mathbf{t} = \mathbf{0} \vee \mathbf{s} = \mathbf{0} \\ 1 & \text{in any other case} \end{cases}$$

Then, we define the inner product between terms recursively by

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Examples

$\langle (\lambda x\ x)\ \mathbf{v} | \mathbf{v} \rangle = 1$

$\langle \mathbf{true} | \mathbf{false} \rangle = 0$

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Examples

$$\langle (\lambda x\ x)\ \mathbf{v} | \mathbf{v} \rangle = 1$$

$$\langle \mathbf{true} | \mathbf{false} \rangle = 0$$

$$\langle \tfrac{1}{\sqrt{2}}.\mathbf{true} + \tfrac{1}{\sqrt{2}}.\mathbf{false} | \mathbf{true} \rangle = \tfrac{1}{\sqrt{2}} \times \langle \mathbf{true} | \mathbf{true} \rangle + \tfrac{1}{\sqrt{2}} \times \langle \mathbf{false} | \mathbf{true} \rangle = \tfrac{1}{\sqrt{2}}$$

Orthogonality

How to check it

Let $\Gamma \vdash \mathbf{u} : T$ and $\Gamma \vdash \mathbf{v} : S$.

Does $\mathbf{u} \perp \mathbf{v} \Leftrightarrow T \perp S$?

(for some definition of orthogonality between types)

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Counterexample

$\vdash \mathrm{true} : \mathit{Bool}$

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Counterexample

$y : W \vdash \lambda x.y : U \to W$

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Workaround

- Church-style $\Rightarrow T \perp S \Rightarrow \mathbf{u} \perp \mathbf{v}$

Summary

- The Scalar TS : Accounts for scalars. Barycentric calculus ✓
- The Additive TS: Intuitions about sums ✓
- The Vectorial TS: Curry not compatible with vectors ✓
- Church-version: subject reduction & orthogonality (work-in-progress)
- May lead to a *quantum computational logic*