

Sums in algebraic lambda-calculi

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Types 2010 • October 13th, 2010 • Warsaw

Motivation

- ▶ Algebraic calculi:
 - ▶ Linear-algebraic λ -calculus [Arrighi, Dowek 2008]
 - ▶ Algebraic λ -calculus [Vaux 2009]
- ▶ CbV-CbN simulation [Díaz-Caro, Perdrix, Tasson, Valiron 2010]

Is it possible to find any interpretation of them in some well established theory?

Motivation

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Is it possible to find any interpretation of them in some well established theory?

- ▶ **In this work we try to answer this question for a simplified version of these calculi**
- ▶ We analyse the role of **sums** in the Linear-algebraic λ -calculus

Linear [Arrighi, Dowek 2008]... without scalars

Higher-order computation

$\mathbf{t}, \mathbf{r}, \mathbf{s} ::= x \mid \lambda x. \mathbf{t} \mid (\mathbf{t}) \mathbf{r} \quad |$

Linear [Arrighi, Dowek 2008]... without scalars

Higher-order computation

$\mathbf{t}, \mathbf{r}, \mathbf{s} ::= x \mid \lambda x. \mathbf{t} \mid (\mathbf{t}) \mathbf{r}$

Linear algebra

$\mathbf{t} + \mathbf{r} \mid \mathbf{0} \mid \mathbf{t}$

Linear [Arrighi, Dowek 2008]... without scalars

Higher-order computation

$\mathbf{t}, \mathbf{r}, \mathbf{s} ::= x \mid \lambda x. \mathbf{t} \mid (\mathbf{t}) \mathbf{r}$

▶ $(\lambda x. \mathbf{t}) \mathbf{b} \rightarrow \mathbf{t}[\mathbf{b}/x]$

\mathbf{b} an abstraction or a variable.

Linear algebra

$\mathbf{t} + \mathbf{r} \mid \mathbf{0} \mid \mathbf{t} \cdot \mathbf{r}$

Linear [Arrighi, Dowek 2008]... without scalars

Higher-order computation

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▶ $(\lambda x. \mathbf{t}) \mathbf{b} \rightarrow \mathbf{t}[\mathbf{b}/x]$

\mathbf{b} an abstraction or a variable.

Linear algebra

$\mathbf{t} + \mathbf{r} \mid \mathbf{0} \mid \mathbf{t}$

▶ Elementary rule:

$\mathbf{t} + \mathbf{0} \rightarrow \mathbf{t}$.

▶ Application rules:

$(\mathbf{t} + \mathbf{r}) \mathbf{s} \rightarrow (\mathbf{t}) \mathbf{s} + (\mathbf{r}) \mathbf{s}$

$(\mathbf{t}) (\mathbf{r} + \mathbf{s}) \rightarrow (\mathbf{t}) \mathbf{r} + (\mathbf{t}) \mathbf{s}$

$(\mathbf{0}) \mathbf{t} \rightarrow \mathbf{0}$

$(\mathbf{t}) \mathbf{0} \rightarrow \mathbf{0}$

▶ AC equivalences

$\mathbf{t} + \mathbf{r} \equiv \mathbf{r} + \mathbf{t}$

$\mathbf{t} + (\mathbf{r} + \mathbf{s}) \equiv (\mathbf{t} + \mathbf{r}) + \mathbf{s}$

Additive type System

Grammar

Types grammar:

$T, R, S := U \mid T + R \mid \bar{0}$ **general** types

$U, V, W := X \mid U \rightarrow T \mid \forall X.U$ **unit** types

Equivalences:

$$T + \bar{0} \equiv T$$

$$T + R \equiv R + T$$

$$T + (R + S) \equiv (T + R) + S$$

Additive type System

Typing rules

$$\frac{}{\Gamma, x: U \vdash x: U} \text{ax} \qquad \frac{}{\Gamma \vdash \mathbf{0}: \bar{0}} \text{ax}_{\bar{0}}$$
$$\frac{\Gamma \vdash \mathbf{t}: \sum_{i=1}^{\alpha} (U \rightarrow T_i) \quad \Gamma \vdash \mathbf{r}: \sum_{j=1}^{\beta} U}{\Gamma \vdash (\mathbf{t}) \mathbf{r}: \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} T_i} \rightarrow_E \qquad \frac{\Gamma, x: U \vdash \mathbf{t}: T}{\Gamma \vdash \lambda x. \mathbf{t}: U \rightarrow T} \rightarrow_I$$
$$\frac{\Gamma \vdash \mathbf{t}: \forall X. U}{\Gamma \vdash \mathbf{t}: U[V/X]} \forall_E \qquad \frac{\Gamma \vdash \mathbf{t}: U \quad X \notin FV(\Gamma)}{\Gamma \vdash \mathbf{t}: \forall X. U} \forall_I$$
$$\frac{\Gamma \vdash \mathbf{t}: T \quad \Gamma \vdash \mathbf{r}: R}{\Gamma \vdash \mathbf{t} + \mathbf{r}: T + R} +_I \qquad \frac{\Gamma \vdash \mathbf{t}: T \quad T \equiv R}{\Gamma \vdash \mathbf{t}: R} \equiv$$

- Strong normalisation ✓
- Subject reduction ✓

System F with pairs

$$\begin{aligned} t, u &:= x \mid \lambda x. t \mid tu \mid \star \mid \langle t, u \rangle \mid \pi_1(t) \mid \pi_2(t) \\ A, B &:= X \mid A \Rightarrow B \mid \forall X. A \mid \mathbf{1} \mid A \times B \end{aligned}$$

System F with pairs

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System F with pairs

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$$\begin{array}{c} \frac{}{\Delta, x : A \vdash_F x : A}^{Ax} \quad \frac{}{\Delta \vdash_F \star : \mathbf{1}}^{\mathbf{1}} \quad \frac{\Delta \vdash_F t : A \times B}{\Delta \vdash_F \pi_1(t) : A}^{\times E_\ell} \\ \\ \frac{\Delta \vdash_F t : A \quad \Delta \vdash_F u : B}{\Delta \vdash_F \langle t, u \rangle : A \times B}^{\times I} \quad \frac{\Delta \vdash_F t : A \times B}{\Delta \vdash_F \pi_2(t) : B}^{\times E_r} \\ \\ \frac{\Delta, x : A \vdash_F t : B}{\Delta \vdash_F \lambda x.t : A \Rightarrow B}^{\Rightarrow I} \quad \frac{\Delta \vdash_F t : A \Rightarrow B \quad \Delta \vdash_F u : A}{\Delta \vdash_F tu : B}^{\Rightarrow E} \\ \\ \frac{\Delta \vdash_F t : A \quad X \notin FV(\Delta)}{\Delta \vdash_F t : \forall X.A}^{\forall I} \quad \frac{\Delta \vdash_F t : \forall X.A}{\Delta \vdash_F t : A[X := B]}^{\forall E} \end{array}$$

Translation of types

Additive		System F with pairs
X	\rightsquigarrow	X
$U \rightarrow T$	\rightsquigarrow	$ U \Rightarrow T $
$\forall X. U$	\rightsquigarrow	$\forall X. U $
$\bar{0}$	\rightsquigarrow	1
$T + S$	\rightsquigarrow	$ T \times S $

Sums as Pairs

$+$, $\bar{0}$ \rightsquigarrow \times , $\mathbf{1}$

$$\begin{aligned}T + S &\equiv S + T \\T + (S + R) &\equiv (T + S) + R \\T + \bar{0} &\equiv T\end{aligned}$$

$$\begin{aligned}A \times B &\neq B \times A \\A \times (B \times C) &\neq (A \times B) \times C \\A \times \mathbf{1} &\neq A\end{aligned}$$

Sums as Pairs

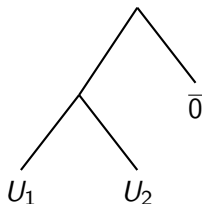
$+, \bar{0} \rightsquigarrow \times, \mathbf{1}$

$$\begin{array}{ll} T + S & \equiv S + T \\ T + (S + R) & \equiv (T + S) + R \\ T + \bar{0} & \equiv T \end{array} \qquad \begin{array}{ll} A \times B & \neq B \times A \\ A \times (B \times C) & \neq (A \times B) \times C \\ A \times \mathbf{1} & \neq A \end{array}$$

$$\begin{array}{l} T, R, S := U \mid T + R \mid \bar{0} \\ U, V, W := X \mid U \rightarrow T \mid \forall X. U \end{array}$$

Type \rightsquigarrow Binary tree (leaf: U or $\bar{0}$)

Example: $T = (U_1 + U_2) + \bar{0}$



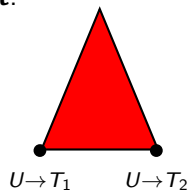
We keep structured sum types

Structured Arrow-elimination

$$\frac{\Gamma \vdash \mathbf{t}: \sum_{i=1}^{\alpha} (U \rightarrow T_i) \quad \Gamma \vdash \mathbf{r}: \sum_{j=1}^{\beta} U}{\Gamma \vdash (\mathbf{t}) \mathbf{r}: \sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} T_i}$$

$$\frac{\Gamma \vdash \mathbf{t}: \mathcal{A}[l \mapsto (U \rightarrow T_\ell)] \quad \Gamma \vdash \mathbf{r}: \mathcal{A}'[l' \mapsto U]}{\Gamma \vdash (\mathbf{t}) \mathbf{r}: \mathcal{A} \circ \mathcal{A}'[ll' \mapsto T_\ell]}$$

t:

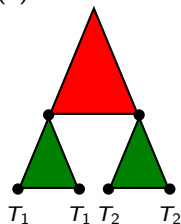


r:



\rightsquigarrow

(t) r:



Equivalence in System F

What about associativity, commutativity and neutral element in system F with pairs?

Theorem

$$T \equiv T' \quad \text{implies} \quad |T| \Leftrightarrow |T'|$$

Where $A \Leftrightarrow B$ means

$$\vdash_F \varepsilon_{A,B} : A \Rightarrow B \quad \text{and} \quad \vdash_F \varepsilon_{B,A} : B \Rightarrow A$$

for some terms $\varepsilon_{A,B}, \varepsilon_{B,A}$ s.t.

$$\varepsilon_{A,B} \circ \varepsilon_{B,A} \approx id_A \quad \text{and} \quad \varepsilon_{B,A} \circ \varepsilon_{A,B} \approx id_B$$

Translation of terms

What happens with linearity?

$$\mathbf{t} + \mathbf{r} \rightsquigarrow \langle [\mathbf{t}], [\mathbf{r}] \rangle$$

$$\begin{array}{l} (\mathbf{t}_1 + \mathbf{t}_2) (\mathbf{r}_1 + \mathbf{r}_2) \rightarrow^* \\ \mathbf{t}_1 \mathbf{r}_1 + \mathbf{t}_1 \mathbf{r}_2 + \mathbf{t}_2 \mathbf{r}_1 + \mathbf{t}_2 \mathbf{r}_2 \end{array} \quad \text{but} \quad \begin{array}{l} \langle t_1, t_2 \rangle \langle r_1, r_2 \rangle \nrightarrow \\ \langle \langle t_1 r_1, t_1 r_2 \rangle, \langle t_2 r_1, t_2 r_2 \rangle \rangle \end{array}$$

No linearity in System F with pairs

Translation of terms

What happens with linearity?

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No linearity in System F with pairs

Main ideas:

- ▶ The sum is distributed during the translation of application

$$[(\mathbf{t}) \mathbf{r}] = \langle \langle [\mathbf{t}_1][\mathbf{r}_1], [\mathbf{t}_1][\mathbf{r}_2] \rangle, \langle [\mathbf{t}_2][\mathbf{r}_1], [\mathbf{t}_2][\mathbf{r}_2] \rangle \rangle$$

if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$

Translation of terms

What happens with linearity?

$$\mathbf{t} + \mathbf{r} \rightsquigarrow \langle [\mathbf{t}], [\mathbf{r}] \rangle$$

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No linearity in System F with pairs

Main ideas:

- ▶ The sum is distributed during the translation of application
$$[(\mathbf{t}) \mathbf{r}] = \langle \langle [\mathbf{t}_1][\mathbf{r}_1], [\mathbf{t}_1][\mathbf{r}_2] \rangle, \langle [\mathbf{t}_2][\mathbf{r}_1], [\mathbf{t}_2][\mathbf{r}_2] \rangle \rangle$$
if $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2$ and $\mathbf{r} = \mathbf{r}_1 + \mathbf{r}_2$
- ▶ The “sum structure” of a term is known thanks to its type
$$\Gamma \vdash \mathbf{t} : (T_1 + T_2) + T_3 \rightsquigarrow \mathbf{t} \text{ “}\simeq\text{” } (\mathbf{t}_1 + \mathbf{t}_2) + \mathbf{t}_3$$
with $\Gamma \vdash \mathbf{t}_i : T_i$

Translation of terms

$$\Gamma \vdash \mathbf{t} : T \quad \rightsquigarrow \quad |\Gamma| \vdash_F [\mathbf{t}]_{\mathcal{D}} : |T|$$

$$\Gamma, x : T \vdash x : T \quad \rightsquigarrow \quad [x]_{\mathcal{D}} = x$$

$$\Gamma \vdash \mathbf{0} : \bar{0} \quad \rightsquigarrow \quad [\mathbf{0}]_{\mathcal{D}} = \star$$

Translation of terms

$$\Gamma \vdash \mathbf{t} : T \quad \rightsquigarrow \quad |\Gamma| \vdash_F [\mathbf{t}]_{\mathcal{D}} : |T|$$

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$$\Gamma \vdash \mathbf{t} : T \quad T \equiv T'$$

$$\rightsquigarrow [\mathbf{t}]_{\mathcal{D}} = \varepsilon_{|T|, |T'|} [\mathbf{t}]_{\mathcal{D}'}$$

$$\hline \Gamma \vdash \mathbf{t} : T'$$

Translation of terms

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$$\frac{}{\Gamma \vdash \mathbf{t} : T'}$$

$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : S}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + S}$$

$$\rightsquigarrow [\mathbf{t} + \mathbf{r}]_{\mathcal{D}} = \langle [\mathbf{t}]_{\mathcal{D}_1}, [\mathbf{r}]_{\mathcal{D}_2} \rangle$$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x. \mathbf{t} : U \rightarrow T}$$

$$\rightsquigarrow [\lambda x. \mathbf{t}]_{\mathcal{D}} = \lambda x. [\mathbf{t}]_{\mathcal{D}'}$$

Translation of terms

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$$\frac{\Gamma \vdash \mathbf{t} : T \quad \Gamma \vdash \mathbf{r} : S}{\Gamma \vdash \mathbf{t} + \mathbf{r} : T + S} \quad \rightsquigarrow \quad [\mathbf{t} + \mathbf{r}]_{\mathcal{D}} = \langle [\mathbf{t}]_{\mathcal{D}_1}, [\mathbf{r}]_{\mathcal{D}_2} \rangle$$

$$\frac{\Gamma, x : U \vdash \mathbf{t} : T}{\Gamma \vdash \lambda x. \mathbf{t} : U \rightarrow T} \quad \rightsquigarrow \quad [\lambda x. \mathbf{t}]_{\mathcal{D}} = \lambda x. [\mathbf{t}]_{\mathcal{D}'}$$

$$\frac{\Gamma \vdash \mathbf{t} : \mathcal{A}[w \mapsto (U \rightarrow T_w)] \quad \Gamma \vdash \mathbf{r} : \mathcal{A}'[v \mapsto U]}{\Gamma \vdash (\mathbf{t}) \mathbf{r} : \mathcal{A} \circ \mathcal{A}'[wv \mapsto T_w]} \quad \rightsquigarrow \quad [(\mathbf{t}) \mathbf{r}]_{\mathcal{D}} = \mathcal{A} \circ \mathcal{A}'[wv \mapsto \pi_{\bar{w}}([\mathbf{t}]_{\mathcal{D}_1}) \pi_{\bar{v}}([\mathbf{r}]_{\mathcal{D}_2})]$$

Translation of terms

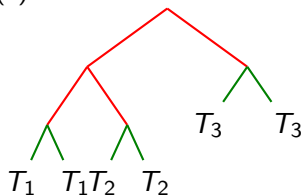
An example

$$\frac{\Gamma \vdash \mathbf{t} : \left((U \rightarrow T_1) + (U \rightarrow T_2) \right) + (U \rightarrow T_3) \quad \Gamma \vdash \mathbf{r} : U + U}{\Gamma \vdash (\mathbf{t} \ \mathbf{r}) : \left((T_1 + T_1) + (T_2 + T_2) \right) + (T_3 + T_3)}$$

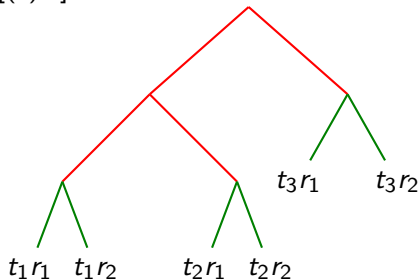
$$t_1 = \pi_{11}([\mathbf{t}]); t_2 = \pi_{12}([\mathbf{t}]); t_3 = \pi_2([\mathbf{t}]);$$

$$r_1 = \pi_1([\mathbf{r}]); r_2 = \pi_2([\mathbf{r}]);$$

$(\mathbf{t} \ \mathbf{r})$:



$[(\mathbf{t} \ \mathbf{r})]_{\mathcal{D}} =$



Correctness

Correctness of typing:

Theorem

$$\Gamma \vdash t : T \quad \Longrightarrow \quad |\Gamma| \vdash_F [t]_{\mathcal{D}} : |T|$$

Correctness of reduction:

Theorem

$$\Gamma \vdash t : T \text{ and } t \rightarrow t' \quad \text{implies} \quad [t]_{\mathcal{D}} \rightarrow^* [t']_{\mathcal{D}'} \quad \text{for some } \mathcal{D}'$$

Conclusions and Future work

- ▶ Sums in algebraic calculi \rightarrow AC pairs + application ✓
- ▶ Can we find an interpretation for the scalars?

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- ▶ Sums in algebraic calculi \rightarrow AC pairs + application ✓
- ▶ Can we find an interpretation for the scalars?
 - ▶ Natural numbers \rightarrow sums ✓
 - ▶ Positive reals¹: Taking the floor (or ceiling), possible **abstract interpretation** using this result
 - ▶ Any general ring?² **no clue**

¹FullAdditive (work-in-progress) [Buiras, Díaz-Caro, Jaskelioff 2010]

²Vectorial type system (work-in-progress) [Arrighi, Díaz-Caro, Valiron 2010]