Signals for Cellular Automata in dimension 2 or higher

Jean-Christophe Dubacq – LRI, Orsay, France
Véronique Terrier – GREYC, Caen, France

LATIN'02 – Cancún, Mexico — Friday, April 5th

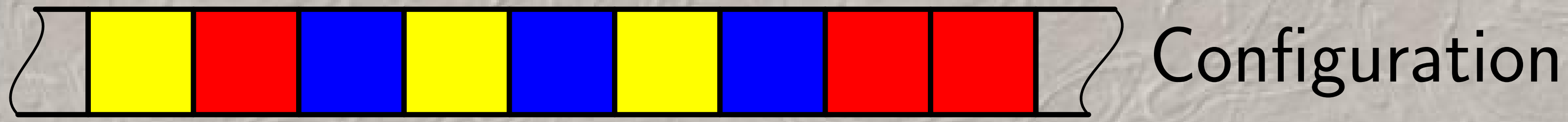
Cellular Automata

A cellular automaton is a local synchronous system, that evolves on a discrete regular space (line, grid).

Cellular Automata

A cellular automaton is a local synchronous system, that evolves on a discrete regular space (line, grid).

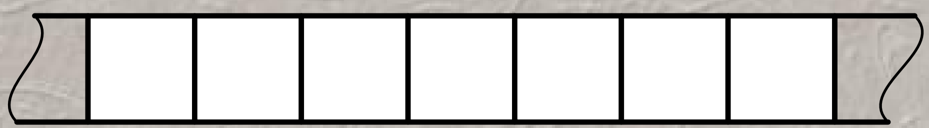
Definition 1 (Automate cellulaire) *A cellular automata of dimension* d *is a triple* $(\mathcal{S}, V, f)$ *where* $\mathcal{S}$ *is a set of states,* V *a subset of* $\mathbf{Z}^d$ *and* f *a mapping of* $\mathcal{S}^{|V|}$ *into* $\mathcal{S}$. *A* configuration *is an element of* $\mathcal{S}^{\mathbf{Z}^d}$.



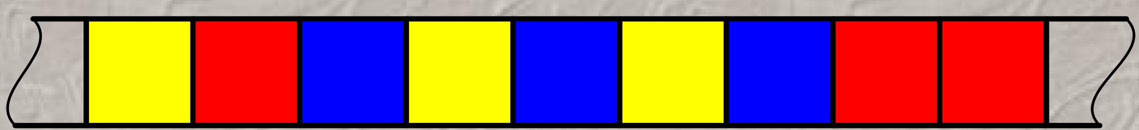
Cellular Automata

A cellular automaton is a local synchronous system, that evolves on a discrete regular space (line, grid).

Definition 1 (Automate cellulaire) *A cellular automata of dimension d is a triple $(\mathcal{S}, V, f)$ where $\mathcal{S}$ is a set of states, V a subset of $\mathbf{Z}^d$ and f a mapping of $\mathcal{S}^{|V|}$ into $\mathcal{S}$. A* configuration *is an element of $\mathcal{S}^{\mathbf{Z}^d}$.*



New configuration

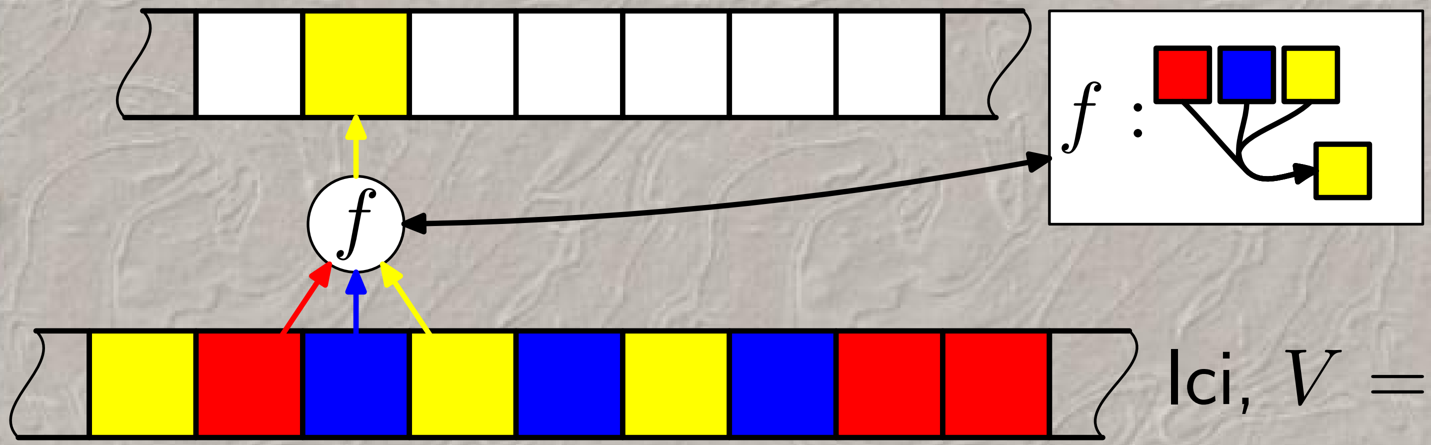


Configuration

Cellular Automata

A cellular automaton is a local synchronous system, that evolves on a discrete regular space (line, grid).

Definition 1 (Automate cellulaire) *A cellular automata of dimension* d *is a triple* $(\mathcal{S}, V, f)$ *where* $\mathcal{S}$ *is a set of states,* V *a subset of* $\mathbf{Z}^d$ *and* f *a mapping of* $\mathcal{S}^{|V|}$ *into* $\mathcal{S}$. *A* configuration *is an element of* $\mathcal{S}^{\mathbf{Z}^d}$.

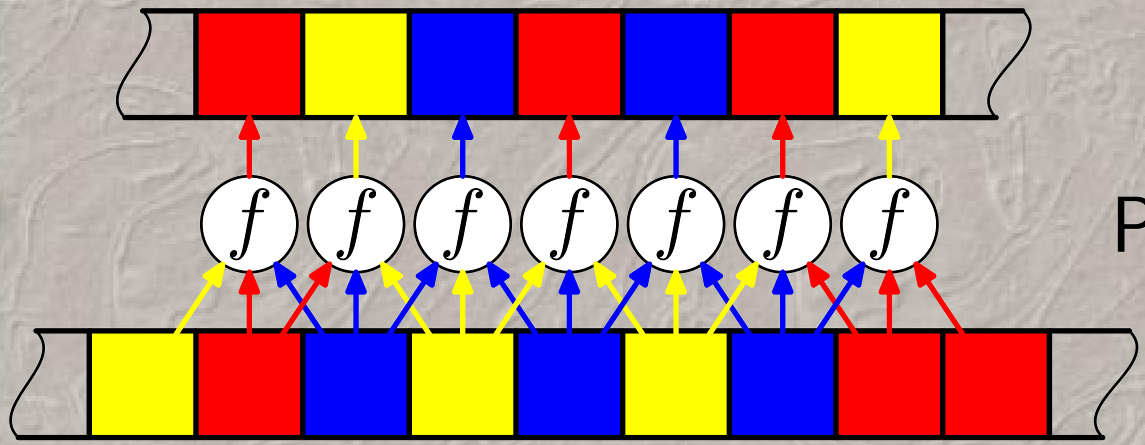


Ici, $V = \{-1, 0, 1\}$

Cellular Automata

A cellular automaton is a local synchronous system, that evolves on a discrete regular space (line, grid).

Definition 1 (Automate cellulaire) *A cellular automata of dimension* d *is a triple* $(\mathcal{S}, V, f)$ *where* $\mathcal{S}$ *is a set of states,* V *a subset of* $\mathbf{Z}^d$ *and* f *a mapping of* $\mathcal{S}^{|V|}$ *into* $\mathcal{S}$. *A* configuration *is an element of* $\mathcal{S}^{\mathbf{Z}^d}$.



Parallelism

Space-Time Diagram

Definition 2 (Global function) *The global function of a cellular automata is the mapping* $\tilde{f} : \mathcal{S}^{\mathbf{Z}^d} \to \mathcal{S}^{\mathbf{Z}^d}$ *that maps a configuration* $\mathcal{C}$ *onto the image configuration* $\tilde{f}(\mathcal{C})$*:*

$$\langle \mathbf{u} \rangle = f\left(\left\langle \mathbf{u} + \mathbf{x}^1 \right\rangle, \ldots, \langle \mathbf{u} + \mathbf{x}^v \rangle \right).$$

Space-Time Diagram

Definition 2 (Global function) *The global function of a cellular automata is the mapping* $\tilde{f} : \mathcal{S}^{\mathbf{Z}^d} \to \mathcal{S}^{\mathbf{Z}^d}$ *that maps a configuration* $\mathcal{C}$ *onto the image configuration* $\tilde{f}(\mathcal{C})$*:*

$$\langle \mathbf{u} \rangle = f\left(\left\langle \mathbf{u} + \mathbf{x}^1 \right\rangle, \ldots, \langle \mathbf{u} + \mathbf{x}^v \rangle\right).$$

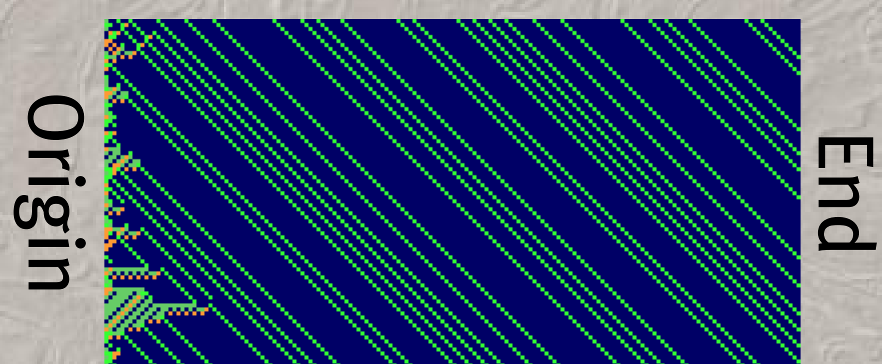
Thus, a sequence of configurations can be defined by iteration. This sequence is called the space-time diagram of the cellular automaton.

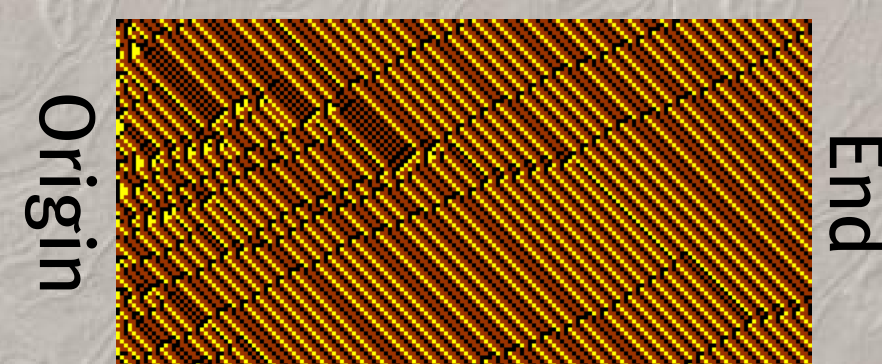
Space-Time Diagram

Definition 2 (Global function) *The global function of a cellular automata is the mapping* $\tilde{f} : \mathcal{S}^{\mathbf{Z}^d} \to \mathcal{S}^{\mathbf{Z}^d}$ *that maps a configuration* $\mathcal{C}$ *onto the image configuration* $\tilde{f}(\mathcal{C})$*:*

$$\langle \mathbf{u} \rangle = f\left(\left\langle \mathbf{u} + \mathbf{x}^1 \right\rangle, \ldots, \left\langle \mathbf{u} + \mathbf{x}^v \right\rangle \right).$$

Thus, a sequence of configurations can be defined by iteration. This sequence is called the space-time diagram of the cellular automaton.



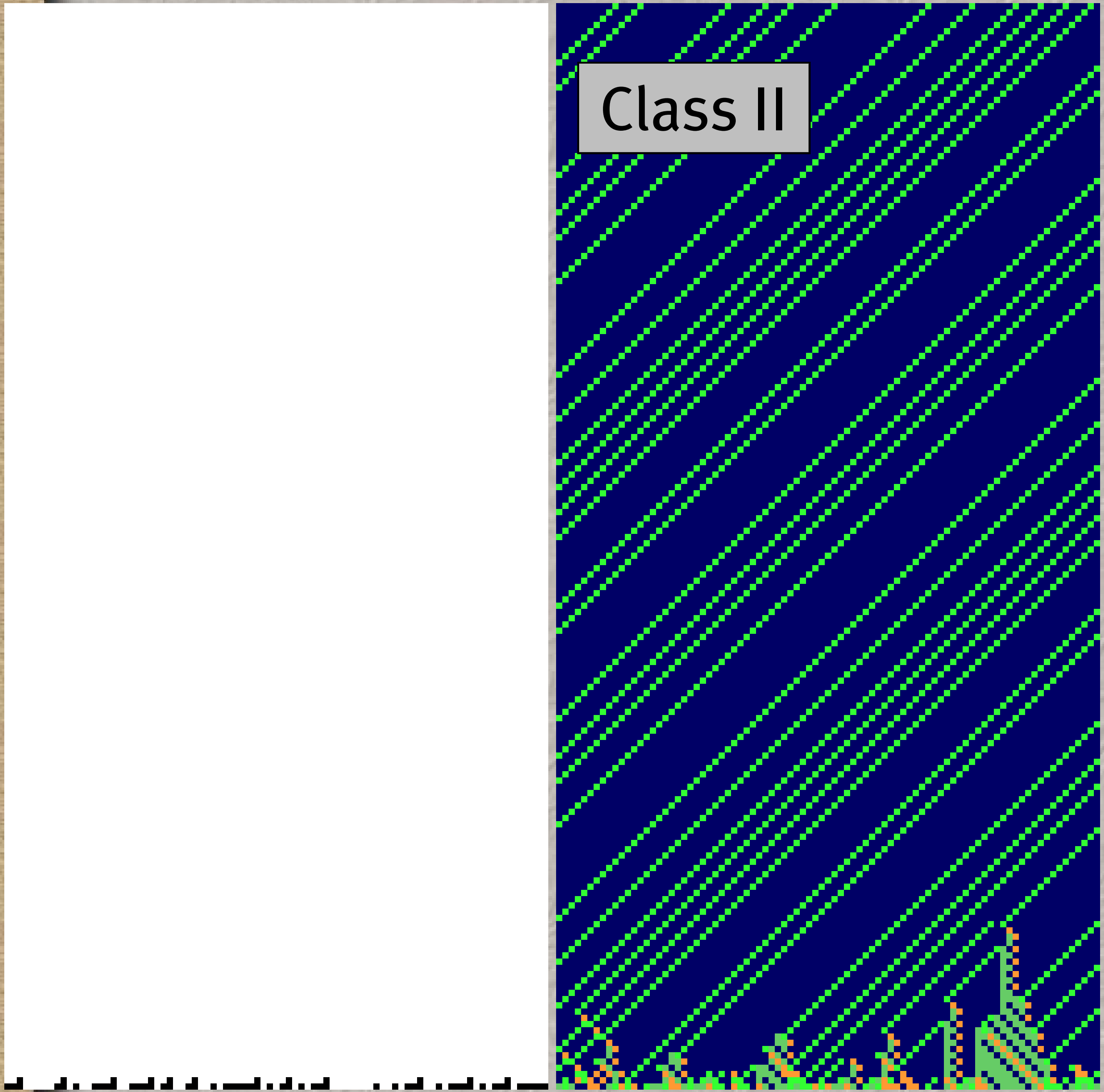


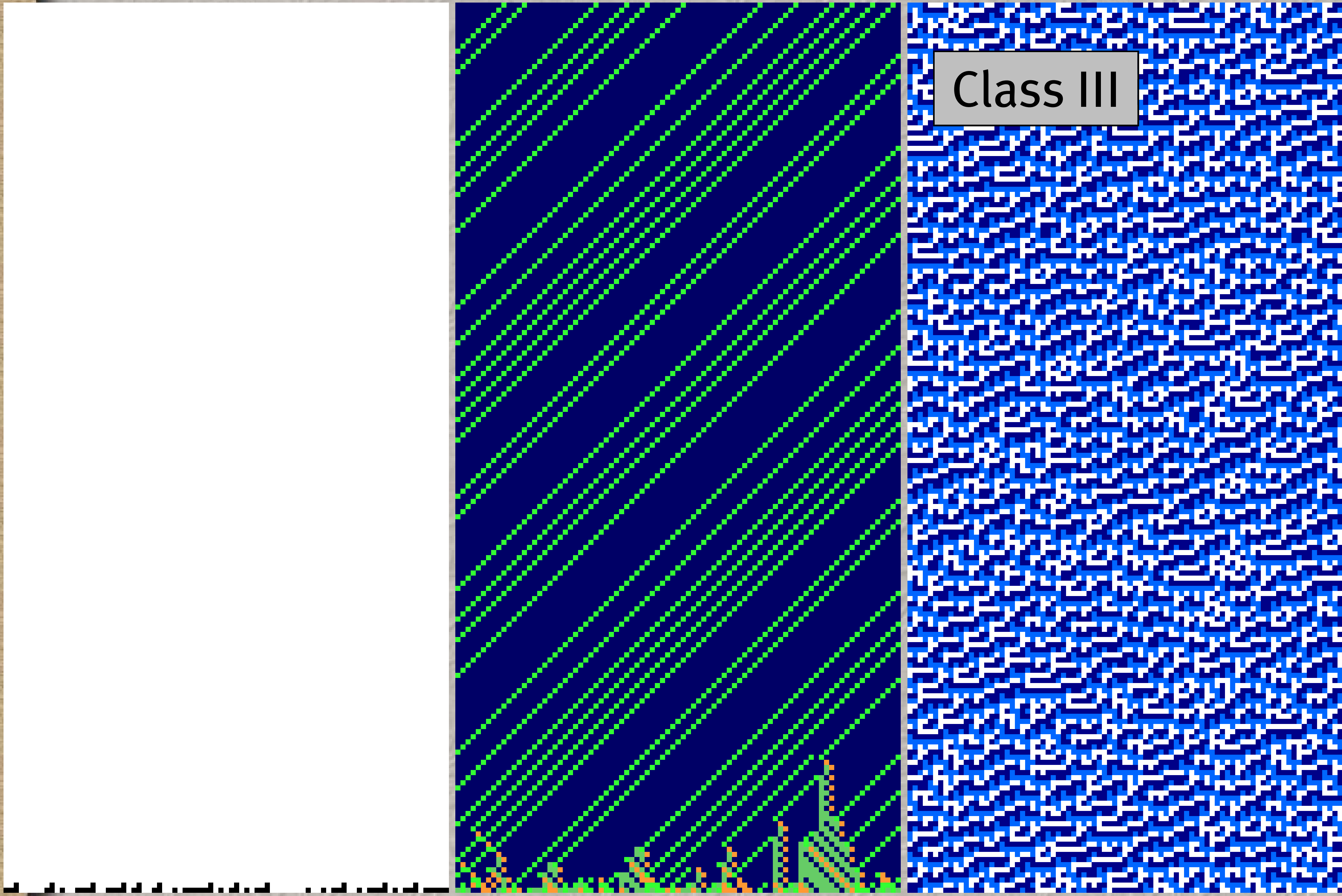
Why signals?

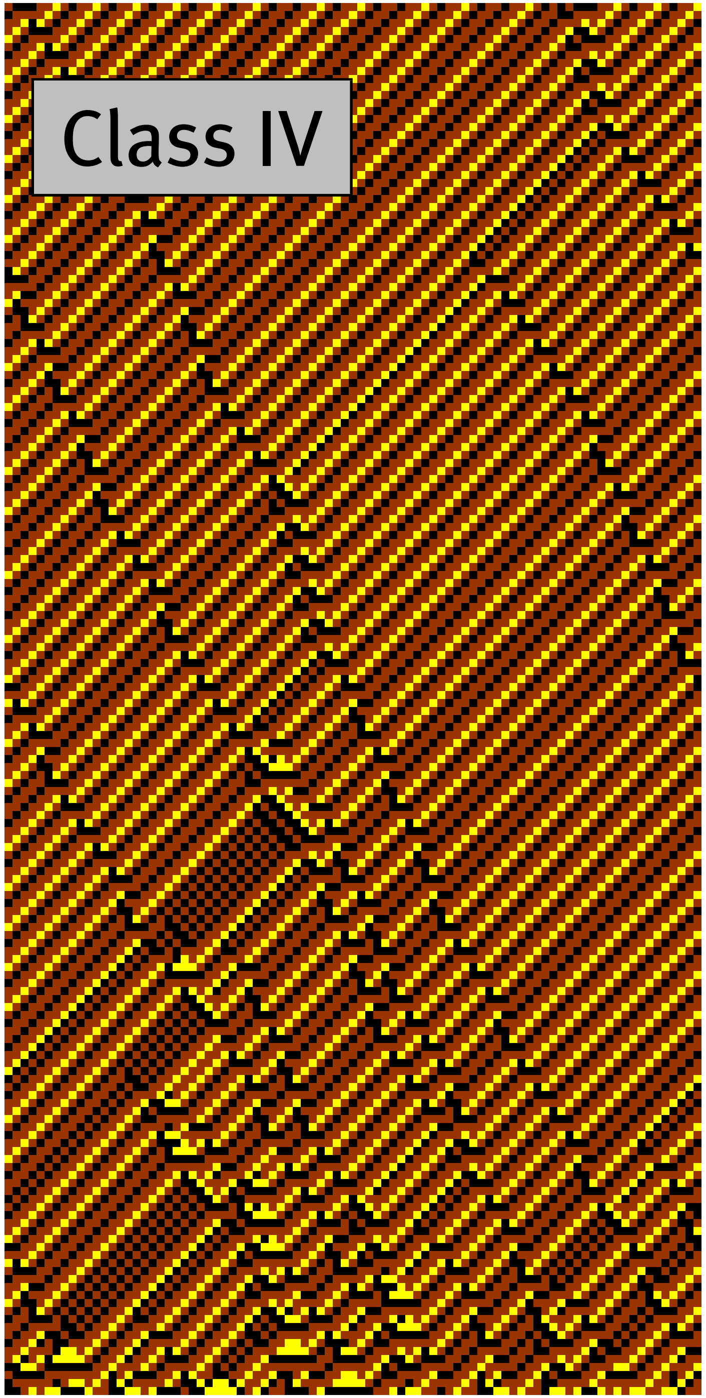
Wolfram created an empirical classification of CA in 4 classes: nilpotents, periodical behavior, random and complex.

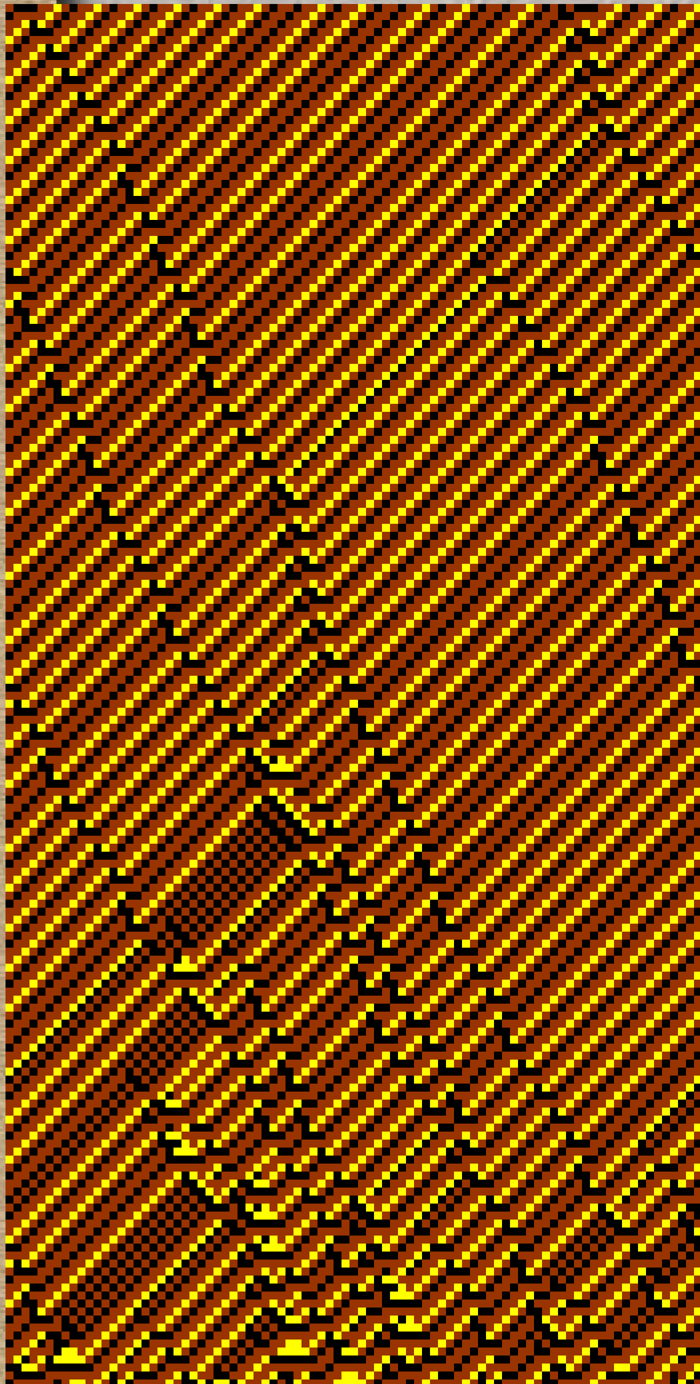
This classification was meant to help studying dynamical behaviour of CA (especially in class IV).

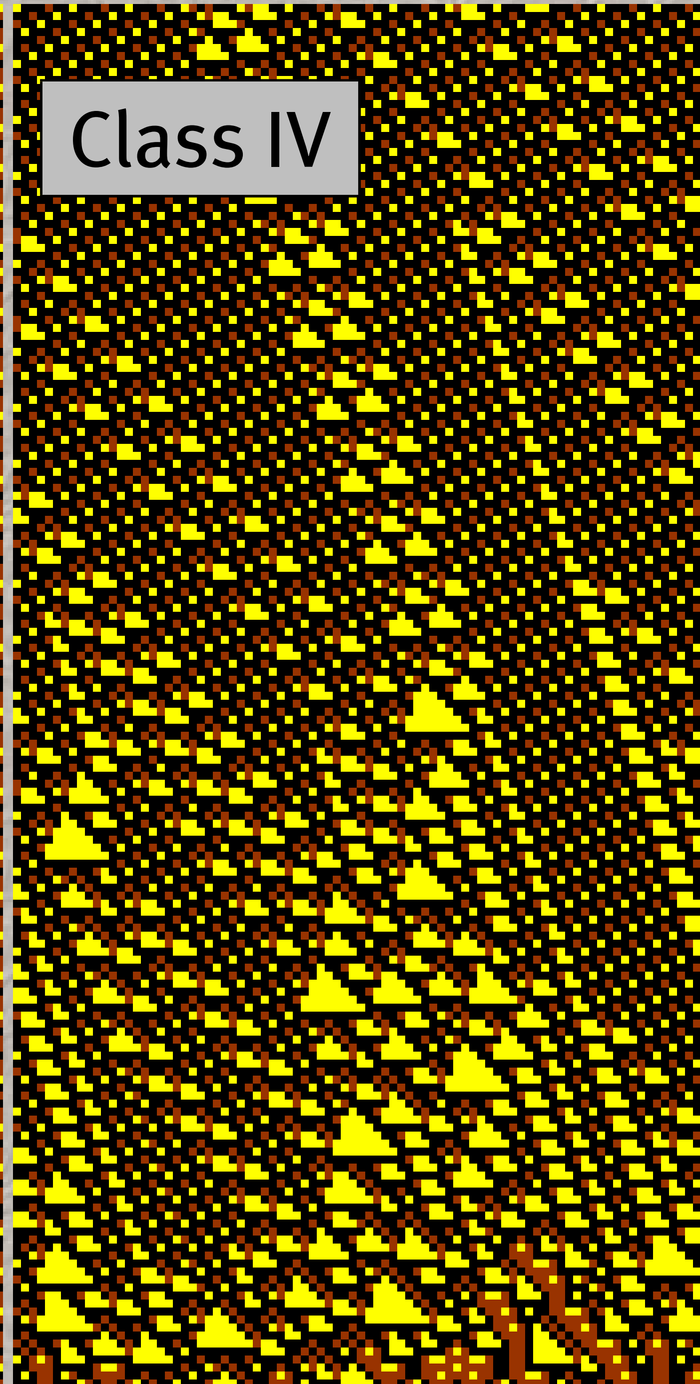
Class I

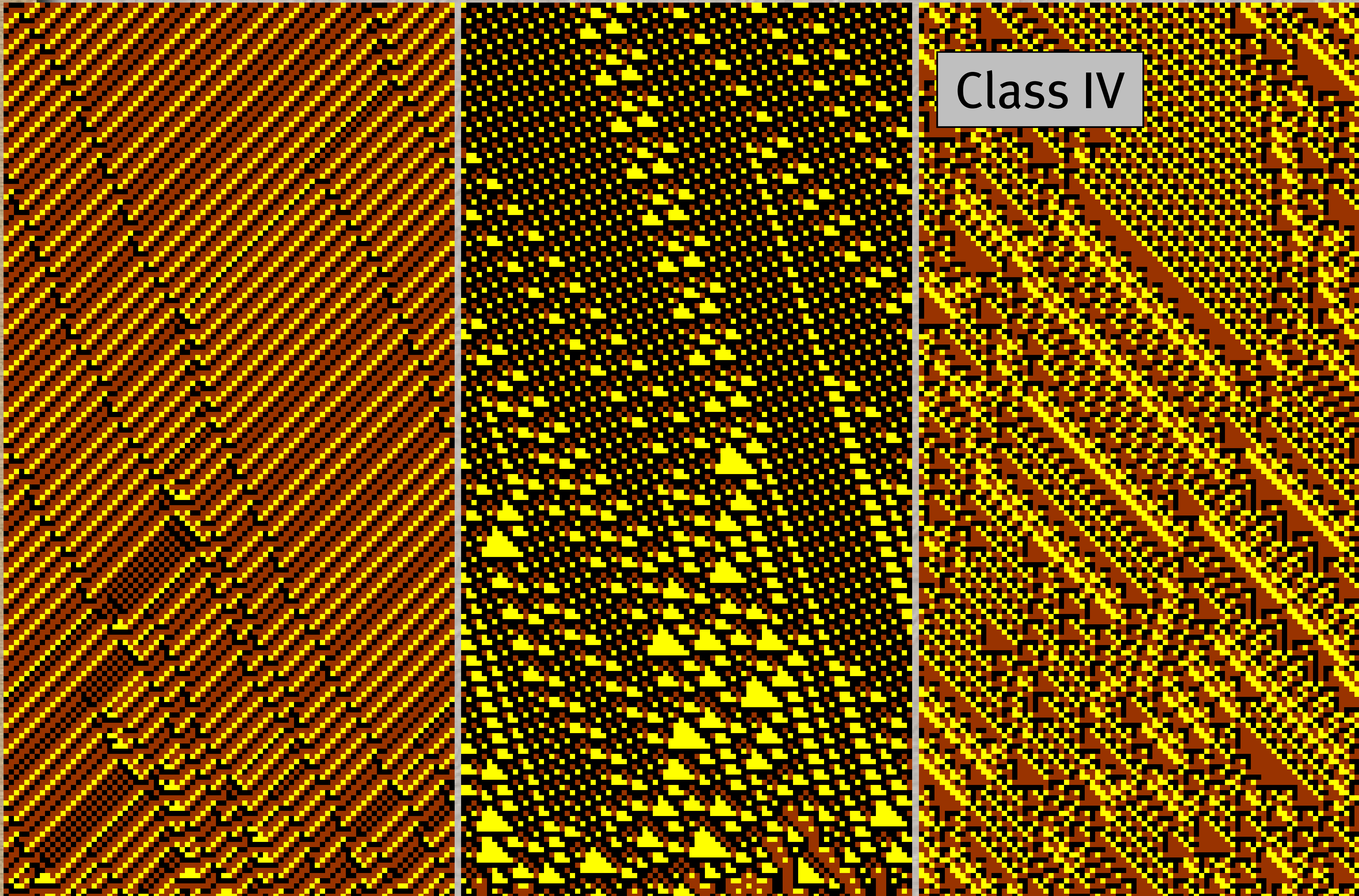

Class II


Class III


Class IV



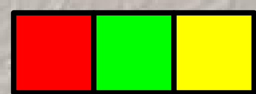
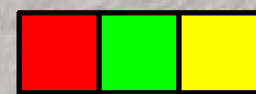
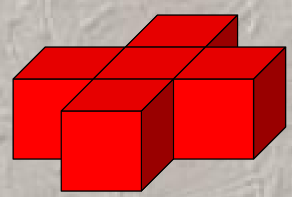
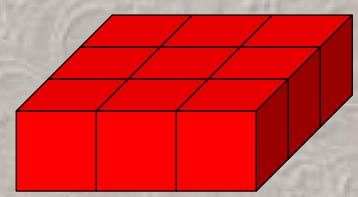
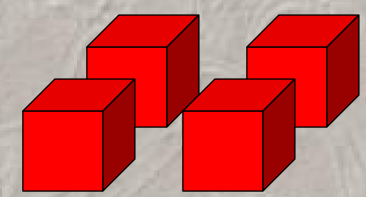

Class IV


Class IV

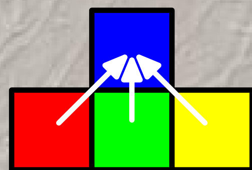
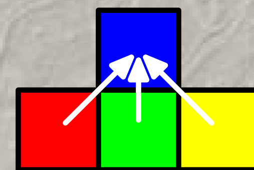
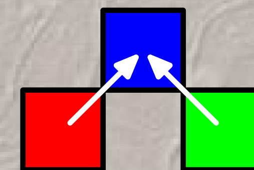
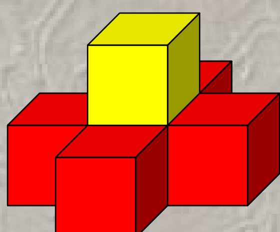
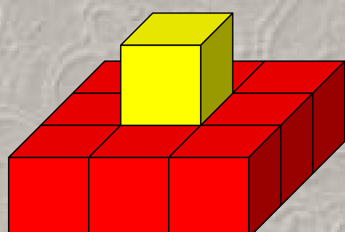
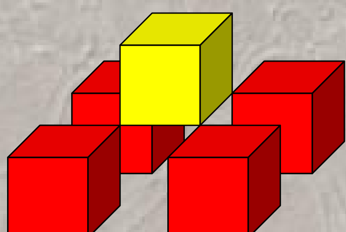
Different neighbourhoods

von Neumann	Moore	Trellis
Dimension 1		
Dimension 2		
Mathematically		
$\sum \lvert x_i \rvert \leq 1$	$\lvert x_i \rvert \leq 1$	$\lvert x_i \rvert = 1$

Different neighbourhoods

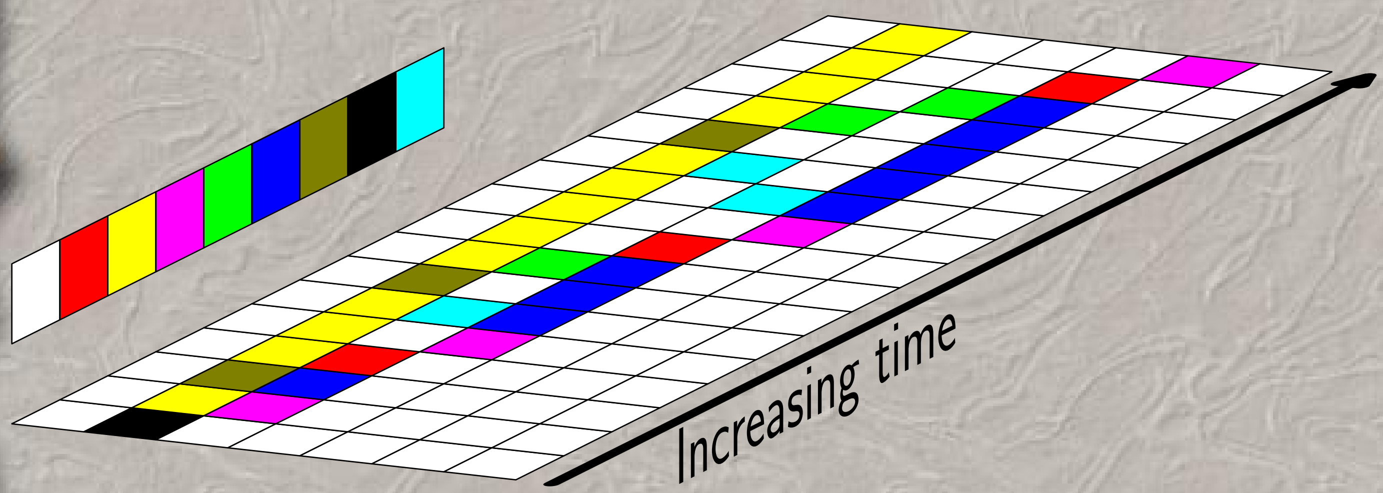
von Neumann	Moore	Trellis
	Dimension 1	
		
	Dimension 2	
		
	Mathematically	
$\sum \lvert x_i \rvert \leq 1$	$\lvert x_i \rvert \leq 1$	$\lvert x_i \rvert = 1$

Different neighbourhoods

von Neumann	Moore	Trellis
Dimension 1		
		
Dimension 2		
		
Mathematically		
$\sum \lvert x_i \rvert \leq 1$	$\lvert x_i \rvert \leq 1$	$\lvert x_i \rvert = 1$

Notion of signal

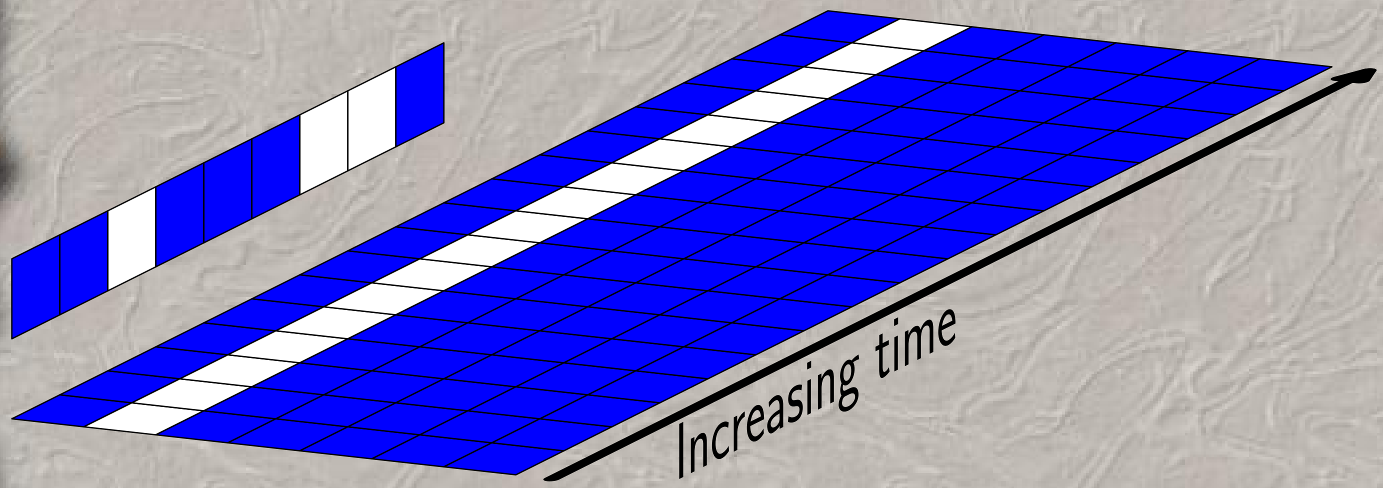
Sample space-time diagram of a CA (9 states):



Notion of signal

Sample space-time diagram of a CA (9 states):

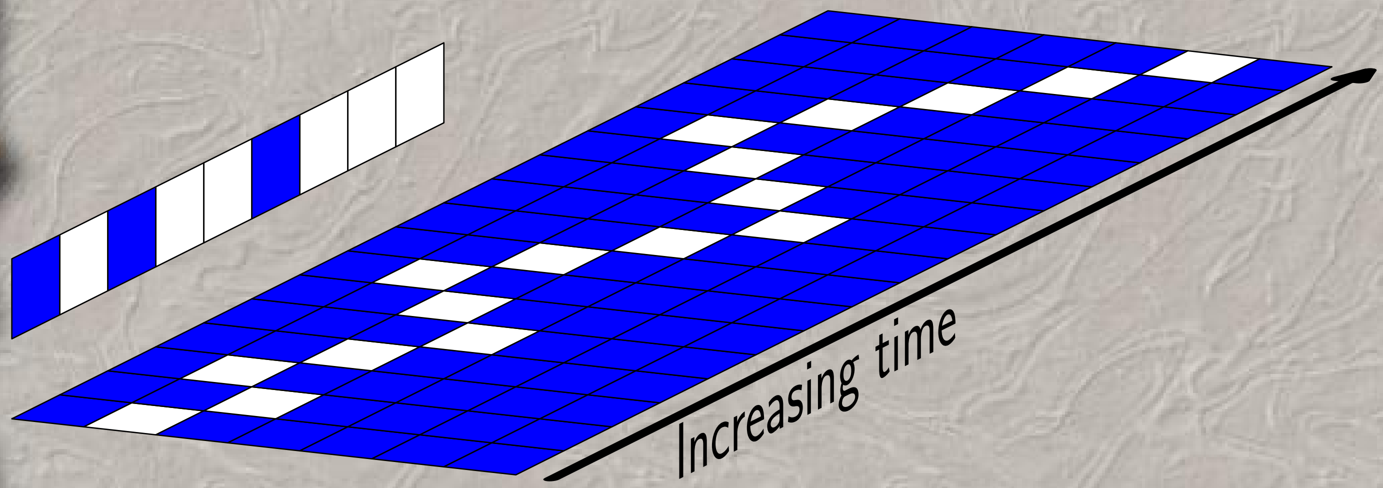
Construction Partition the states.



Notion of signal

Sample space-time diagram of a CA (9 states):

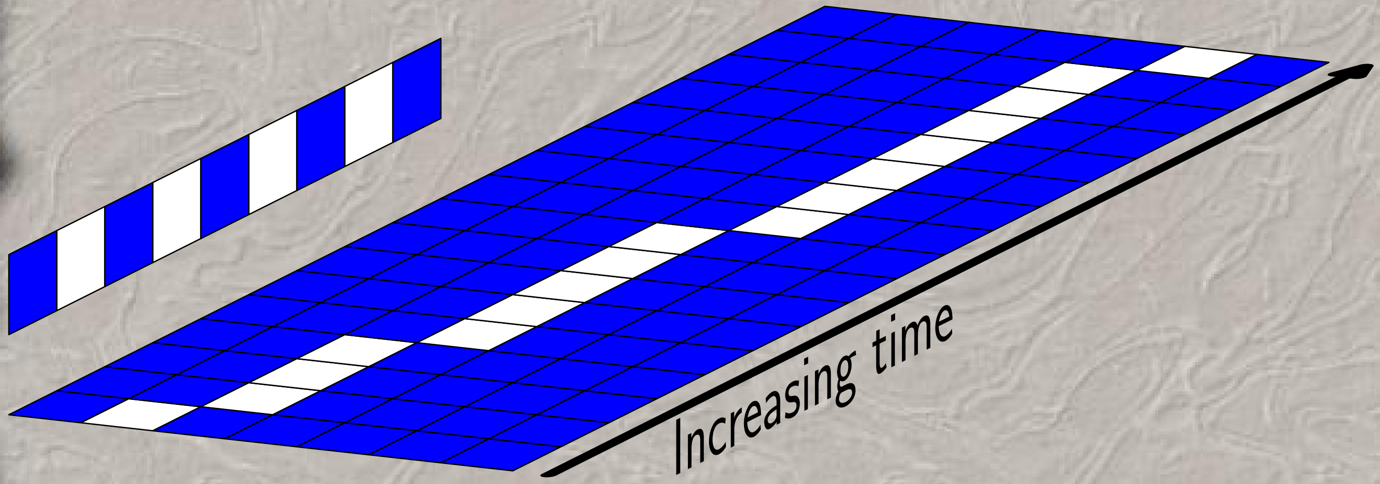
Construction Partition the states.



Notion of signal

Sample space-time diagram of a CA (9 states):

Construction Partition the states.



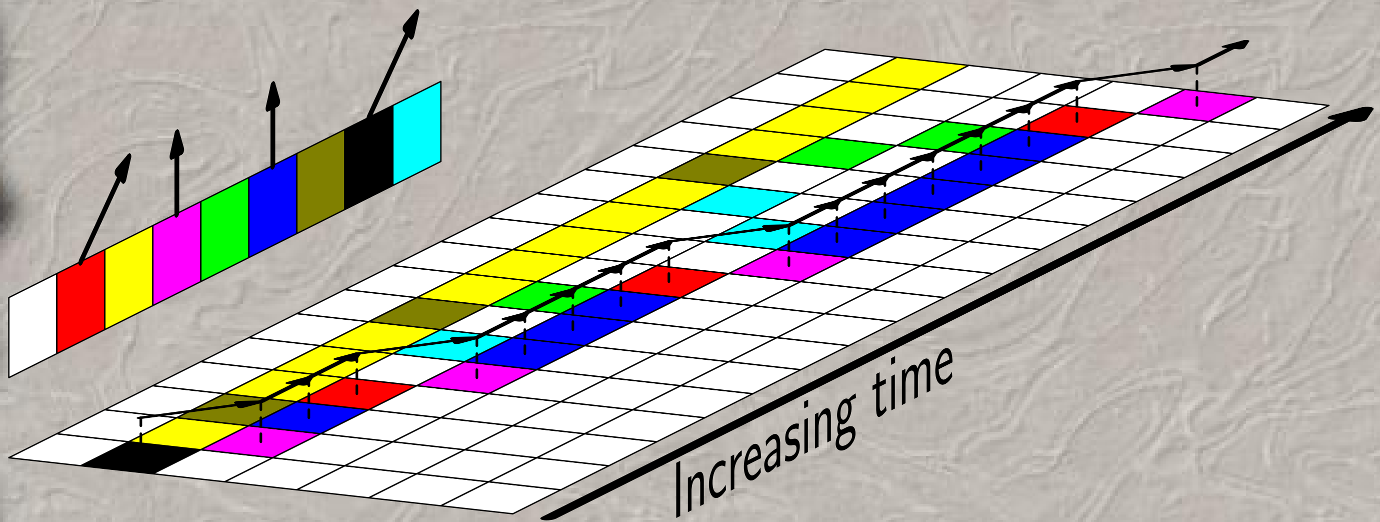
Notion of signal

Sample space-time diagram of a CA (9 states):

Construction Partition the states.

Detection Decide of a direction.

Support Use a finite automaton.



Definition 3 (Impulse CA) *We shall use CA $\mathcal{A}$ with two distinguished states, $\color{red}{\blacksquare}$ and $\square$ such that $f(\square, \ldots, \square) = \square$. We study the space-time diagram of $\mathcal{A}$ applied to* ${}^{\infty}\square\square\square\square\color{red}{\blacksquare}\color{black}\square\square\square\square^{\infty}$.

Definition 3 (Impulse CA) *We shall use CA $\mathcal{A}$ with two distinguished states, $\blacksquare$ and $\square$ such that $f(\square, \ldots, \square) = \square$. We study the space-time diagram of $\mathcal{A}$ applied to* $^{\infty}$$^{\infty}$.

Definition 4 (Signal) *A V-signal Γ is a sequence of sites $\{(\mathbf{u}(t), t)\}_{t \geq 0}$ such that*

- $\mathbf{u}(0) = \mathbf{0}$.
- $\forall t \geq 0$: $\mathbf{u}(t+1) - \mathbf{u}(t) \in V$.

Definition 3 (Impulse CA) *We shall use CA $\mathcal{A}$ with two distinguished states, $\blacksquare$ and $\square$ such that $f(\square, \ldots, \square) = \square$. We study the space-time diagram of $\mathcal{A}$ applied to ${}^{\infty}\square\square\square\blacksquare\square\square\square{}^{\infty}$.*

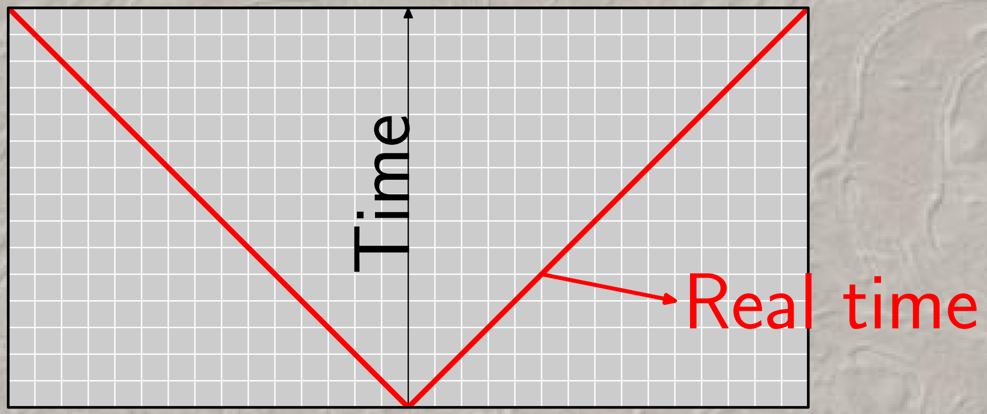
Definition 4 (Signal) *A V-signal Γ is a sequence of sites $\{(\mathbf{u}(t), t)\}_{t \geq 0}$ such that*

- $\mathbf{u}(0) = \mathbf{0}$.
- $\forall t \geq 0$: $\mathbf{u}(t+1) - \mathbf{u}(t) \in V$.

Definition 5 (Base signals) *Base signals are the ultimately periodic signals.*

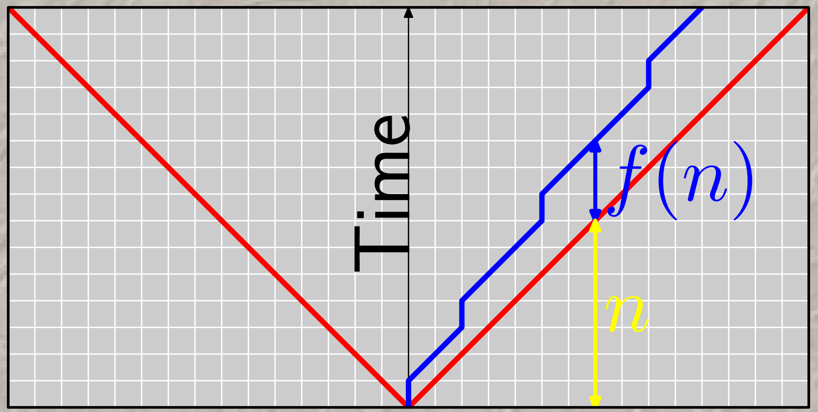
Doing fast signals

The ‘fastest’ signal one can find is the real-time signal:



Doing fast signals

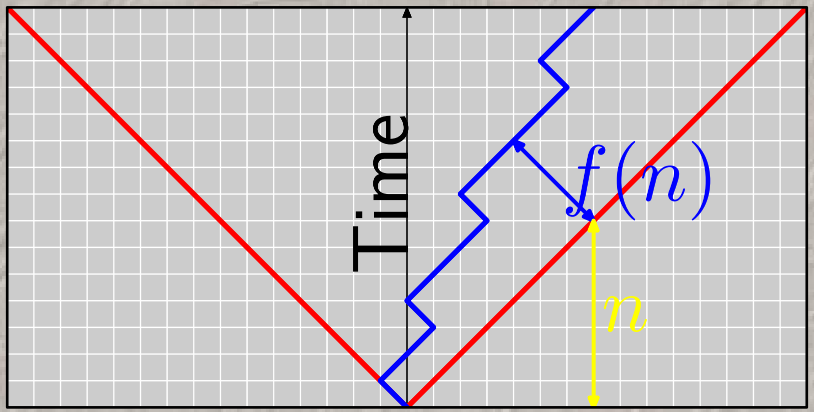
The ‘fastest’ signal one can find is the real-time signal:



A signal ‘defines’ a function f, either by the signal $(n, n + f(n))$,

Doing fast signals

The 'fastest' signal one can find is the real-time signal:



A signal 'defines' a function f, either by the signal $(n, n + f(n))$, or by the signal $(n - f(n), n + f(n))$
$\Longrightarrow$ *ratio* of the signal.

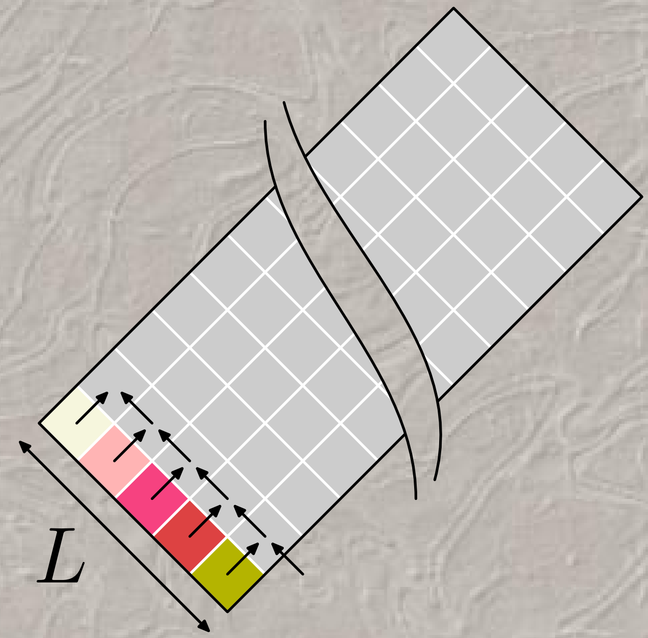
Non-basic fast signals construction

We show that not all signals can be generated.

Theorem 1 *Let $\mathcal{A}$ be a q-states CA. It is not possible to suport a signal which is not ultimately periodic with a ratio smaller than:*

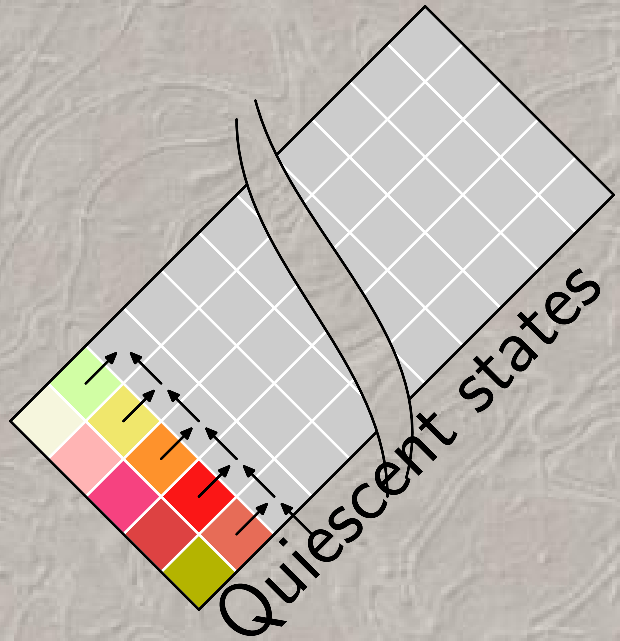
- $\log_q(n)$ *in dimension 1,*
- $\log_{\mathrm{lcm}\,1\ldots q}(n)$ *in higher dimension.*

Dimension 1: periodic strips



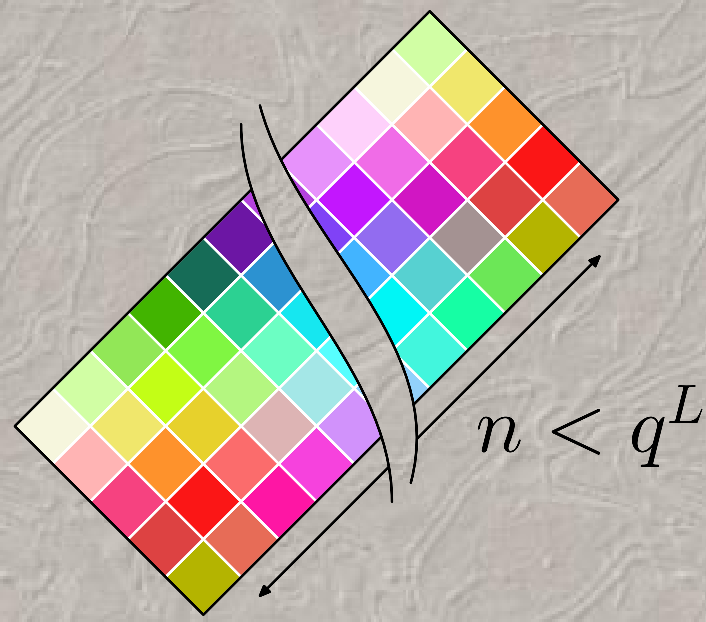
We consider a strip of cells of width L. The strip can be cut into layers.

Dimension 1: periodic strips



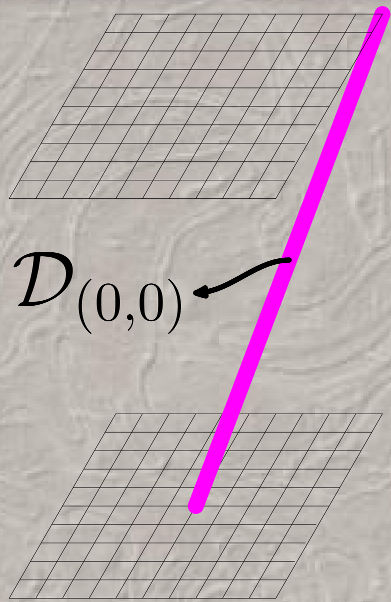
We consider a strip of cells of width L. The strip can be cut into layers. Each layer is completely determined by the former layer.

Dimension 1: periodic strips



We consider a strip of cells of width L. The strip can be cut into layers. Each layer is completely determined by the former layer. Hence the periodicity.

Higher dimension: generalisation



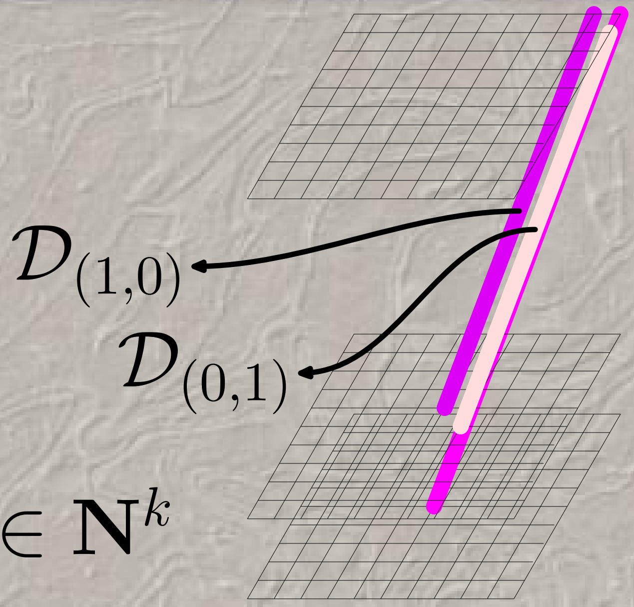
$\mathcal{D}_{\mathbf{i}}$ is defined as follows:

$$\mathcal{D}^t_{\mathbf{i}} = \langle t \cdot \mathbf{1} - \mathbf{i}, t \rangle$$

$$\mathcal{D}_{\mathbf{i}} = \begin{cases} (\mathcal{D}^t_{\mathbf{i}})_{t \geq \lceil \max(i_1,\ldots,i_k)/2 \rceil} & \text{if } \mathbf{i} \in \mathbf{N}^k \\ \lambda^\infty & \text{else.} \end{cases}$$

- $\mathcal{D}_{\mathbf{0}}$ is the real-time signal.

Higher dimension: generalisation



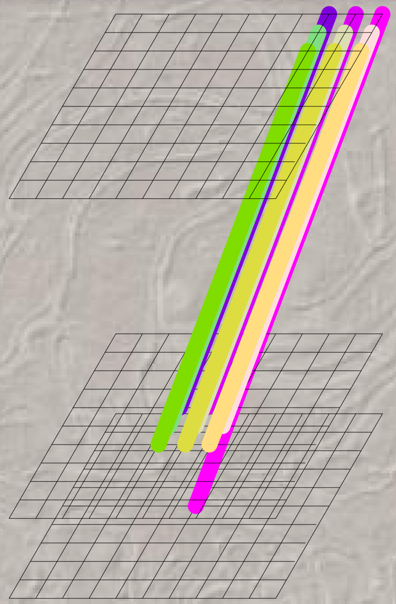
$\mathcal{D}_{\mathbf{i}}$ is defined as follows:

$$\mathcal{D}^t_{\mathbf{i}} = \langle t \cdot \mathbf{1} - \mathbf{i}, t \rangle$$

$$\mathcal{D}_{\mathbf{i}} = \begin{cases} (\mathcal{D}^t_{\mathbf{i}})_{t \geq \lceil \max(i_1, \ldots, i_k)/2 \rceil} & \text{if } \mathbf{i} \in \mathbf{N}^k \\ \lambda^\infty & \text{else.} \end{cases}$$

- $\mathcal{D}_{\mathbf{0}}$ is the real-time signal.

Higher dimension: generalisation



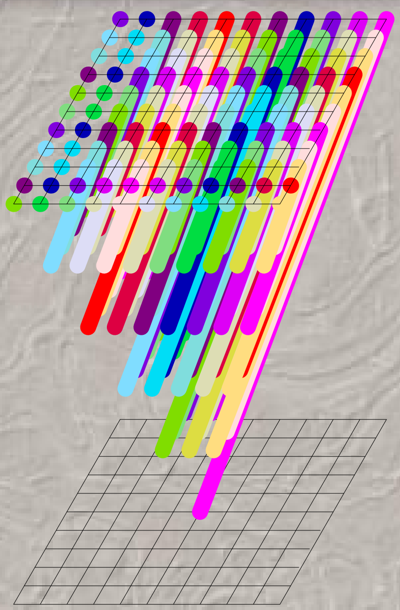
$\mathcal{D}_{\mathbf{i}}$ is defined as follows:

$$\mathcal{D}_{\mathbf{i}}^t = \langle t \cdot \mathbf{1} - \mathbf{i}, t \rangle$$

$$\mathcal{D}_{\mathbf{i}} = \begin{cases} (\mathcal{D}_{\mathbf{i}}^t)_{t \geq \lceil \max(i_1, \ldots, i_k)/2 \rceil} & \text{if } \mathbf{i} \in \mathbf{N}^k \\ \lambda^\infty & \text{else.} \end{cases}$$

- $\mathcal{D}_{\mathbf{0}}$ is the real-time signal.
- Infinite words, defined starting with a potentially non-quiescent state.

Higher dimension: generalisation



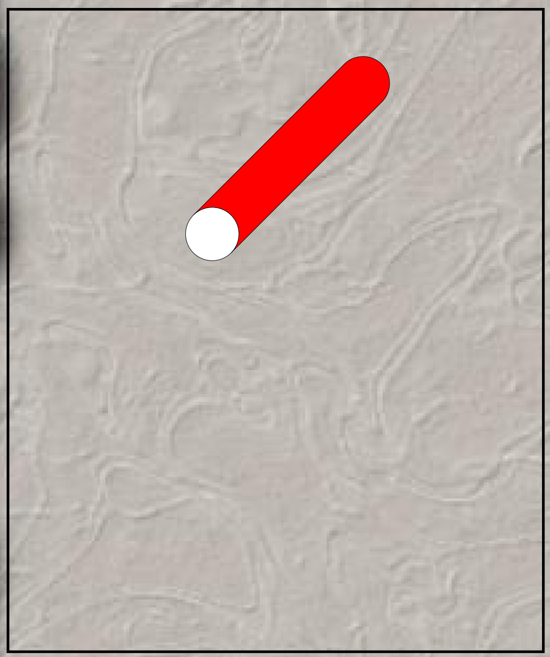
$\mathcal{D}_{\mathbf{i}}$ is defined as follows:

$$\mathcal{D}_{\mathbf{i}}^t = \langle t \cdot \mathbf{1} - \mathbf{i}, t \rangle$$

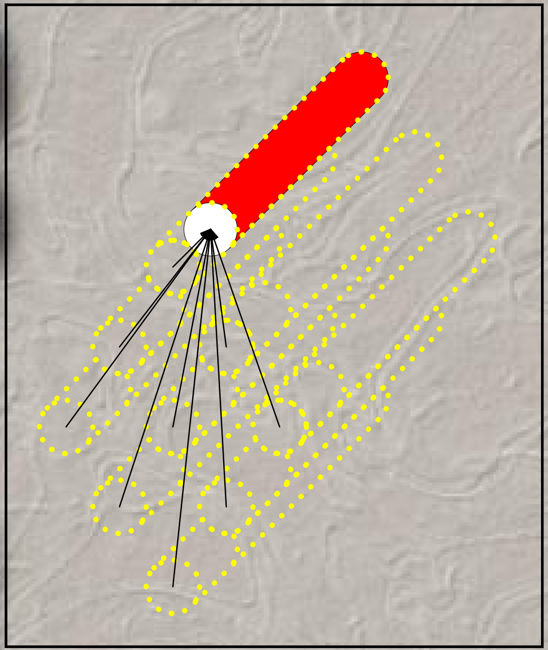
$$\mathcal{D}_{\mathbf{i}} = \begin{cases} (\mathcal{D}_{\mathbf{i}}^t)_{t \geq \lceil \max(i_1,\ldots,i_k)/2 \rceil} & \text{if } \mathbf{i} \in \mathbf{N}^k \\ \lambda^\infty & \text{else.} \end{cases}$$

- $\mathcal{D}_{\mathbf{0}}$ is the real-time signal.
- Infinite words, defined starting with a potentially non-quiescent state.

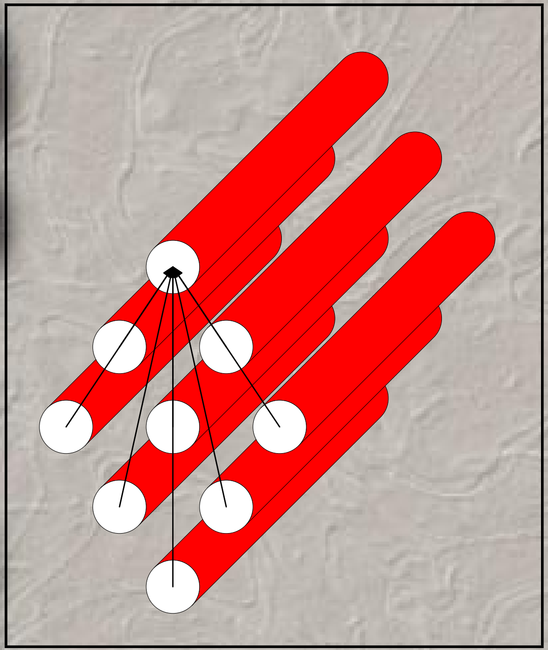
As $\mathcal{D}_{\mathbf{i}}^{t+1}$ is a function of $\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}^{t}$ for $\mathbf{x} \in V_{\text{Moore}}$,



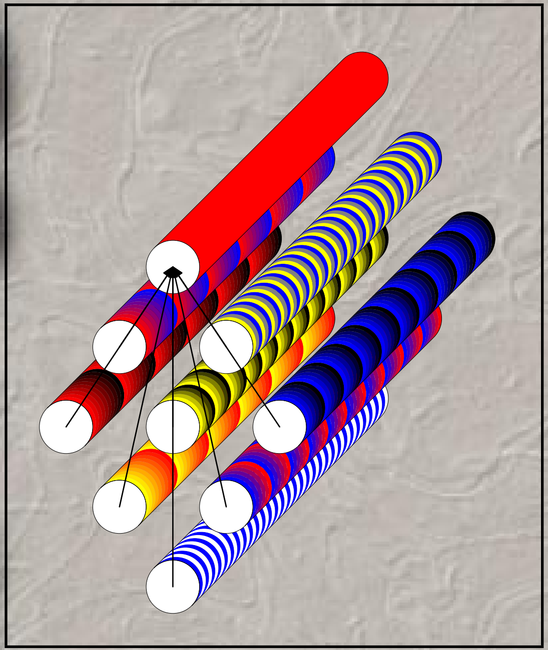
As $\mathcal{D}^{t+1}_{\mathbf{i}}$ is a function of $\mathcal{D}^{t}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}$ for $\mathbf{x} \in V_{\text{Moore}}$,



As $\mathcal{D}_{\mathbf{i}}^{t+1}$ is a function of $\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}^{t}$ for $\mathbf{x} \in V_{\text{Moore}}$, there is a relation between the period of $\mathcal{D}_{\mathbf{i}}$ and the period of $\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}$ for $\mathbf{x} \in V_{\text{Moore}}$.

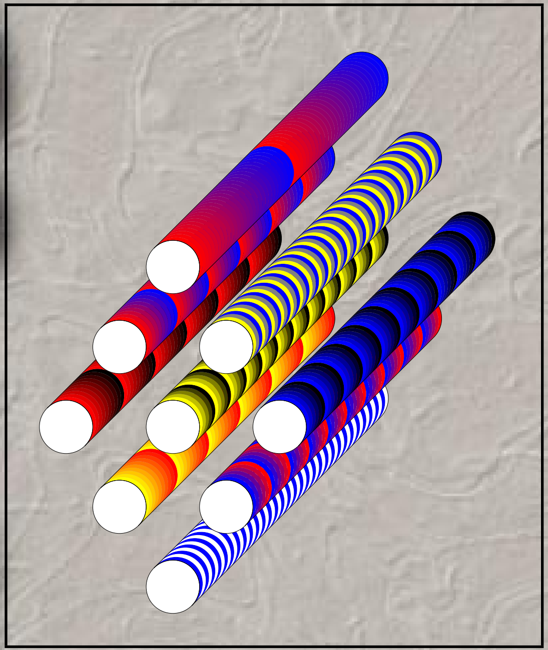


As $\mathcal{D}_{\mathbf{i}}^{t+1}$ is a function of $\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}^{t}$ for $\mathbf{x} \in V_{\text{Moore}}$, there is a relation between the period of $\mathcal{D}_{\mathbf{i}}$ and the period of $\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}$ for $\mathbf{x} \in V_{\text{Moore}}$.



$\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}$ is periodic ($\mathbf{x} \in V_{\text{Moore}}$)
$\Rightarrow \mathcal{D}_{\mathbf{i}}$ is periodic.

As $\mathcal{D}_{\mathbf{i}}^{t+1}$ is a function of $\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}^{t}$ for $\mathbf{x} \in V_{\text{Moore}}$, there is a relation between the period of $\mathcal{D}_{\mathbf{i}}$ and the period of $\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}$ for $\mathbf{x} \in V_{\text{Moore}}$.

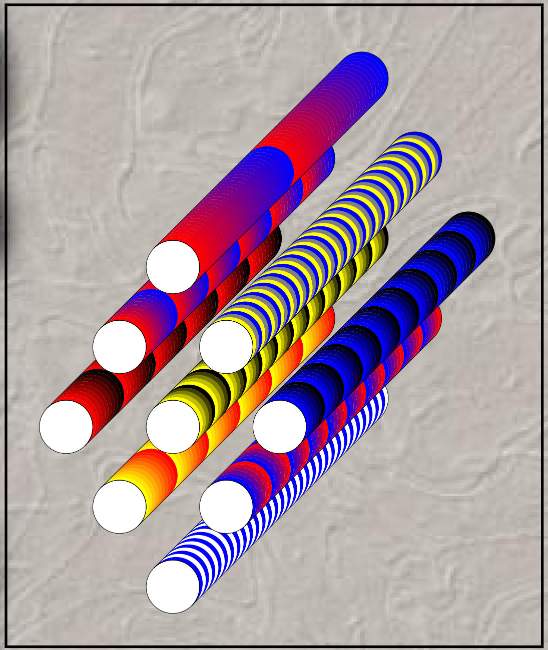


$\mathcal{D}_{\mathbf{i}-\mathbf{x}-\mathbf{1}}$ is periodic ($\mathbf{x} \in V_{\text{Moore}}$)
$\Rightarrow \mathcal{D}_{\mathbf{i}}$ is periodic.

The periods do collapse:
$2, 3, 5, 6, 10 \Rightarrow 30$ and not 1800.

Therefore, the period of the 8-tuple is the lcm of the periods.

As $\mathcal{D}_{\mathbf{i}}^{t+1}$ is a function of $\mathcal{D}_{\mathbf{i-x-1}}^{t}$ for $\mathbf{x} \in V_{\text{Moore}}$, there is a relation between the period of $\mathcal{D}_{\mathbf{i}}$ and the period of $\mathcal{D}_{\mathbf{i-x-1}}$ for $\mathbf{x} \in V_{\text{Moore}}$.



$\mathcal{D}_{\mathbf{i-x-1}}$ is periodic ($\mathbf{x} \in V_{\text{Moore}}$)
$\Rightarrow \mathcal{D}_{\mathbf{i}}$ is periodic.
The periods do collapse:
$2, 3, 5, 6, 10 \Rightarrow 30$ and not 1800.
Therefore, the period of the 8-tuple is the lcm of the periods.

$\Rightarrow$ Thus, each period divides $\mathrm{lcm}(1, \ldots, q)^k$.

States reduction: logarithm

Let $\ell(t) = \lfloor \log_2(t+1) \rfloor$. It is possible to detect the signal $\Gamma = (t-\ell(t), t-\ell(t), t+\ell(t))$ with trellis neighbourhood.

a	b	c	d	$f(a,b,c,d)$	Rule #
λ	λ	λ	λ	λ	#0
1	λ	λ	λ	0	#1
0	λ	λ	λ	1	#2
λ	λ	0	1	1	#3
1	λ	0	1	0	#4
0	λ	0	1	1	#5
1	λ	1	0	1	#6
1	λ	0	0	1	#7
0	λ	1	0	0	#8
0	λ	0	0	0	#9
$\star$	1	λ	$\star$	1	#10
$\star$	1	1	$\star$	1	#11
$\star$	1	0	$\star$	0	#12
$\star$	0	$\star$	$\star$	0	#13
$\star$	$\star$	$\star$	$\star$	λ	#14

a, b, c and d are the cells with relative coordinates:

- a is $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$,
- b is $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$,
- c is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$,
- d is $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

The rules are sorted by order of precedence.

States reduction: logarithm

Let $\ell(t) = \lfloor \log_2(t+1) \rfloor$. It is possible to detect the signal $\Gamma = (t-\ell(t), t-\ell(t), t+\ell(t))$ with trellis neighbourhood.

a	b	c	d	$f(a,b,c,d)$	Rule #
λ	λ	λ	λ	λ	#0
1	λ	λ	λ	0	#1
0	λ	λ	λ	1	#2
λ	λ	0	1	1	#3
1	λ	0	1	0	#4
0	λ	0	1	1	#5
1	λ	1	0	1	#6
1	λ	0	0	1	#7
0	λ	1	0	0	#8
0	λ	0	0	0	#9
$\star$	1	λ	$\star$	1	#10
$\star$	1	1	$\star$	1	#11
$\star$	1	0	$\star$	0	#12
$\star$	0	$\star$	$\star$	0	#13
$\star$	$\star$	$\star$	$\star$	λ	#14

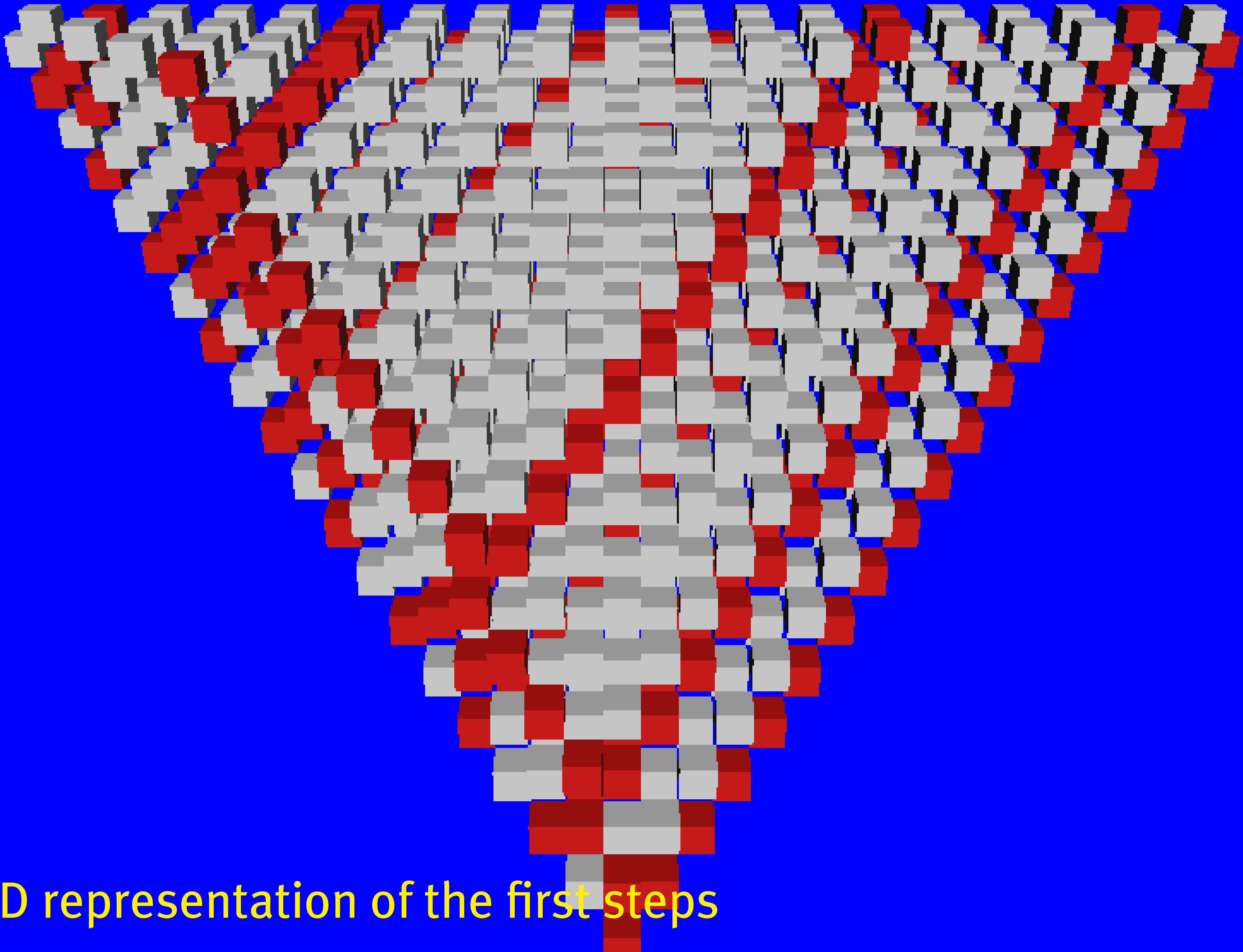
a, b, c and d are the cells with relative coordinates:

- a is $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$,
- b is $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$,
- c is $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$,
- d is $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

The rules are sorted by order of precedence.

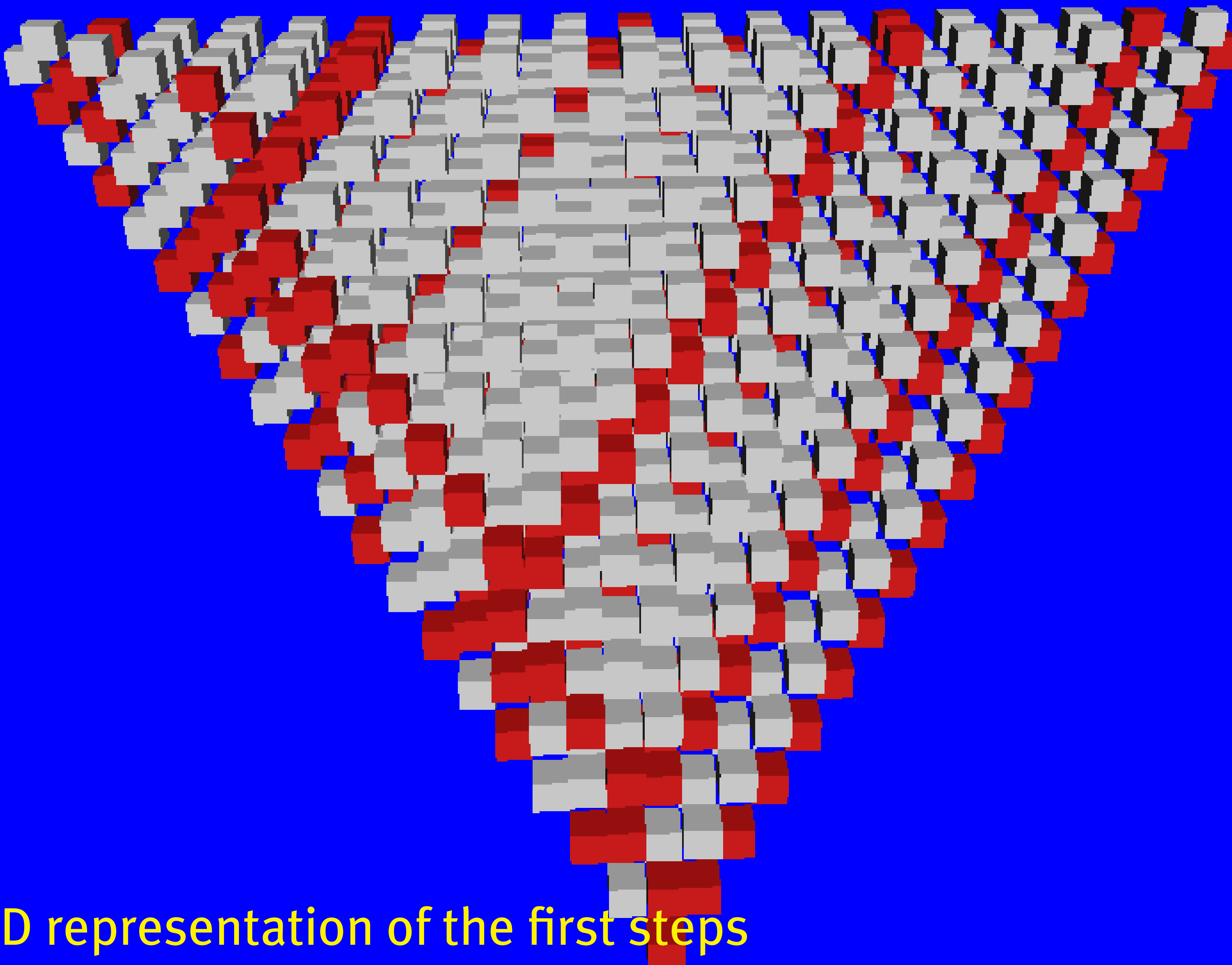
$$\left\{ \begin{array}{l} \text{2 states} + \lambda \\ \text{(dimension 2)} \\ \hline \text{3 states} + \lambda \\ \text{(dimension 1)} \end{array} \right.$$

Time=20



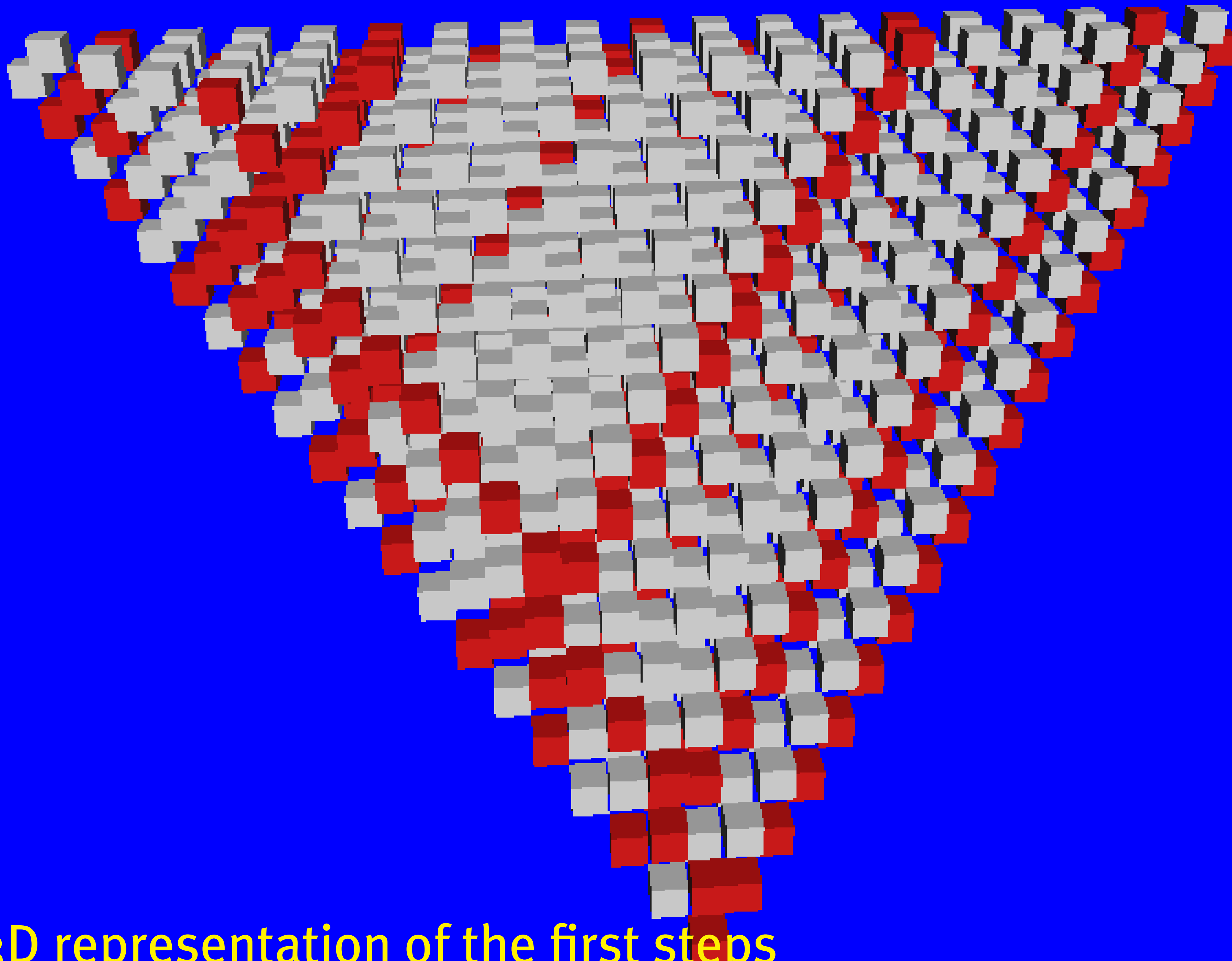
3D representation of the first steps

Time=20



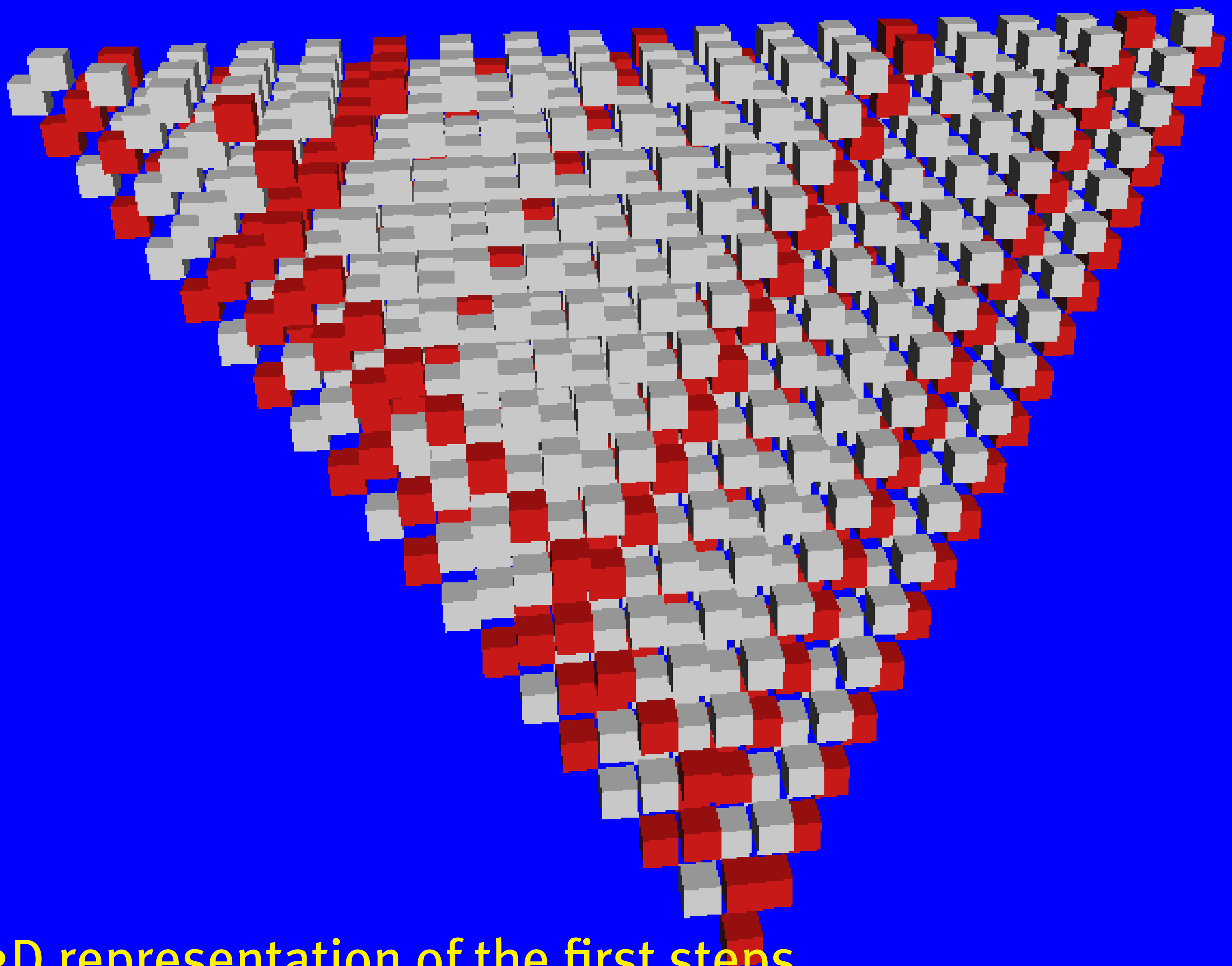
3D representation of the first steps

Time=20



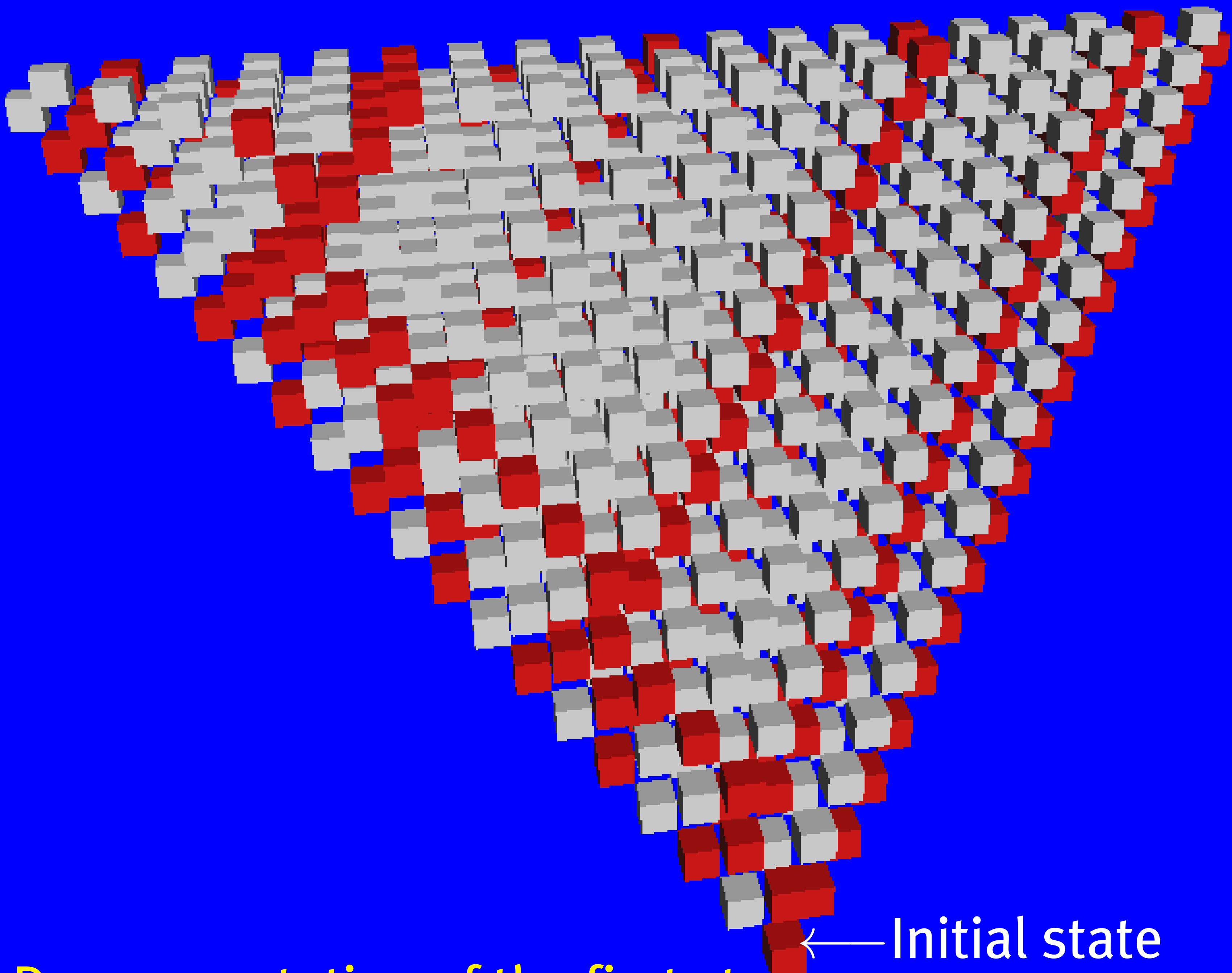
3D representation of the first steps

Time=20



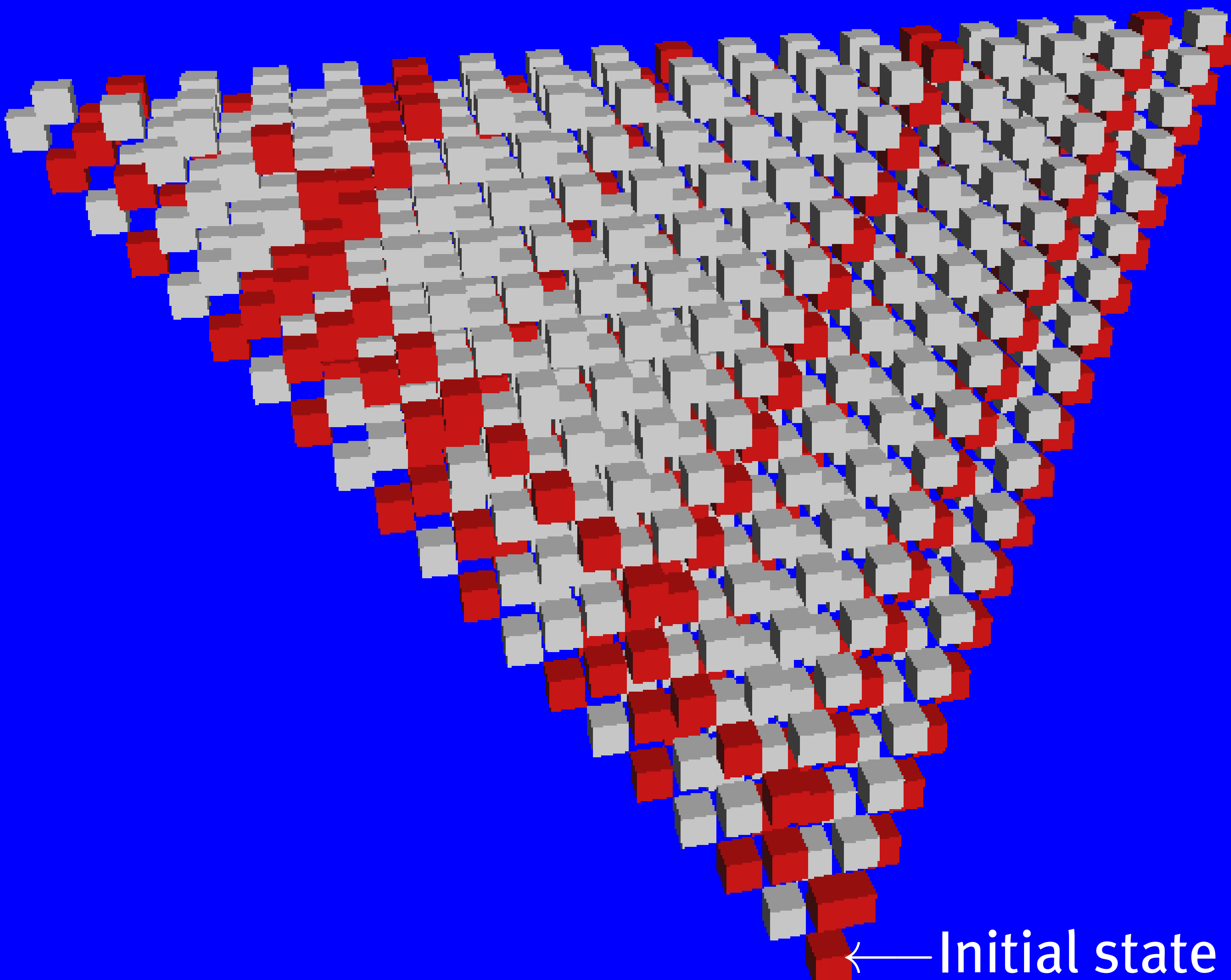
3D representation of the first steps

Time=20



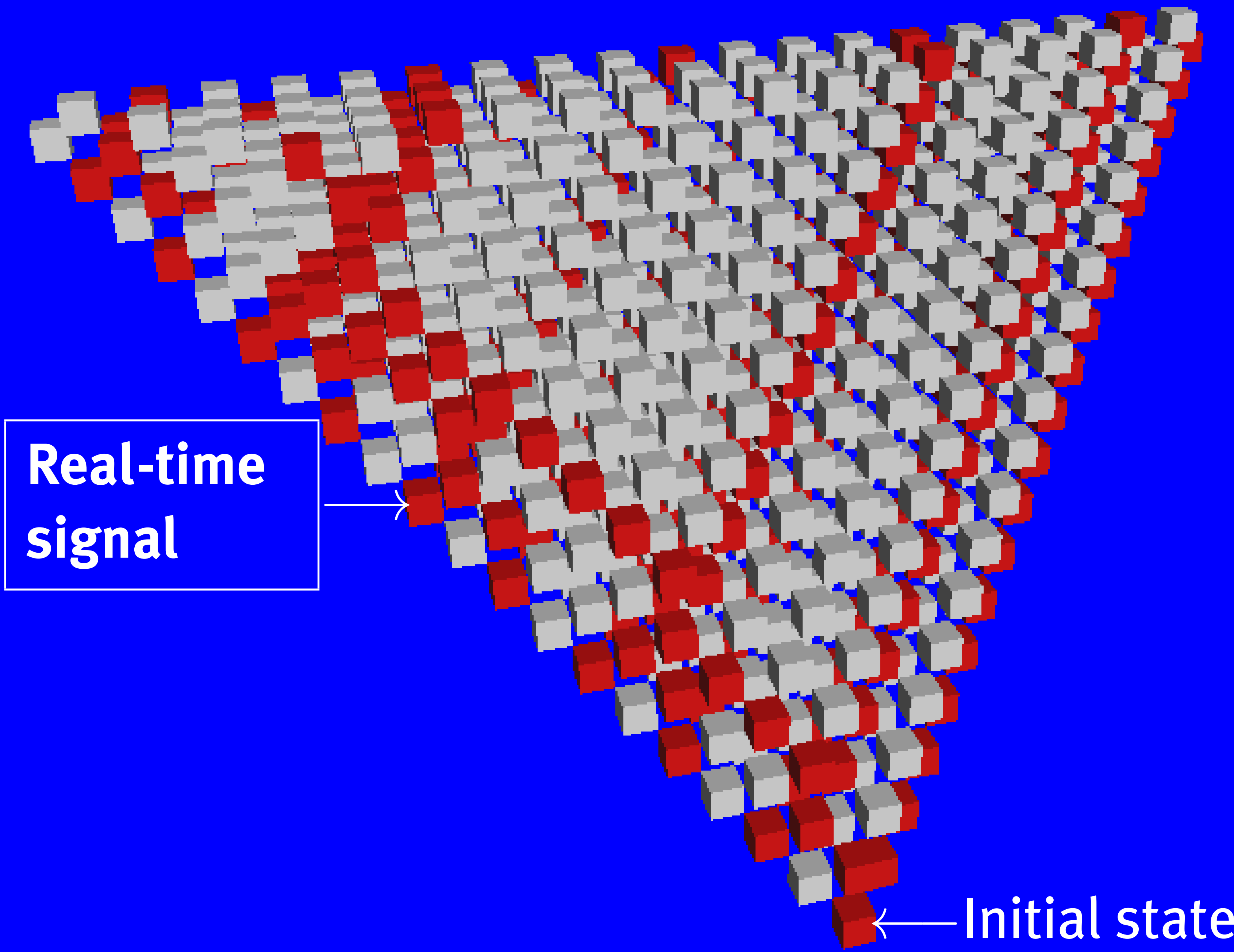
3D representation of the first steps

Time=20



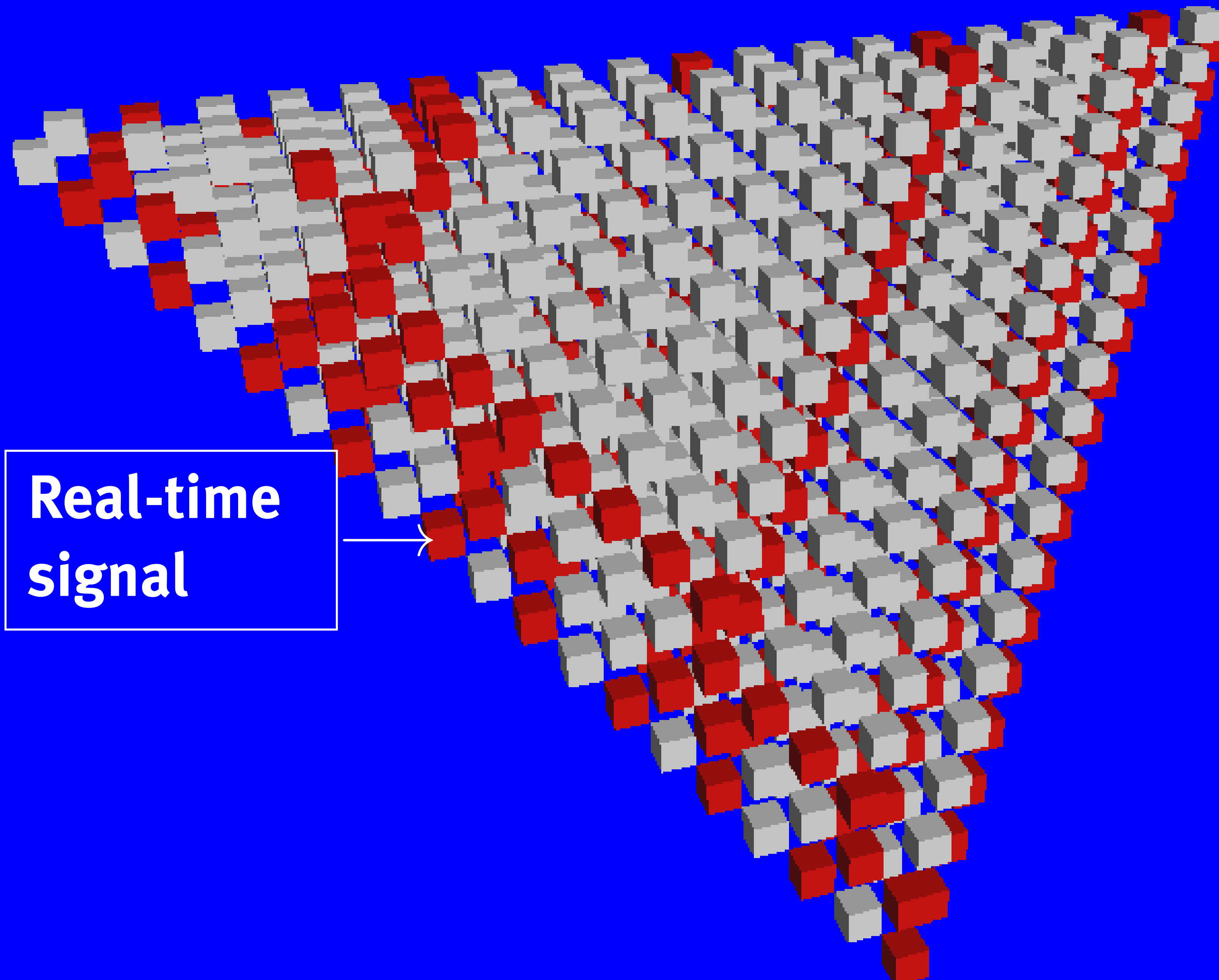
3D representation of the first steps

Time=20

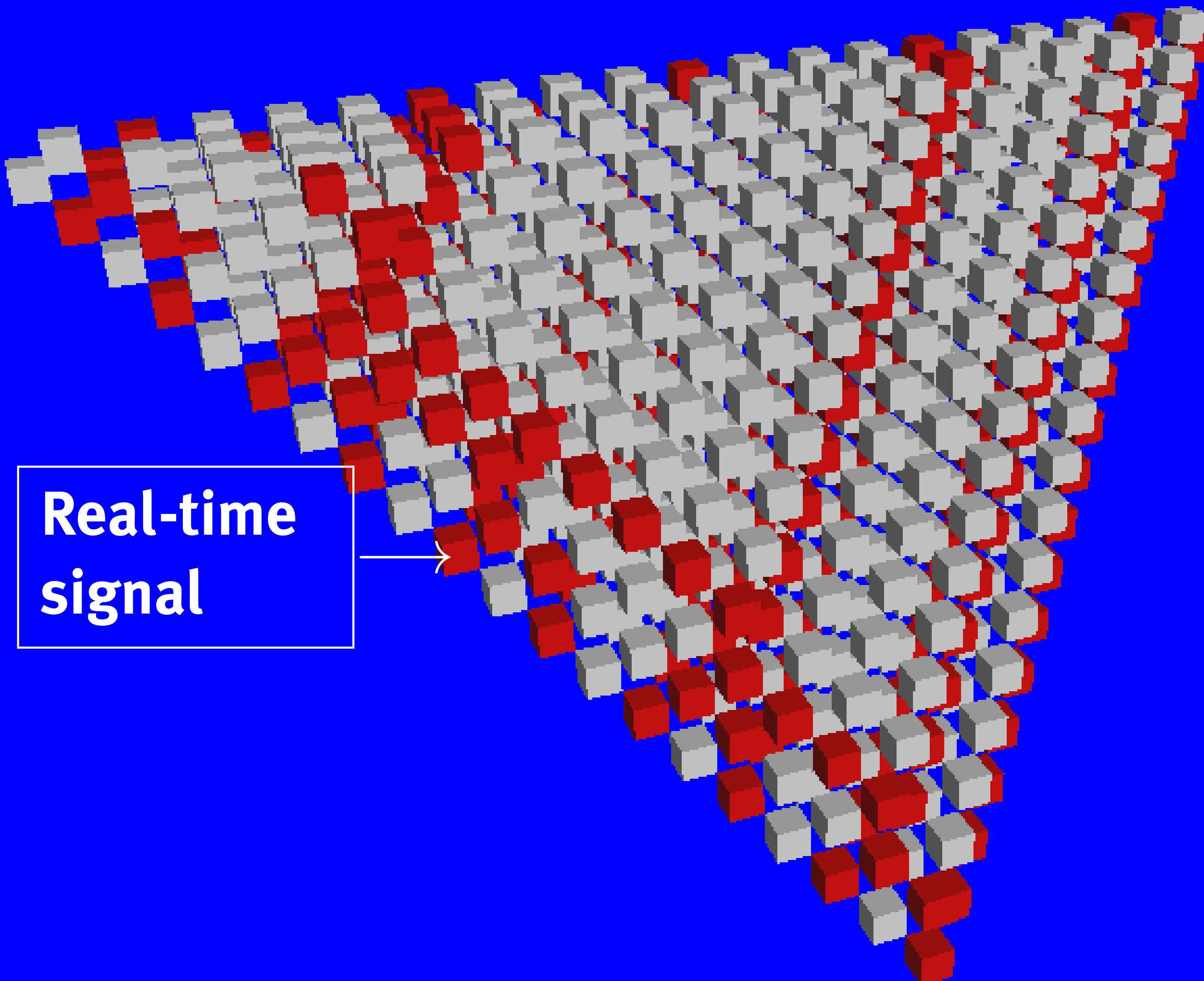


3D representation of the first steps

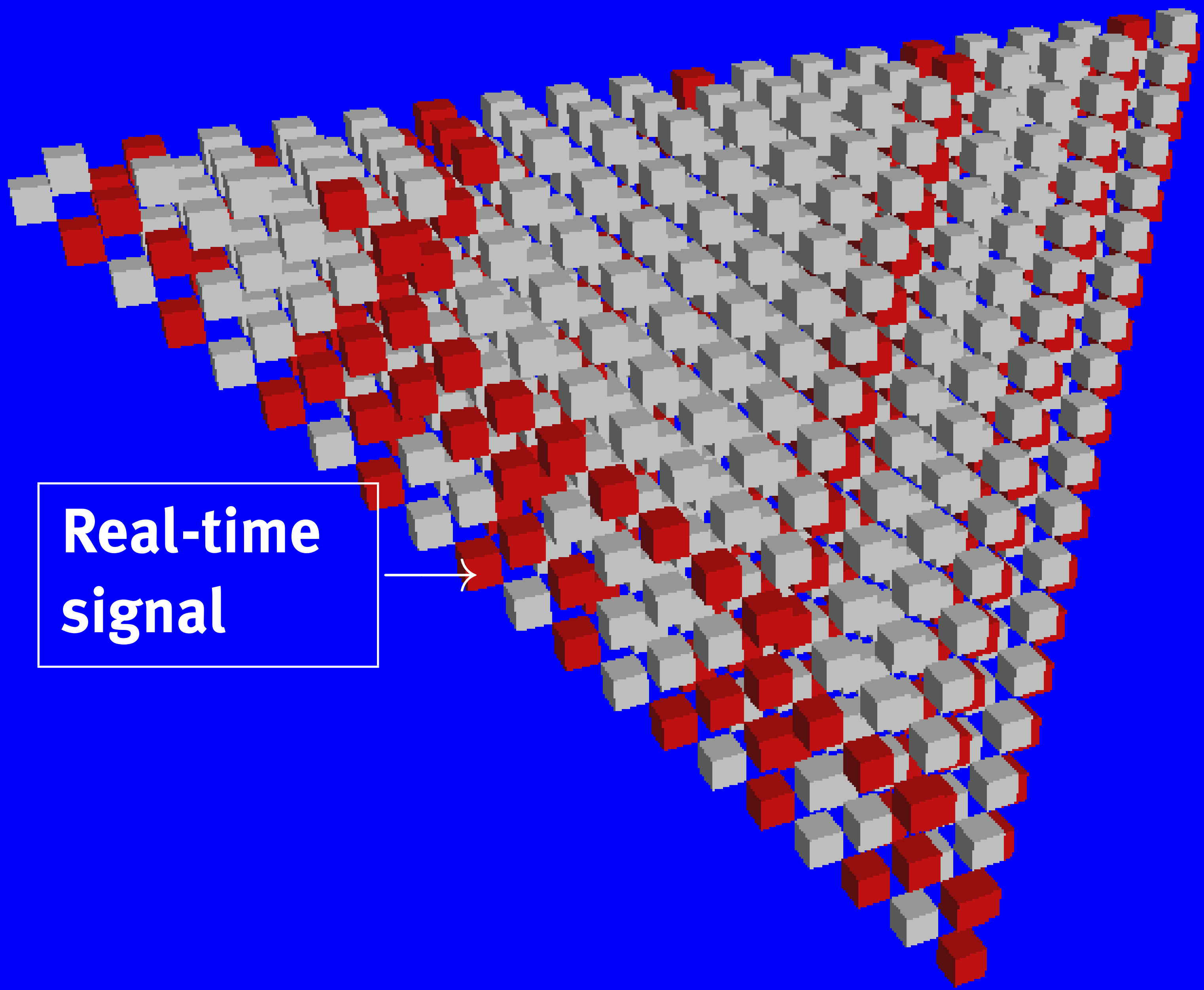
Time=20



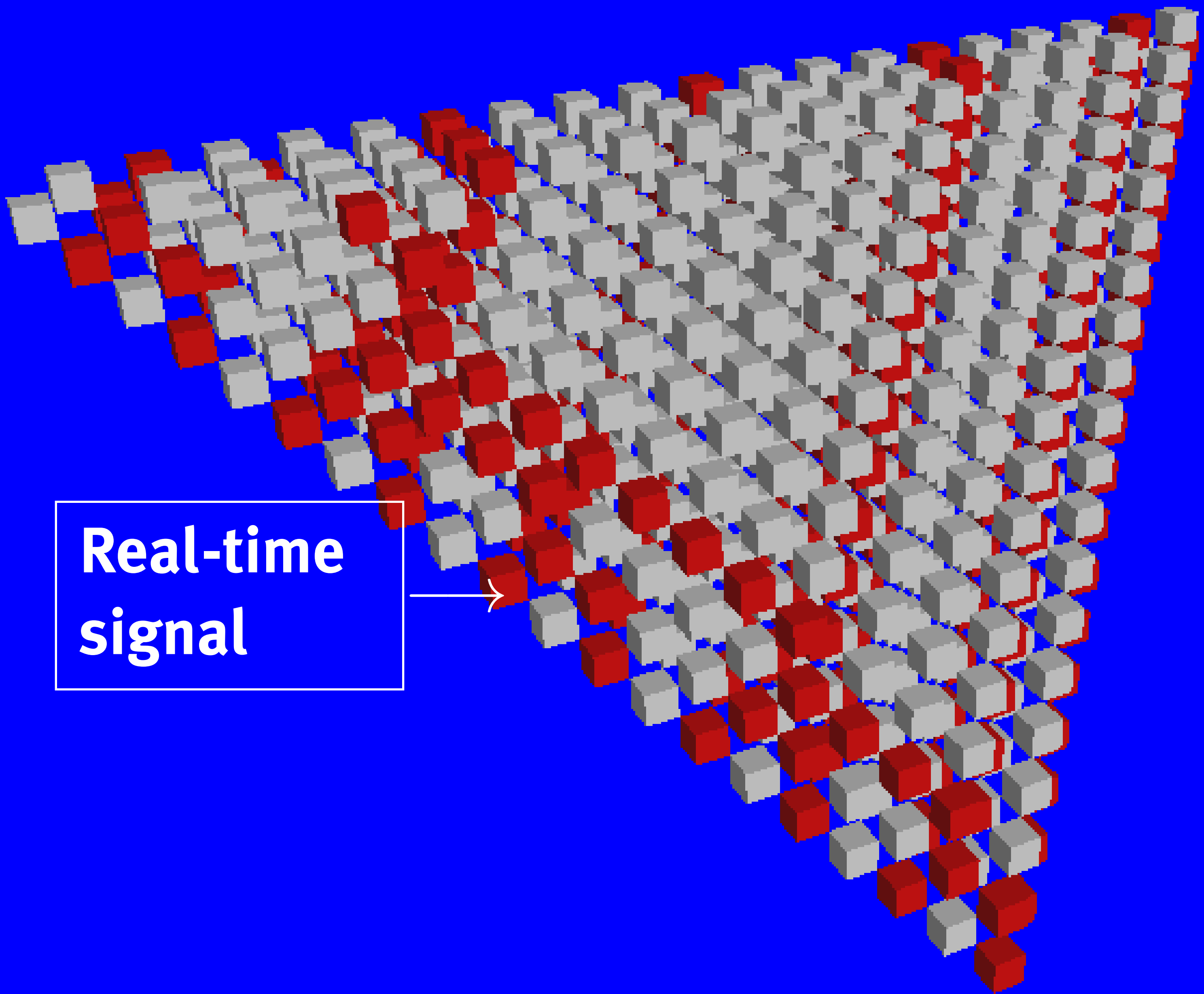
3D representation of the first steps



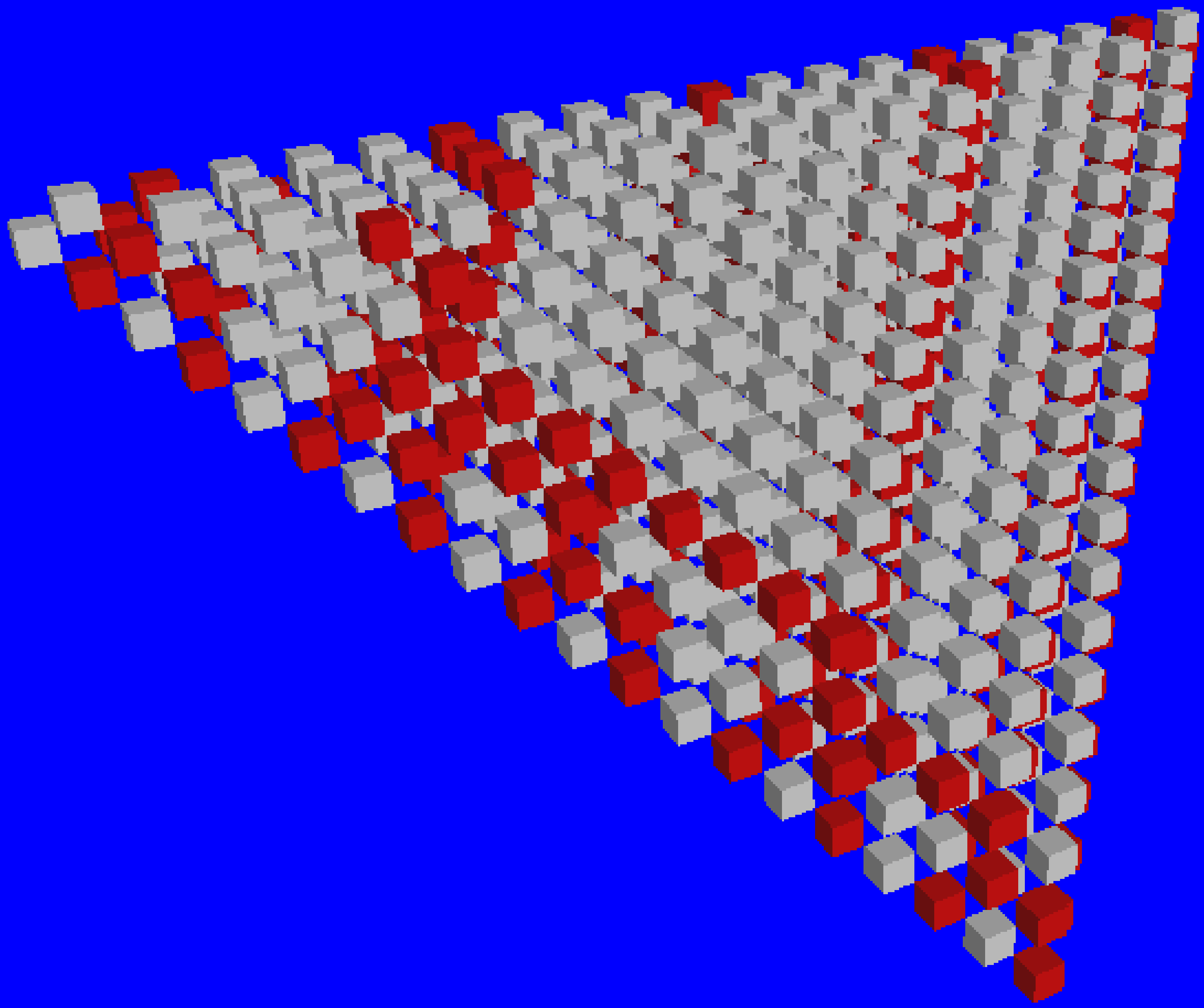
3D representation of the first steps



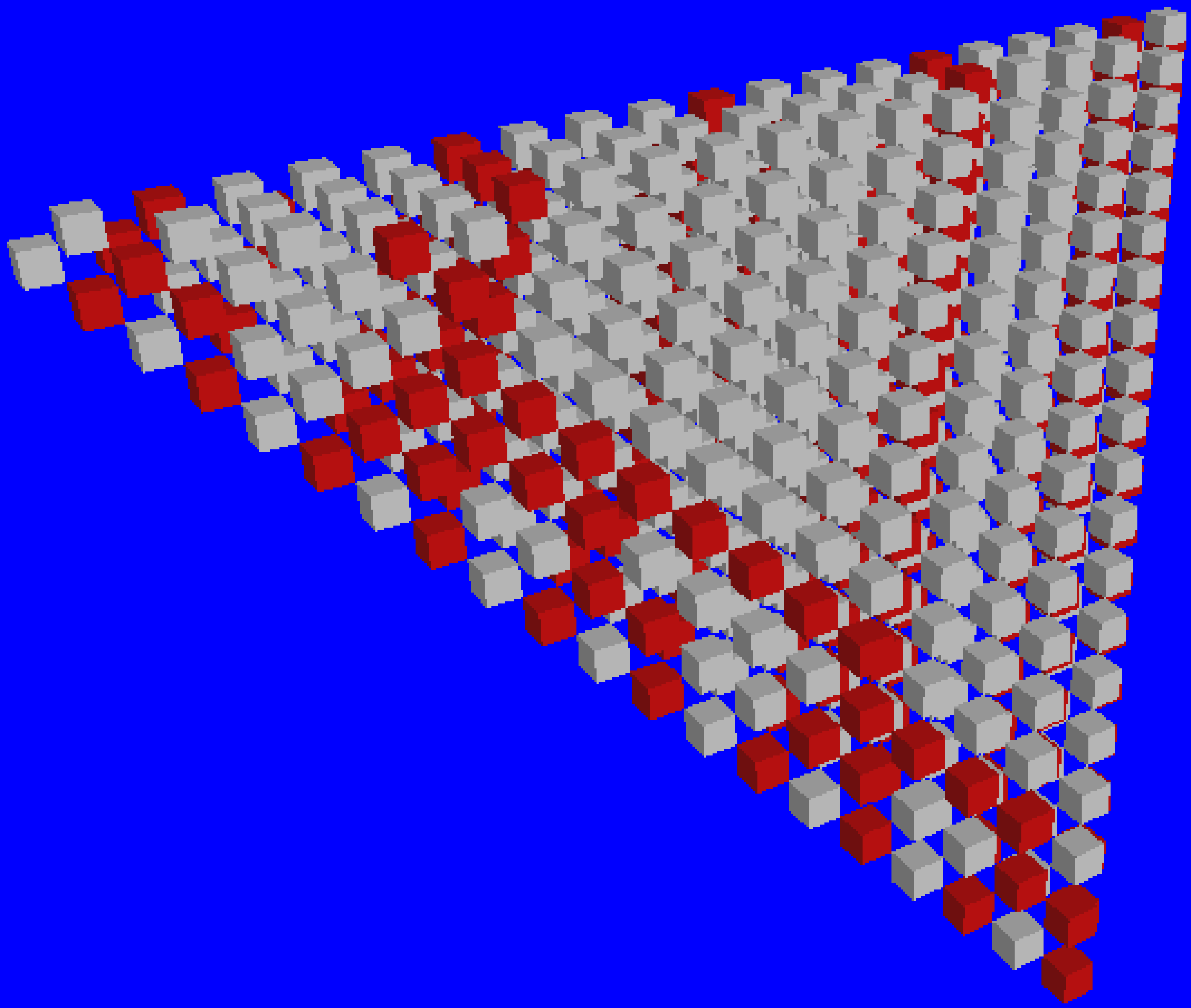
3D representation of the first steps



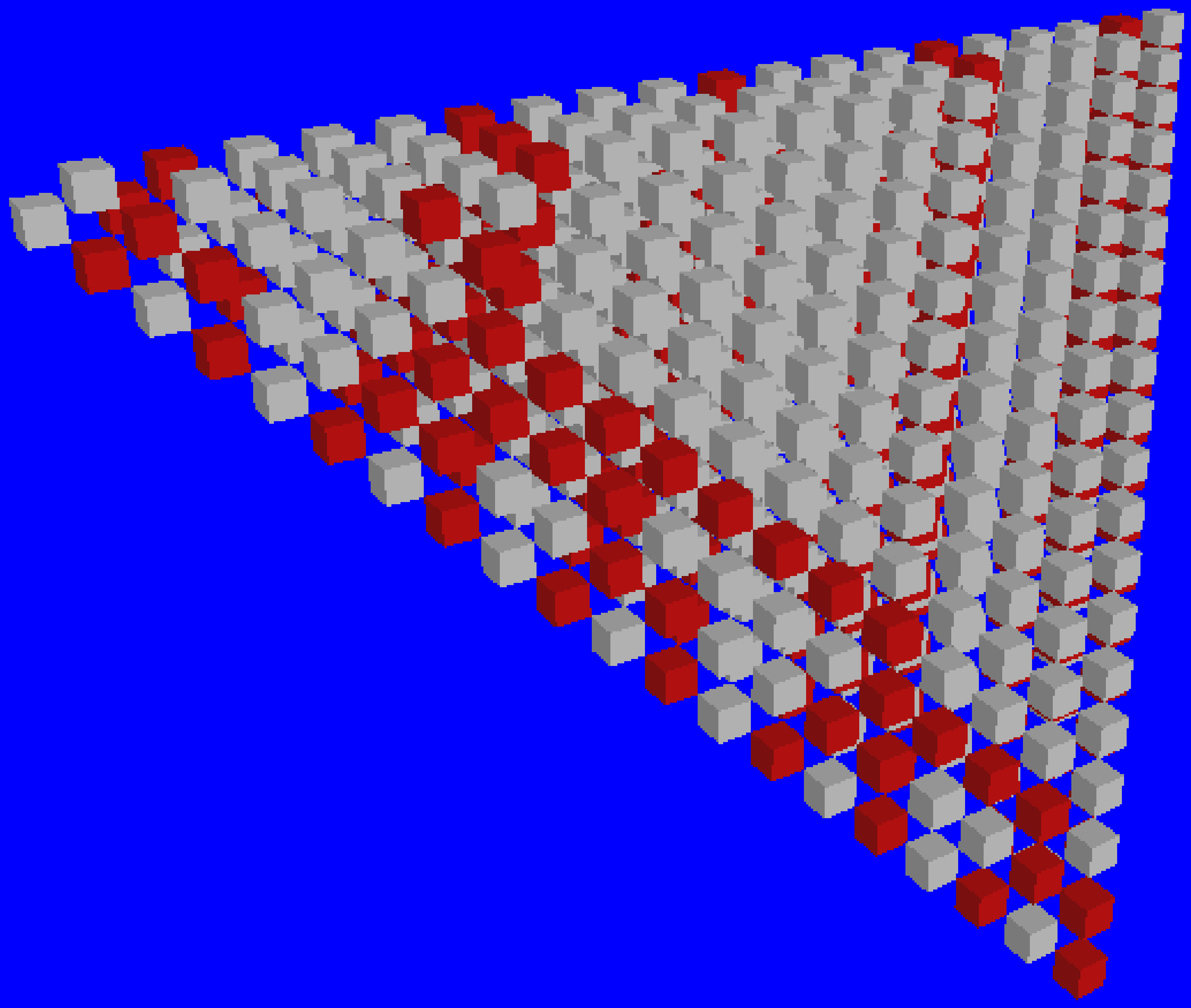
3D representation of the first steps



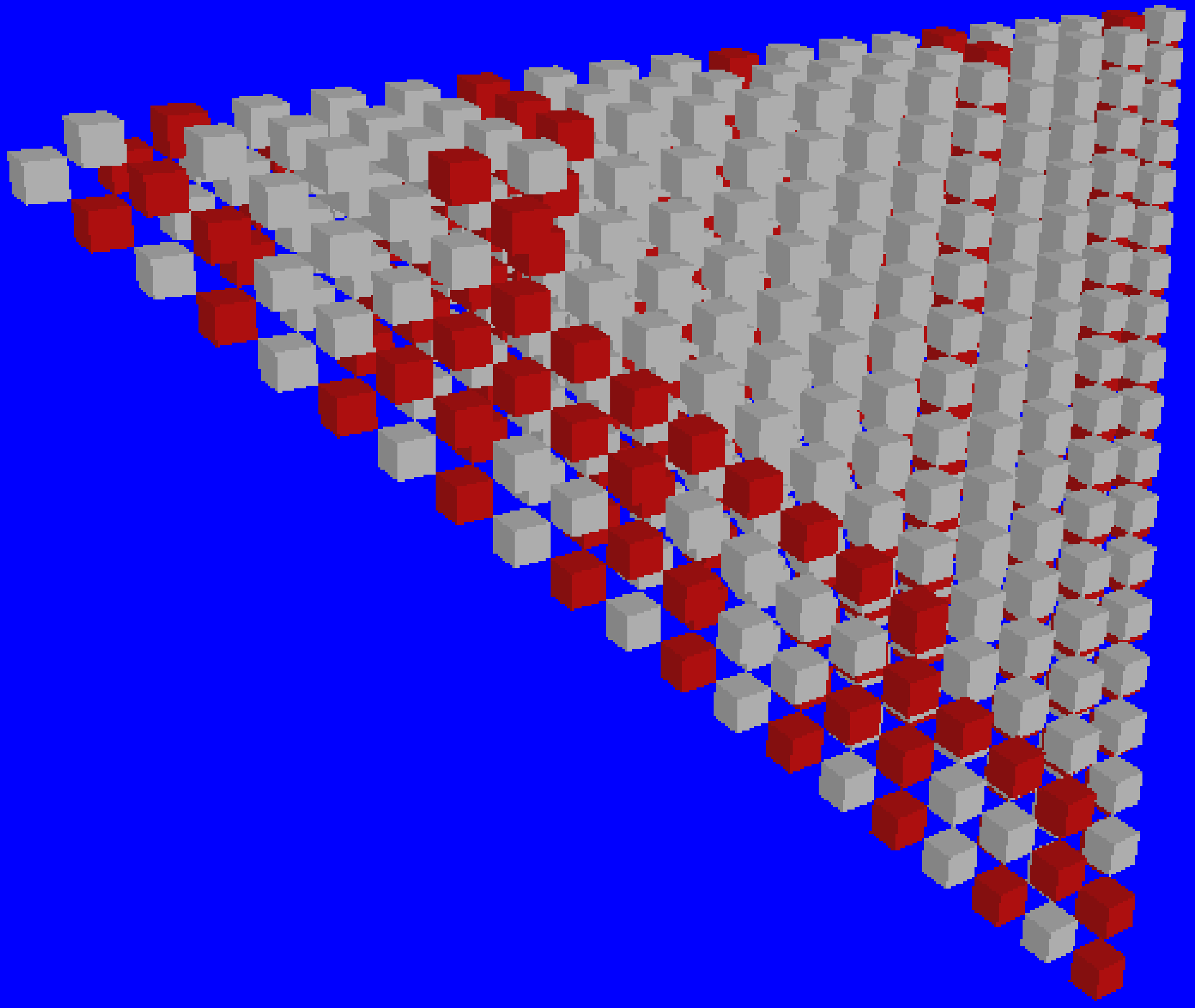
3D representation of the first steps



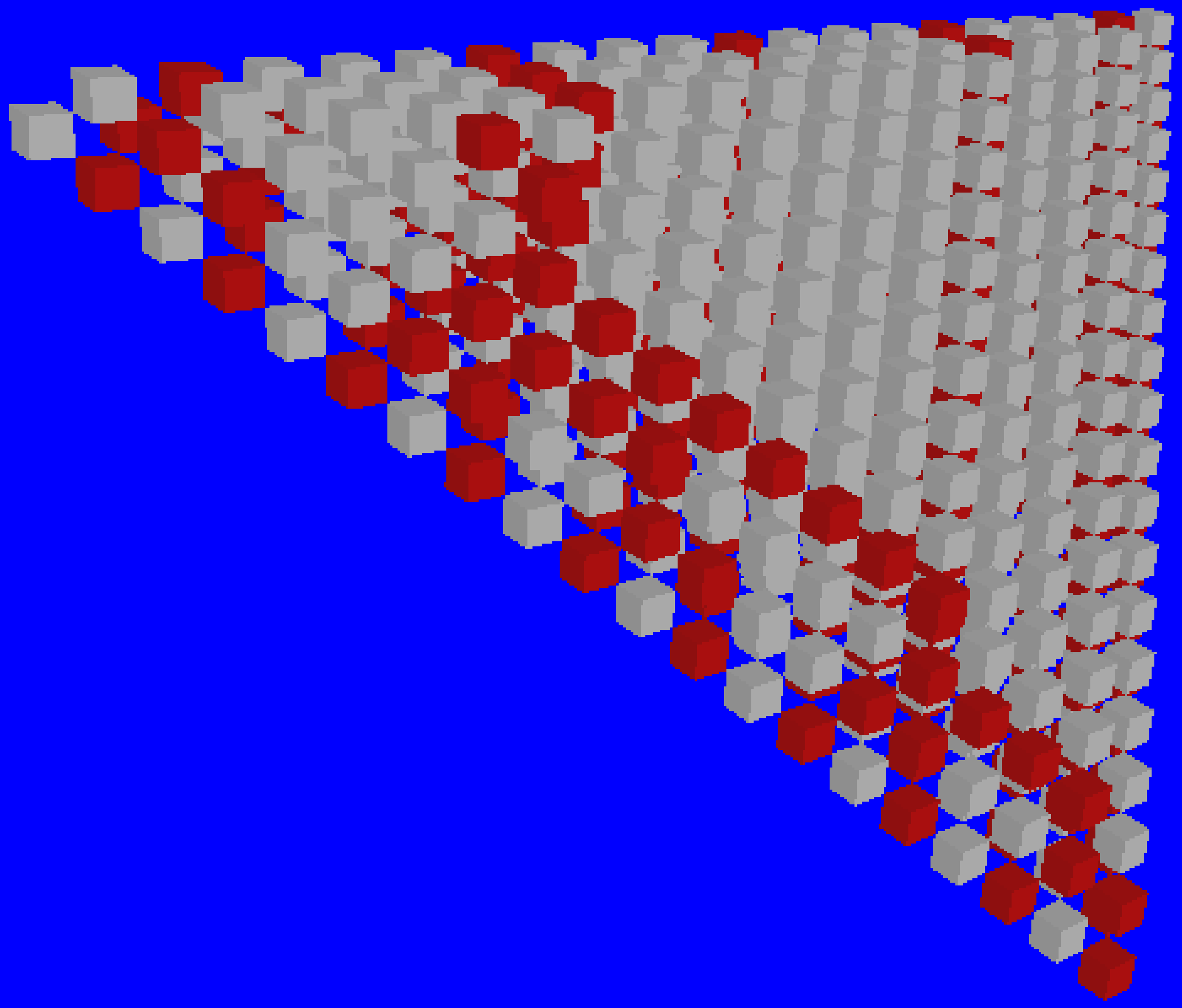
3D representation of the first steps



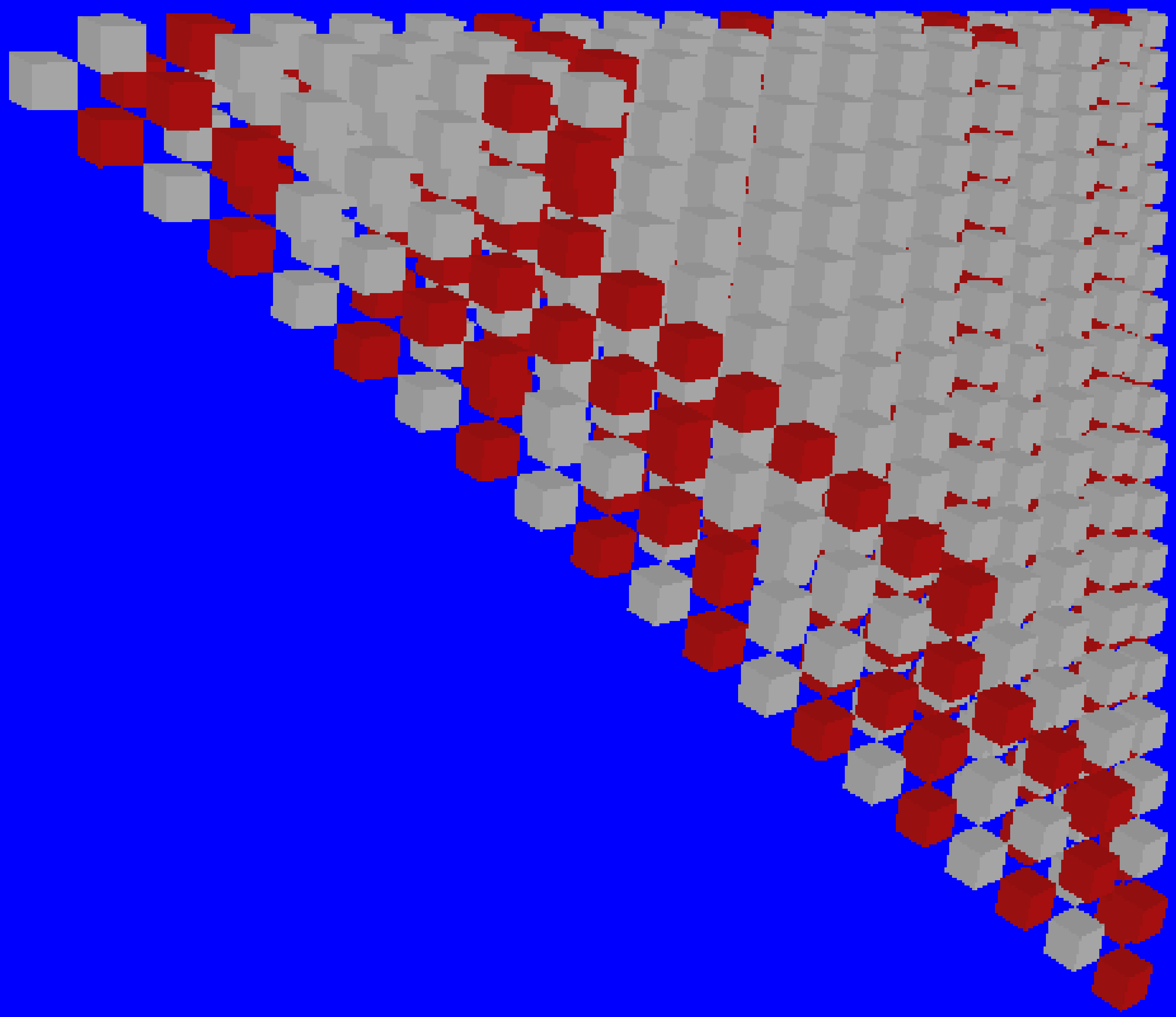
3D representation of the first steps



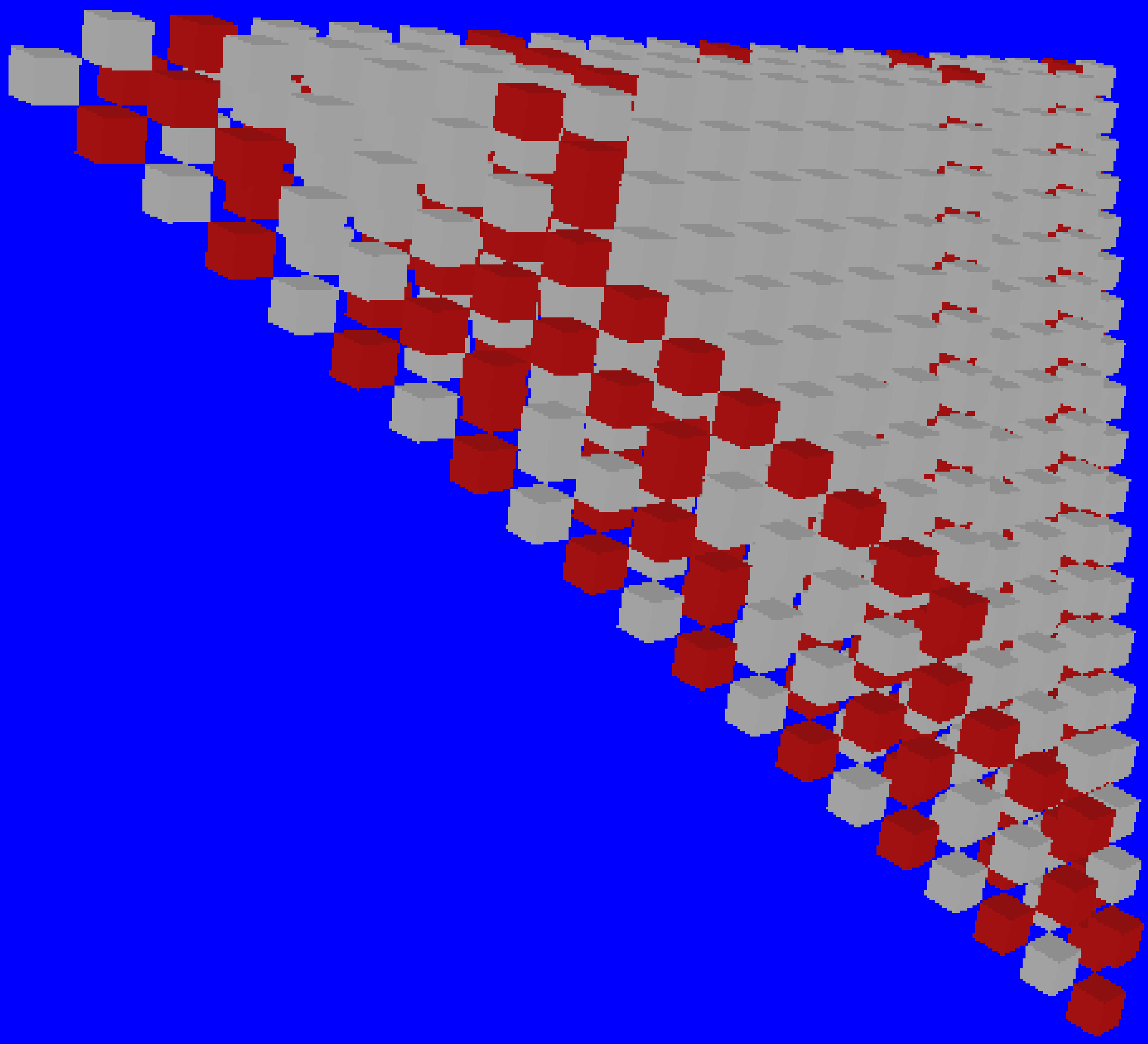
3D representation of the first steps



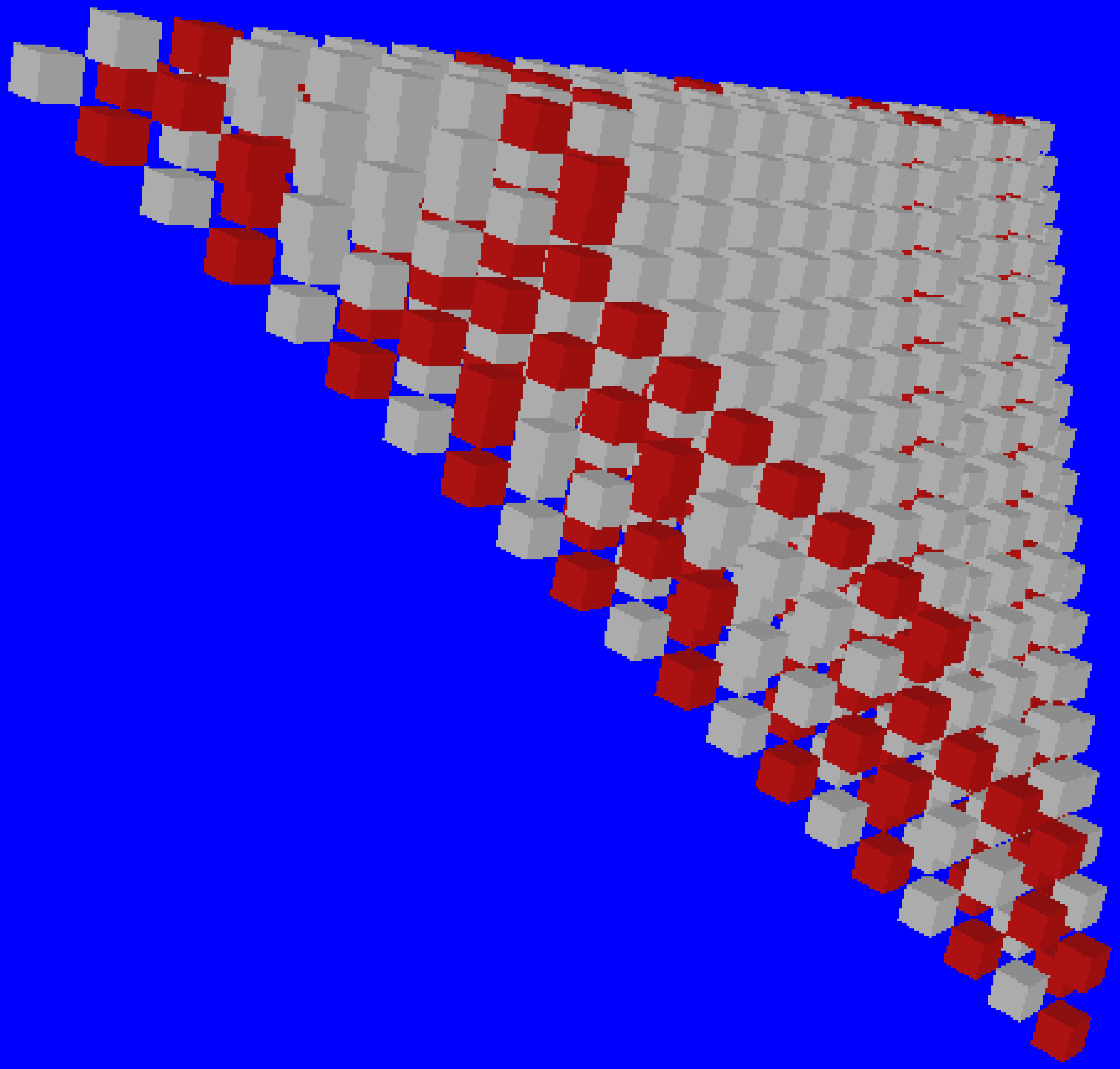
3D representation of the first steps



3D representation of the first steps

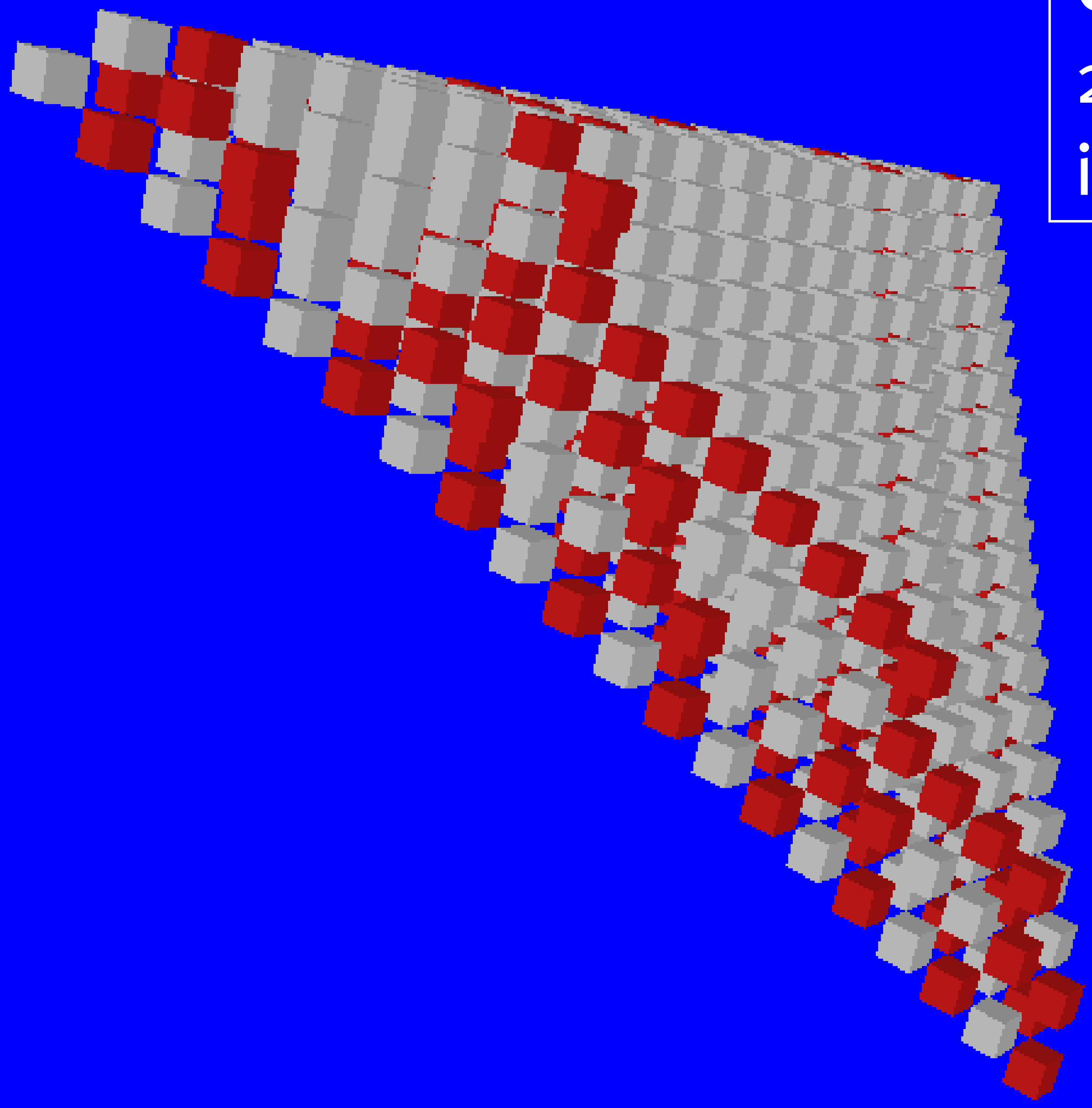


3D representation of the first steps

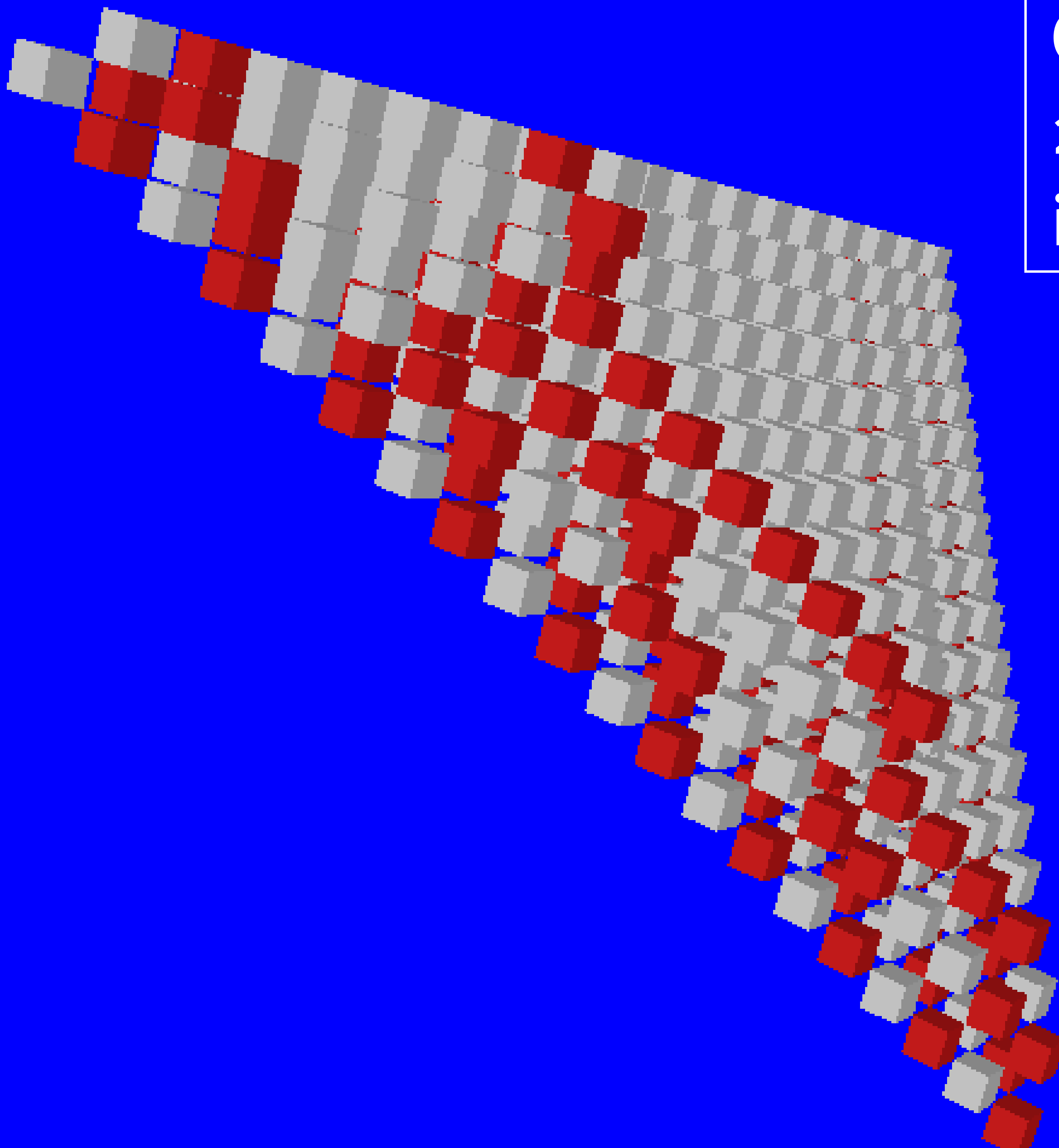


3D representation of the first steps

Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1

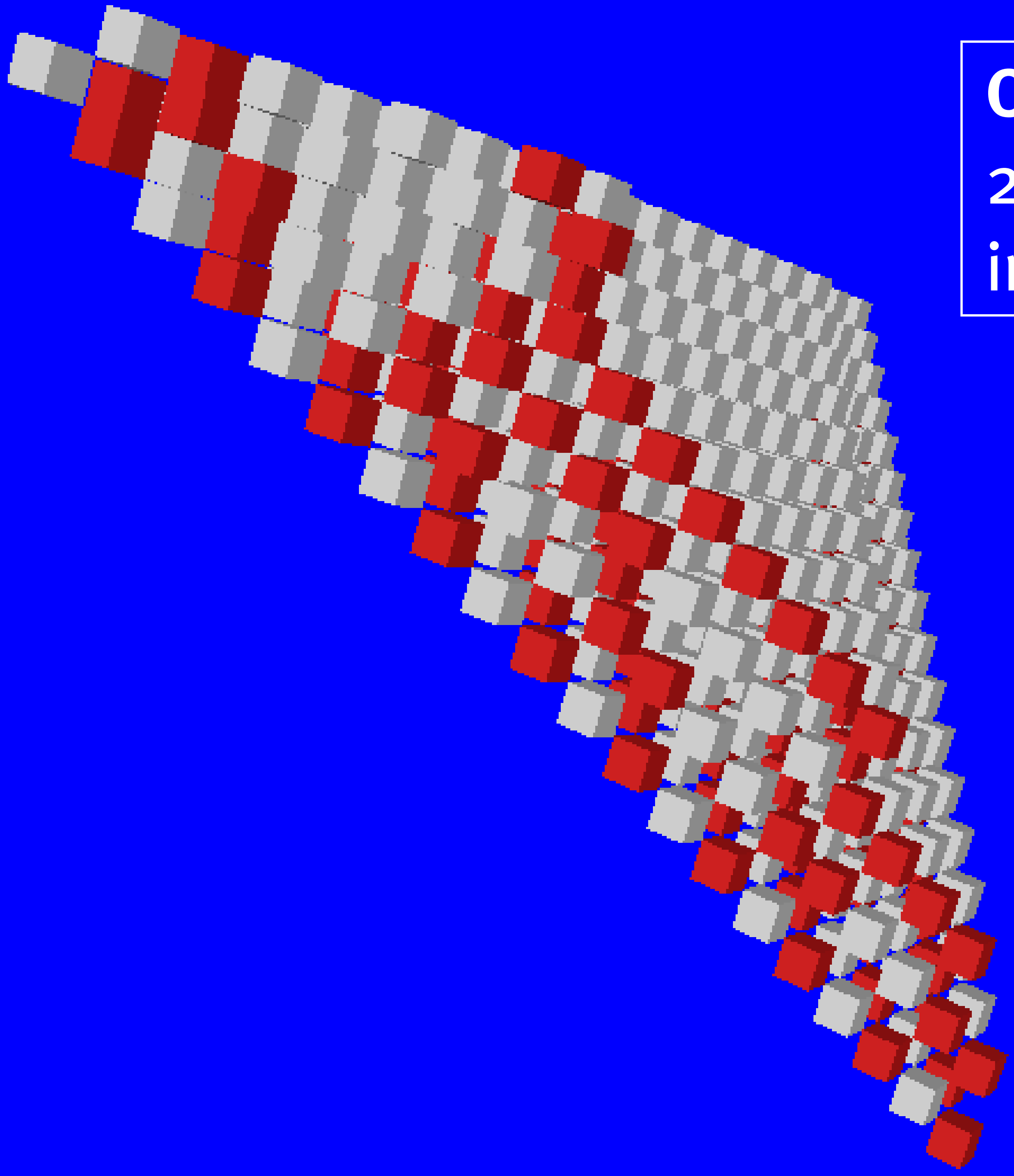


3D representation of the first steps



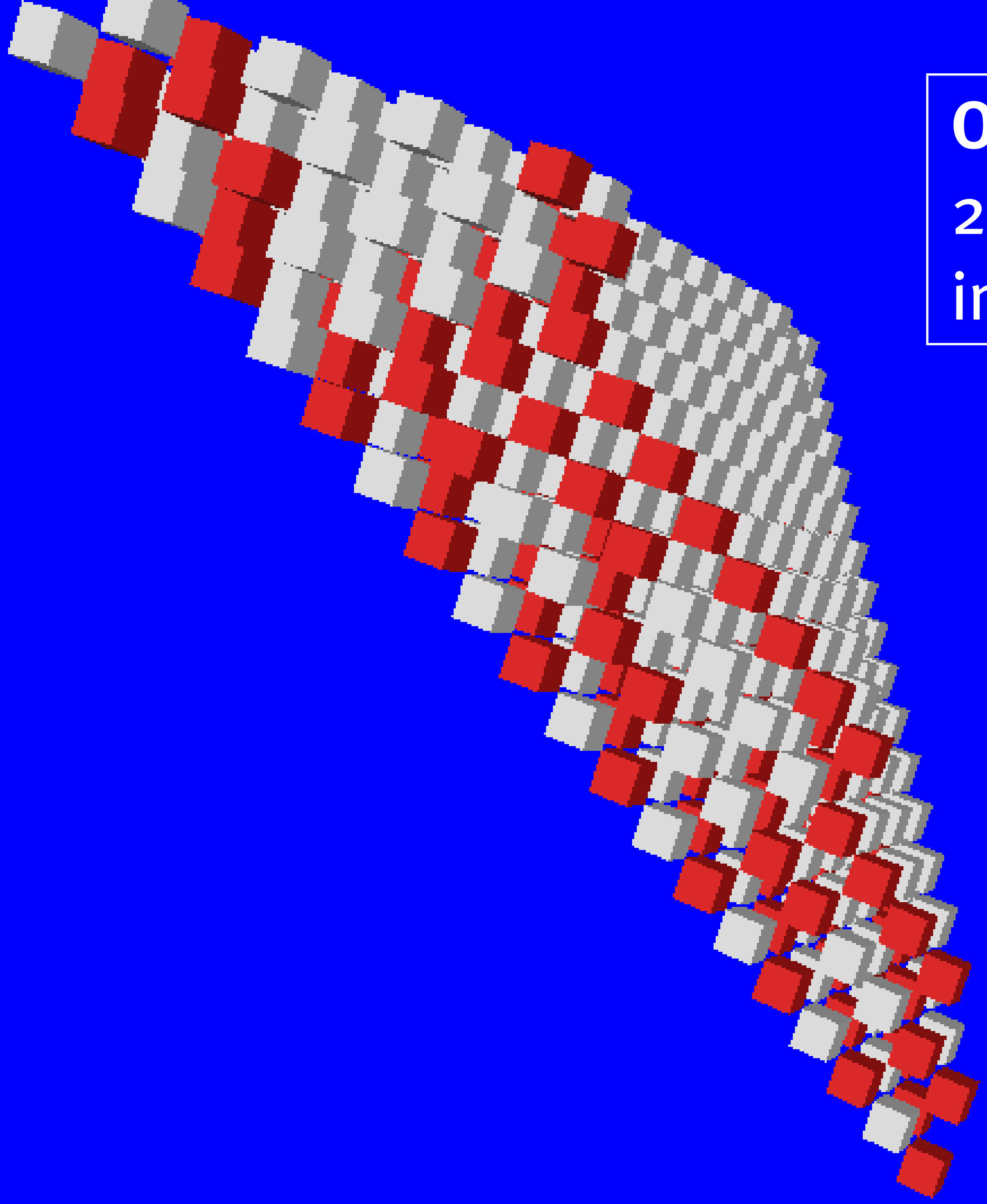
Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1

3D representation of the first steps



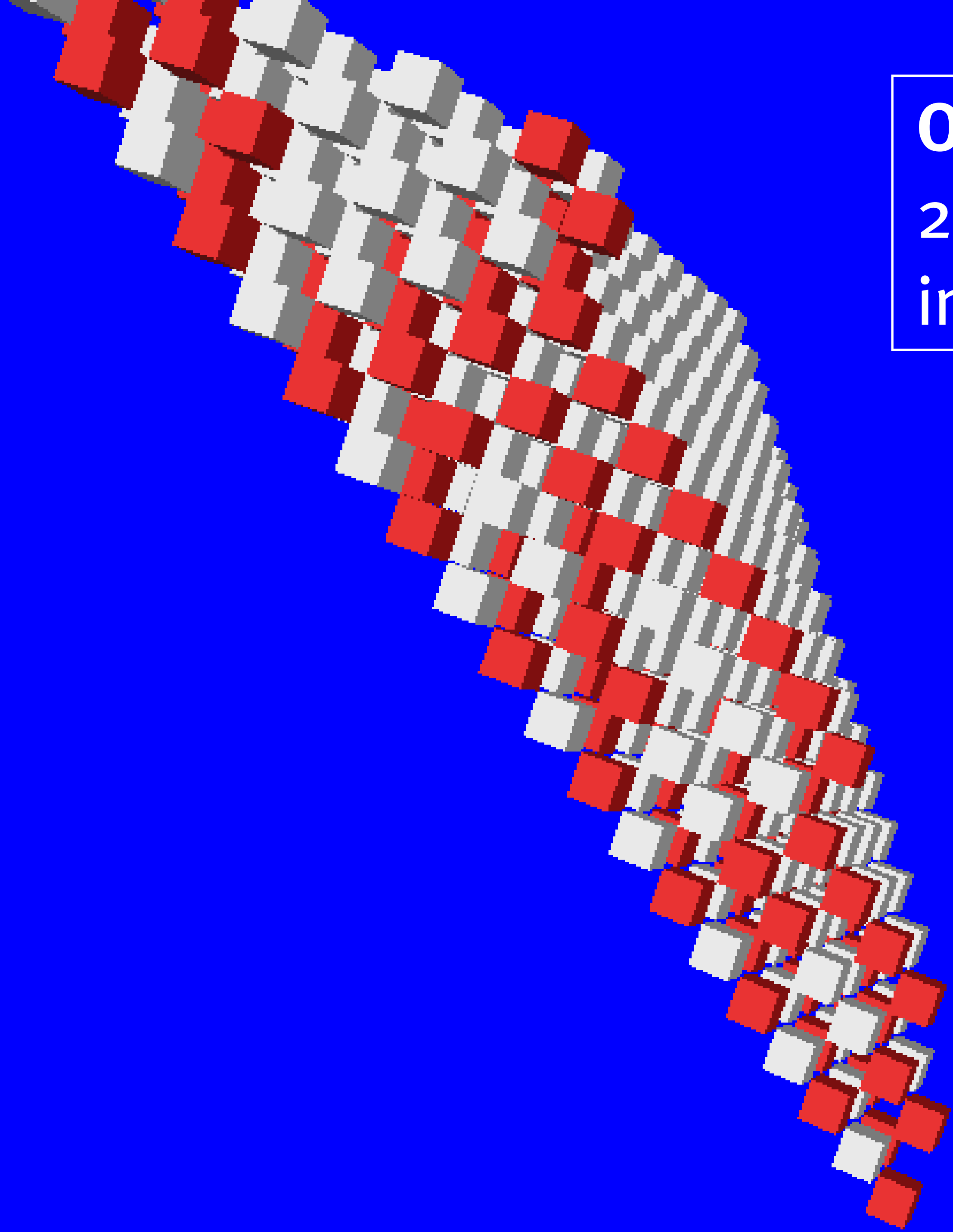
Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1

3D representation of the first steps



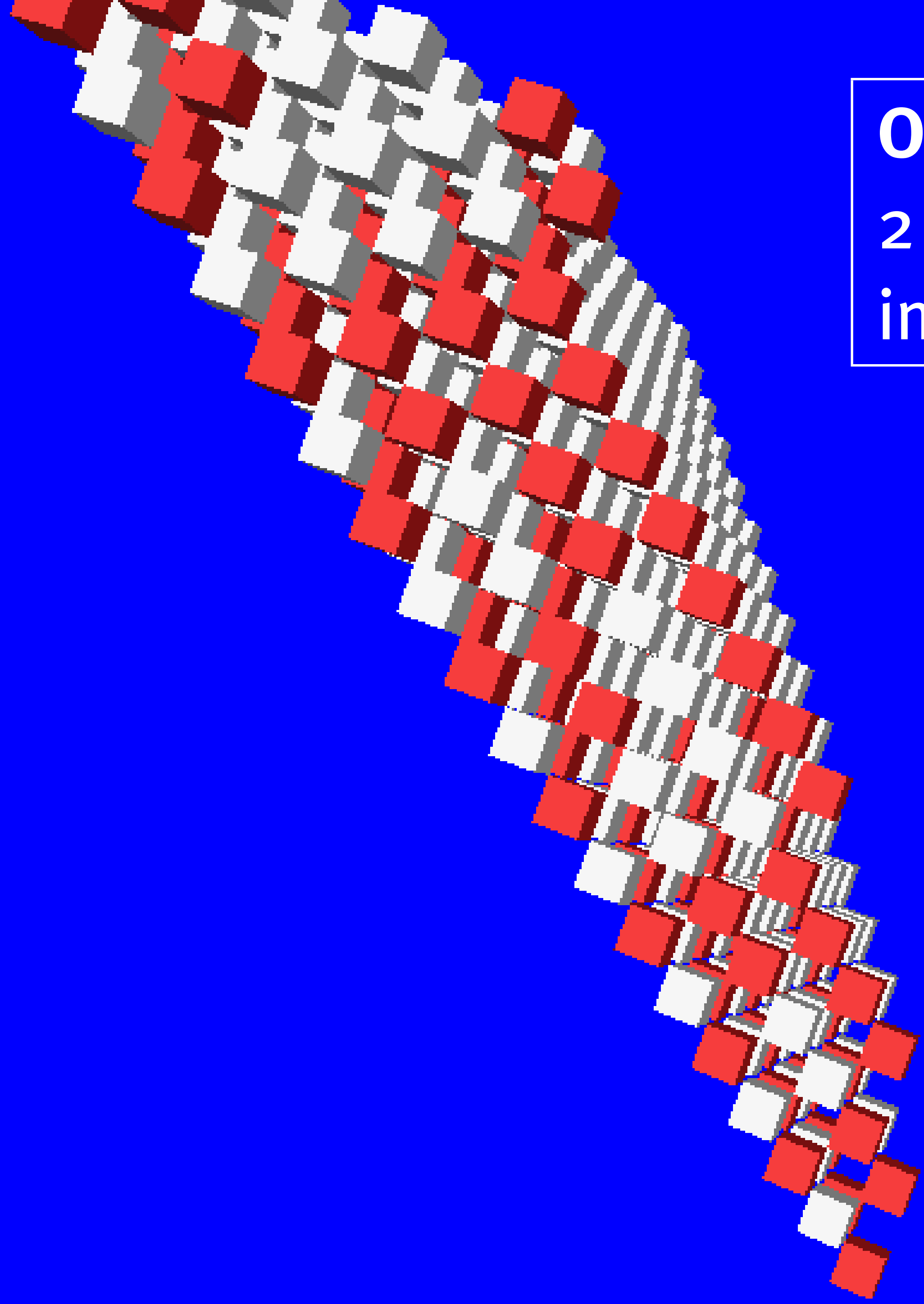
Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1

3D representation of the first steps



Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1

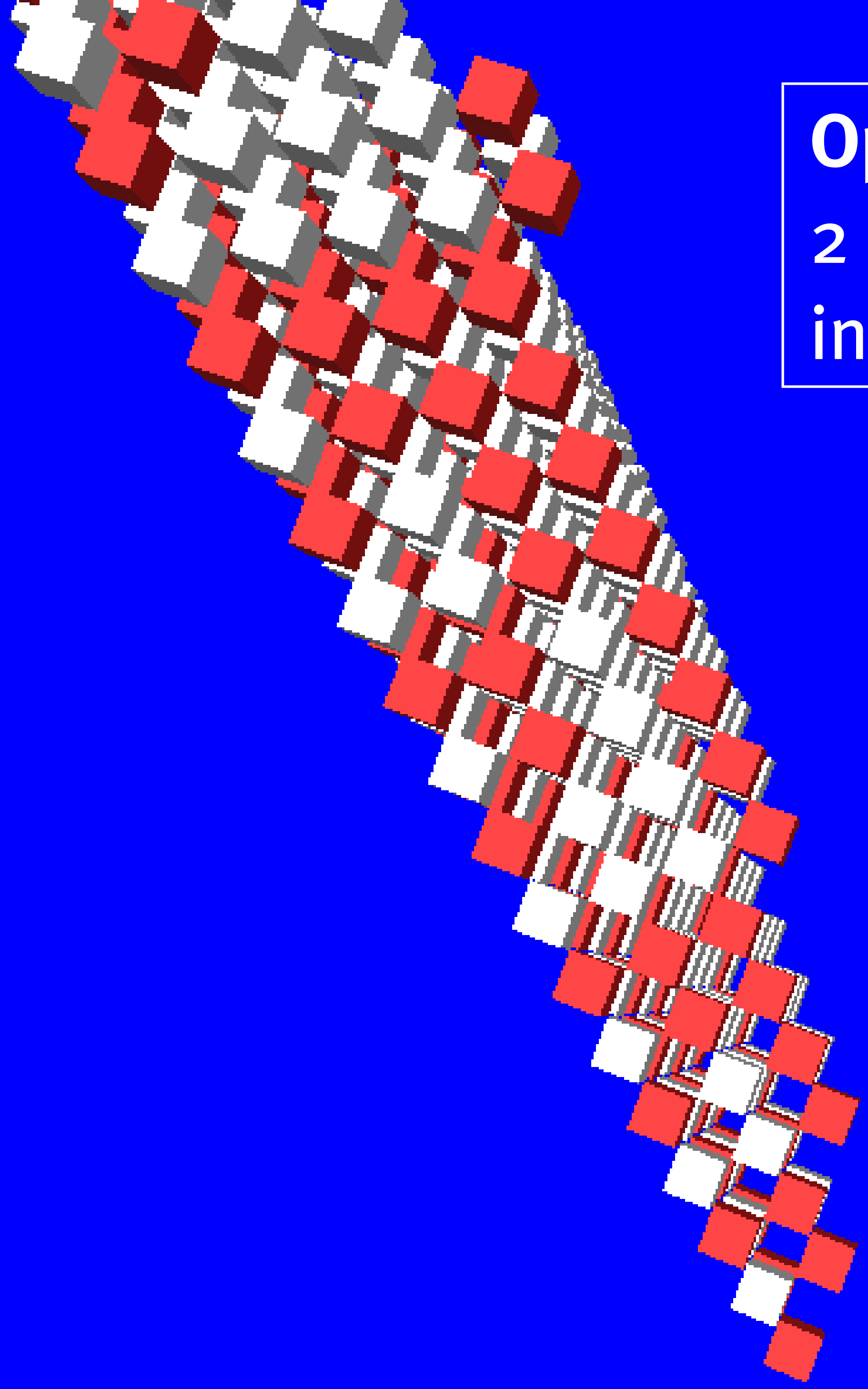
3D representation of the first steps



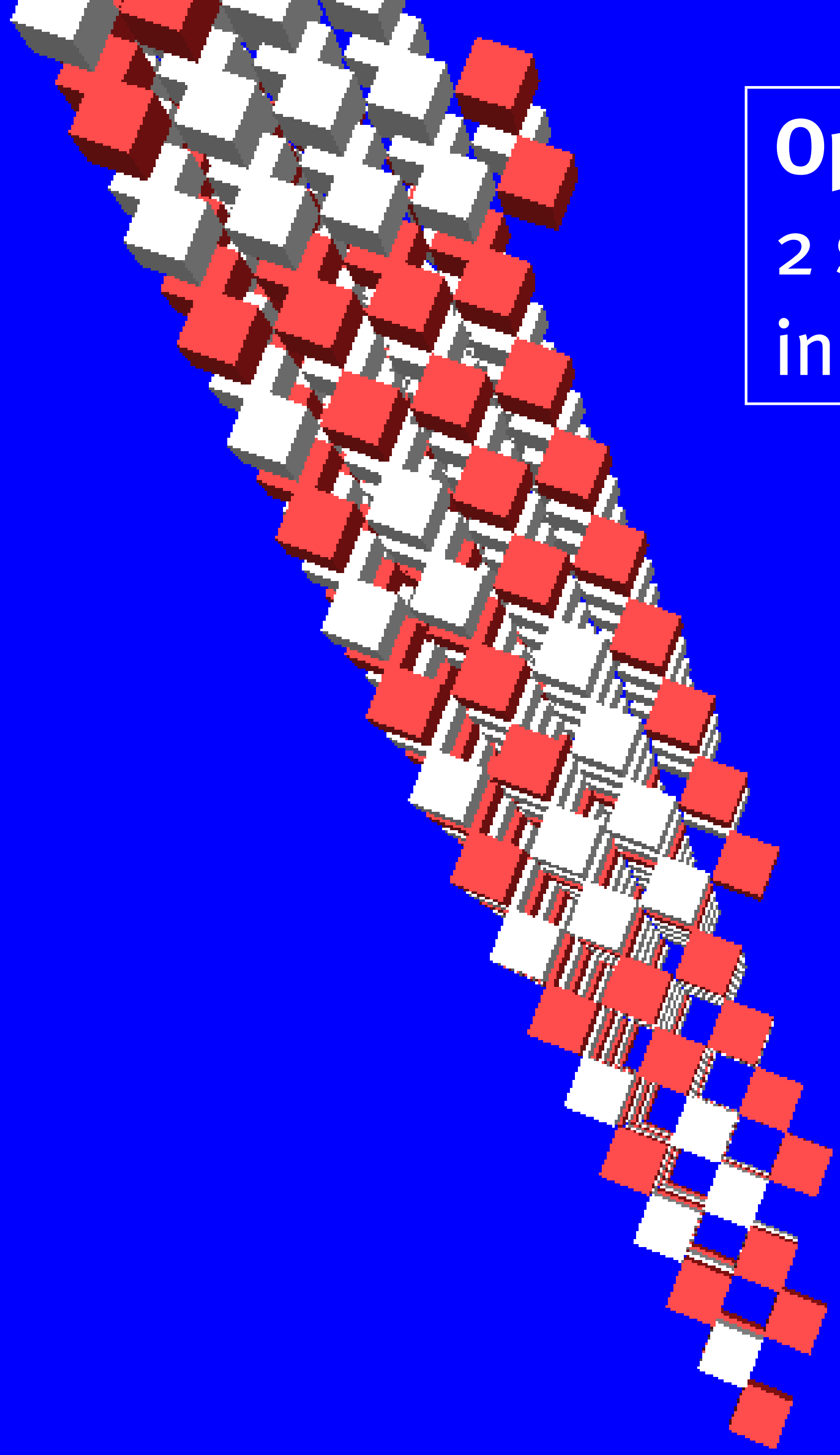
Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1

3D representation of the first steps

Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1



3D representation of the first steps



Optimal result:
2 states $\Rightarrow$ impossible
in dimension 2 or 1

3D representation of the first steps

Enhancement: any basis

$\Rightarrow$ Let x and y be numbers such that $\gcd(x,y)=1$.

a	b	c	d	$f(a,b,c,d)$		Rule #
λ	λ	λ	λ	λ		#0
Rules for $l=0$						
π_j	λ	λ	λ	π_{j+1}	(or π_1 if $j=k$)	#1
π_x	λ	π_k	$\kappa_\star$	π_0	$(k\neq x)$	#2
π_j	λ	π_k	$\kappa_\star$	π_j	$(j,k\neq x)$	#3
π_x	λ	π_x	κ_k	π_0	$(k\neq y-1)$	#4
π_j	λ	π_x	κ_k	π_j	$(j\neq x, k\neq y-1)$	#5
π_j	λ	π_x	κ_{y-1}	π_{j+1}	(or π_1 if $j=k$)	#6
λ	λ	π_x	κ_{y-1}	π_1		#7
Rules for $l=1$						
κ_{y-1}	π_x	λ,κ_y	λ	κ_y		#8
κ_{y-1}	π_k	λ,κ_y	λ	κ_0	$(k\neq x)$	#9
κ_y	$\pi_\star$	λ,κ_y	λ	κ_1		#10
κ_j	$\pi_\star$	λ,κ_y	λ	κ_{j+1}	$(j\neq y-1,y)$	#11
κ_y	$\pi_\star$	κ_k	λ	κ_0	$(k\neq y)$	#12
κ_j	$\pi_\star$	κ_k	λ	κ_j	$(j\neq y, k\neq y)$	#13
λ	π_1	λ,κ_y	λ	κ_1		#14
$\star$	$\star$	$\star$	$\star$	λ		#15

Function $\log_{xy}$ in $\max(x,y)+2$ states (with minor enhancement).

$\Rightarrow$ Clear gain over 1D (at least $xy+1$ states).

Conclusion

- Gap for signal construction, independently of the dimension;

Conclusion

- Gap for signal construction, independently of the dimension;
- Conjecture: the number of states necessary to build $\log_a$ is related to the decomposition of a with prime numbers;

Conclusion

- Gap for signal construction, independently of the dimension;
- Conjecture: the number of states necessary to build $\log_a$ is related to the decomposition of a with prime numbers;
- Possible link between the constructible signals and OCA recognisable languages.

Conclusion

- Gap for signal construction, independently of the dimension;
- Conjecture: the number of states necessary to build $\log_a$ is related to the decomposition of a with prime numbers;
- Possible link between the constructible signals and OCA recognisable languages.
- Functions equivalent to integer logarithm:

Conclusion

- Gap for signal construction, independently of the dimension;
- Conjecture: the number of states necessary to build $\log_a$ is related to the decomposition of a with prime numbers;
- Possible link between the constructible signals and OCA recognisable languages.
- Functions equivalent to integer logarithm:
  - Non-integer logarithms,

Conclusion

- Gap for signal construction, independently of the dimension;
- Conjecture: the number of states necessary to build $\log_a$ is related to the decomposition of a with prime numbers;
- Possible link between the constructible signals and OCA recognisable languages.
- Functions equivalent to integer logarithm:
  - Non-integer logarithms,
  - Other functions (lcm inverse function).