Transcendence in the enumeration of lattice walks

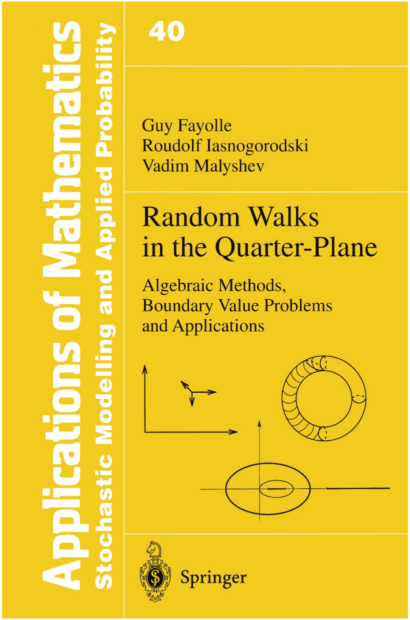
Alin Bostan

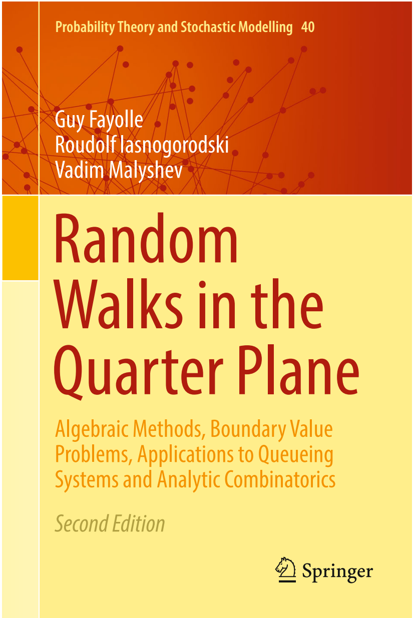
Inria

Combinatorics and Arithmetic for Physics

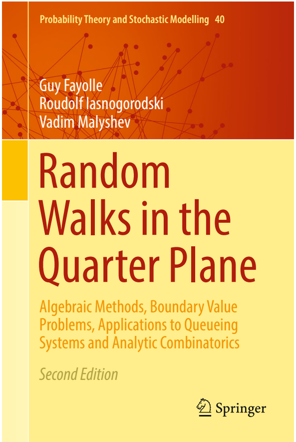
IHES, October 24, 2018

An influential little yellow book





. . . with some combinatorial flavors in the second edition



Lattice walks with small steps in the quarter plane

▷ Nearest-neighbor walks in the quarter plane:
walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Counting sequence $f_{\mathfrak{S}}(n;i,j)$: number of walks of length n ending at (i,j).

▷ Complete generating function:

$$F_{\mathfrak{S}}(t;x,y) = \sum_{i,j,n=0}^{\infty} f_{\mathfrak{S}}(n;i,j) x^i y^j t^n \in \mathbb{Q}[[x,y,t]].$$

Among the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

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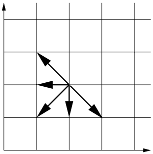

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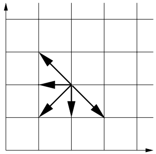

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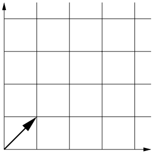

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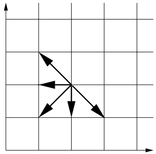

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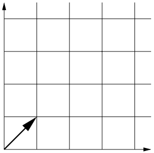

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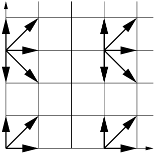

intrinsic to the half plane,

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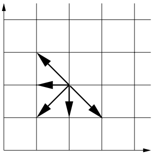

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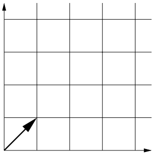

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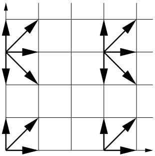

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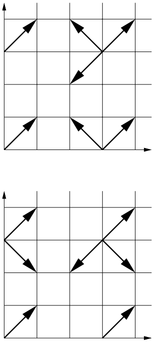
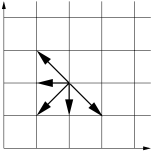


symmetrical.

Small-step models of interest

Among the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:



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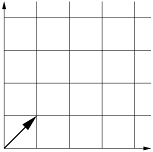

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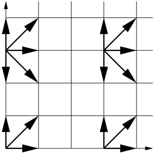

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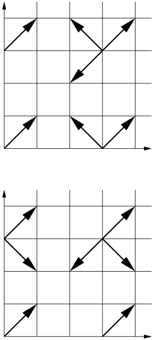
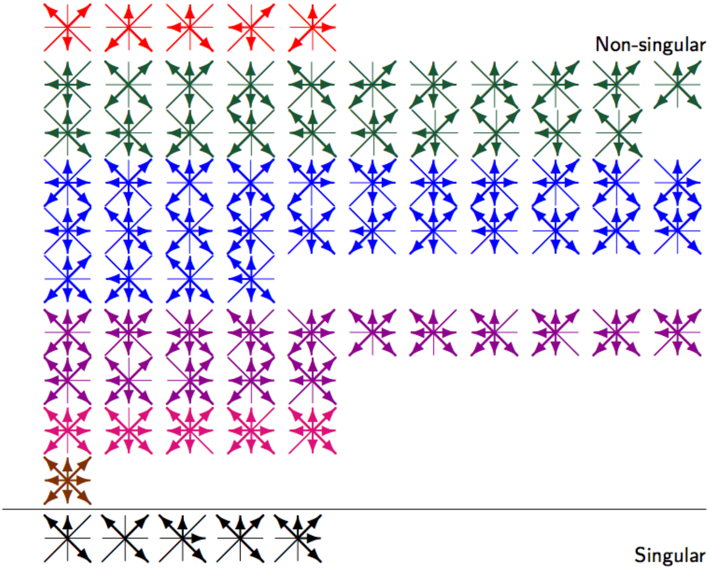


symmetrical.

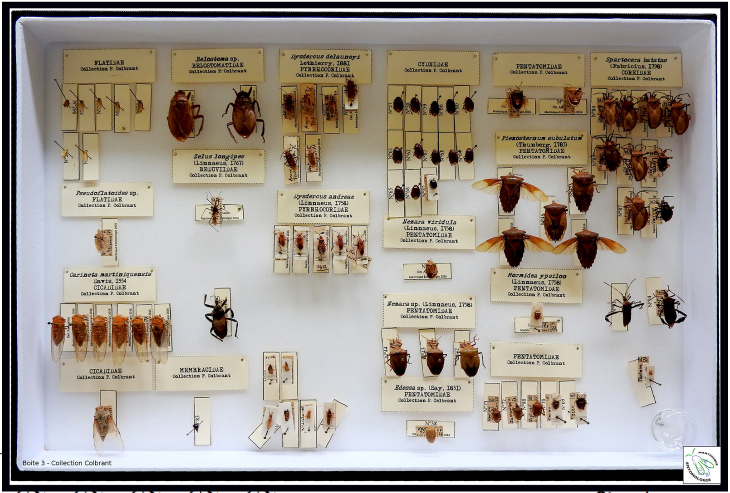
One is left with 79 interesting distinct models.

The 79 models



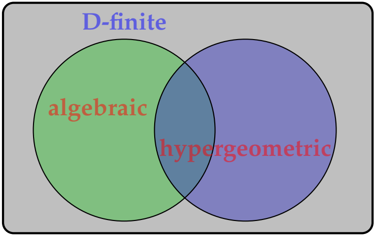
Task: classify their generating functions!

Non-singular



Singular

Important classes of univariate generating functions



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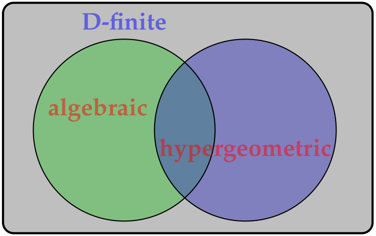

$$S(t) = \sum_{n=0}^{\infty} s_n t^n \in \mathbb{Q}[[t]] \quad \text{is}$$

▷ *algebraic* if $P(t, S(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$;

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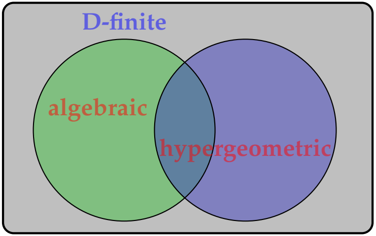

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▷ *algebraic* if $P(t, S(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$;

▷ *D-finite* if $c_r(t) S^{(r)}(t) + \cdots + c_0(t) S(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero;

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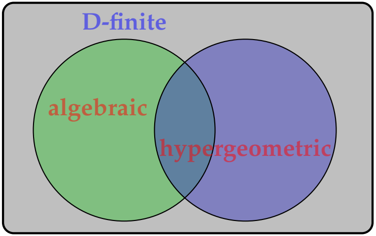

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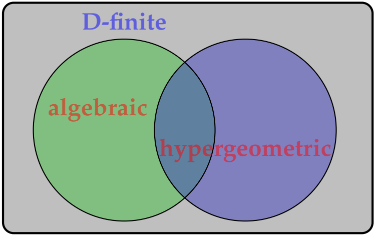

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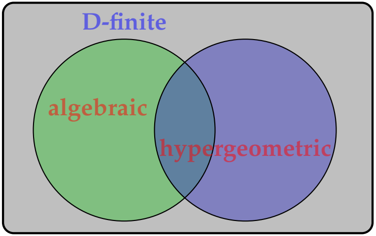
$$\ln(1-t); \quad \frac{\arcsin(\sqrt{t})}{\sqrt{t}}; \quad (1-t)^{\alpha}, \alpha \in \mathbb{Q}$$

Important classes of univariate generating functions



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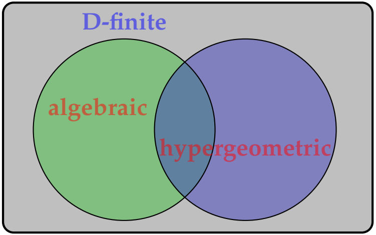
$${}_2F_1\left(\begin{matrix} a\ b \\ c \end{matrix} \middle| t\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Important classes of univariate generating functions



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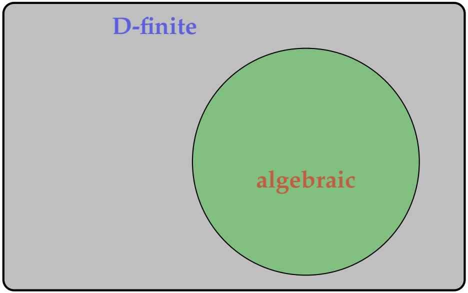
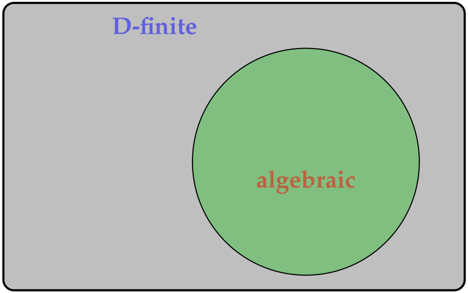
$${}_2F_1\left(\begin{matrix}\frac{1}{2} \ \frac{1}{2} \\ 1\end{matrix} \middle| \, t\right) = \frac{2}{\pi} \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-tx^2)}}.$$

Important classes of multivariate generating functions



▷ $S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a polynomial $P \in \mathbb{Q}[x, y, t, T]$;

Important classes of multivariate generating functions



▷ $S \in \mathbb{Q}[[x, y, t]]$ is *algebraic* if it is the root of a polynomial $P \in \mathbb{Q}[x, y, t, T]$;

▷ $S \in \mathbb{Q}[[x, y, t]]$ is *D-finite* if it satisfies a system of linear partial differential equations with polynomial coefficients

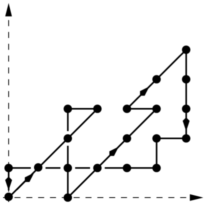
$$\sum_i a_i(t, x, y) \frac{\partial^i S}{\partial x^i} = 0, \quad \sum_i b_i(t, x, y) \frac{\partial^i S}{\partial y^i} = 0, \quad \sum_i c_i(t, x, y) \frac{\partial^i S}{\partial t^i} = 0.$$

Kreweras walks (1965)

- $k(n;i,j)$ = number of n-step $\{\downarrow, \leftarrow, \nearrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Question: What is the nature of the generating function

$$K(t;x,y) = \sum_{i,j,n=0}^{\infty} k(n;i,j)\, x^i y^j t^n \ \in \mathbb{Q}[[x,y,t]]?$$



Theorem $K(t;x,y)$ is an algebraic function.

$\rightarrow$ First proof [Gessel, 1986]: (human) Guess'n'Prove

$\rightarrow$ Similar result for GF of invariant measure† [Flatto and Hahn, 1984]: complex analysis, elliptic functions, Riemann surfaces

$\rightarrow$ Many other proofs, most inspired by the little yellow book!

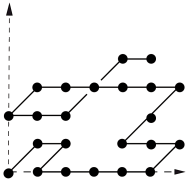
† a system of two parallel queues with two demands (continuous time)

Gessel walks (2000)

- $g(n;i,j)$ = number of n-step $\{\nearrow,\swarrow,\leftarrow,\rightarrow\}$-walks in $\mathbb{N}^2$ from $(0,0)$ to (i,j)

Question: What is the nature of the generating function

$$G(t;x,y) = \sum_{i,j,n=0}^{\infty} g(n;i,j)\, x^i y^j t^n \in \mathbb{Q}[[x,y,t]]?$$



Theorem $G(t;x,y)$ is an algebraic function†.

→ First proof [B.-Kauers, 2010]: effective, computer-driven discovery/proof

→ Several recent (human) proofs, most inspired by the little yellow book!

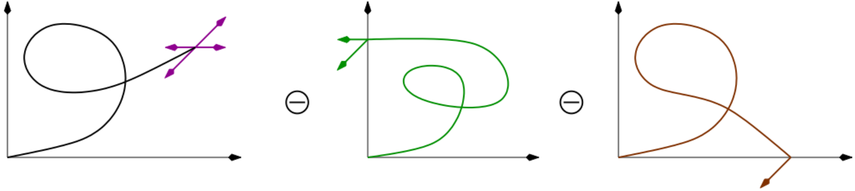
† Minimal polynomial $P(x,y,t,G(t;x,y)) = 0$ has $> 10^{11}$ terms; $\approx 30\,\text{Gb}$ (!)

Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{i,j,n=0}^{\infty} g(n;i,j) x^i y^j t^n \in \mathbb{Q}[[x,y,t]]$

"Kernel equation":

$$G(t;x,y) = 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right) G(t;x,y) - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right) G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right)$$



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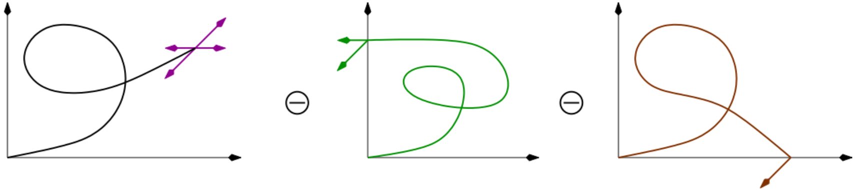

Task: Solve this functional equation!

Algebraic reformulation: solving a functional equation

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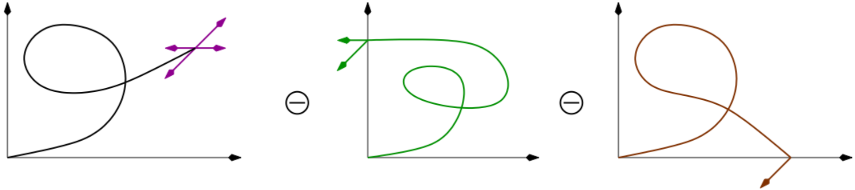
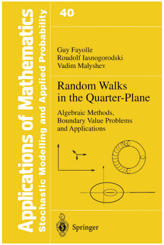
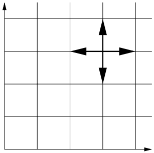
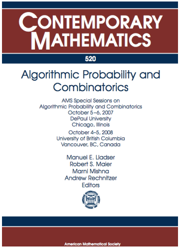


Task: For the other models – solve 78 similar equations!

A crucial tool: the group of a model

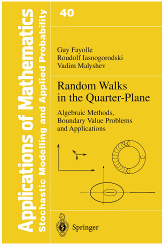


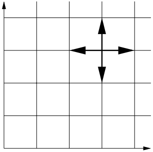


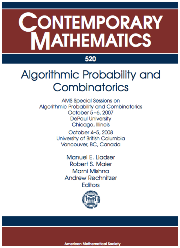


The generating polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

A crucial tool: the group of a model







The generating polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

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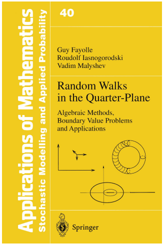

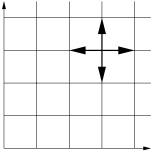

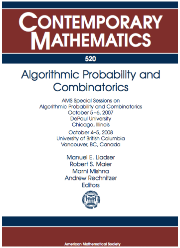

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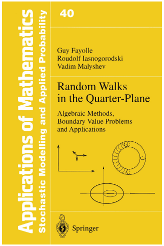
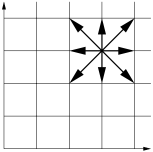
$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

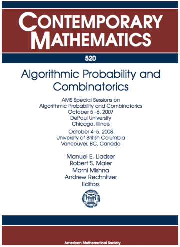
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case

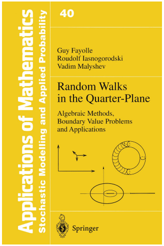


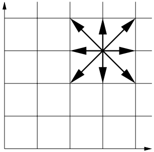


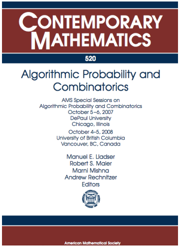


The generating polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

The group of a model: the general case







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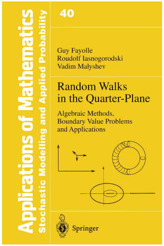

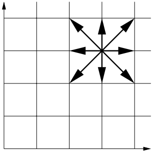

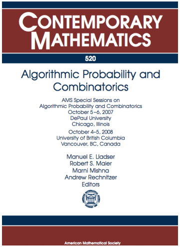

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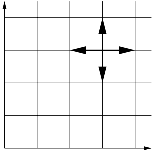
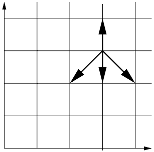
and thus under any element of the (dihedral) group

$$\mathcal{G}_{\mathfrak{S}} := \langle \psi, \phi \rangle.$$

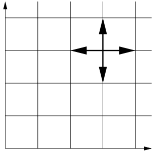
Examples of groups

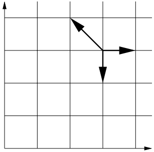


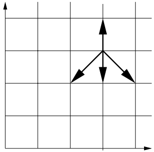


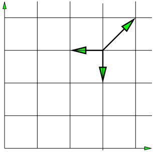
Order 4,

Examples of groups



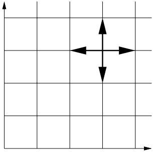
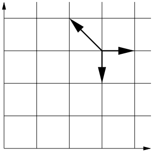
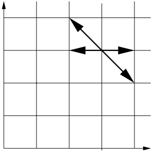
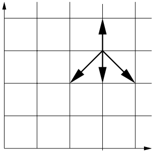
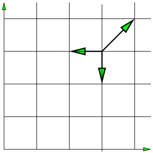
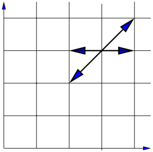






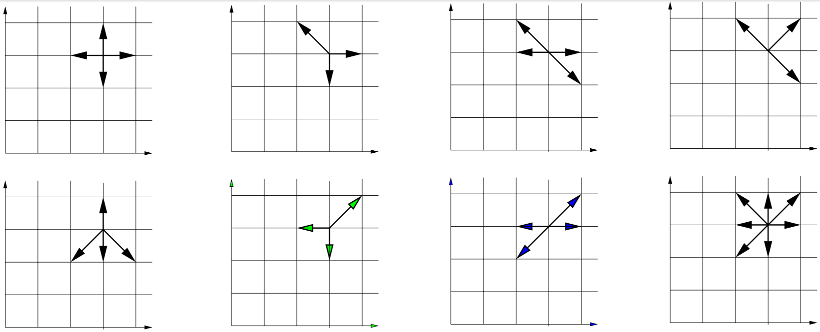
Order 4, order 6,

Examples of groups

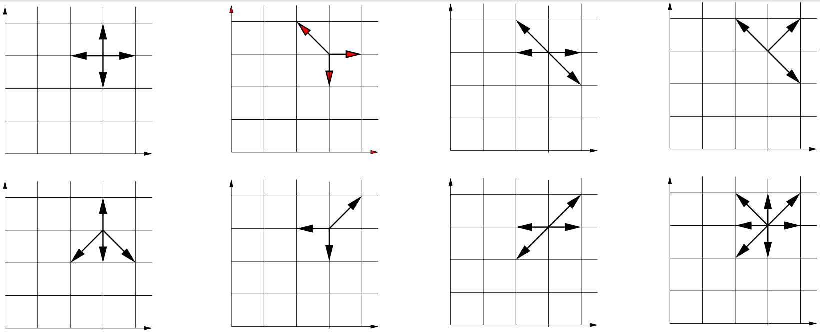
Order 4, order 6, order 8,

Examples of groups



Order 4, order 6, order 8, order ∞.

Examples of groups



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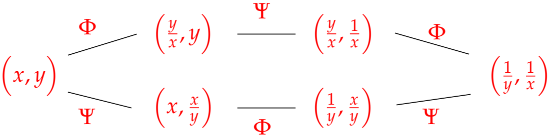
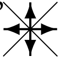


Another important concept: the orbit sum (OS)

When $\mathcal{G}_{\mathfrak{S}}$ is finite, the orbit sum of $\mathfrak{S}$ is the polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

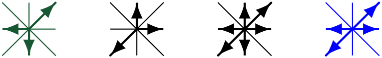
$$\mathrm{OS}_{\mathfrak{S}} := \sum_{\theta \in \mathcal{G}_{\mathfrak{S}}} (-1)^{\theta} \theta(xy)$$

▷ E.g., for the simple walk, with $\mathcal{G}_{\mathfrak{S}} = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}$:

OS 

$$= x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

▷ For 4 models, the orbit sum is zero:



E.g., for the Kreweras model:

OS 

$$= x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} \quad = 0$$

Main classification result

Theorem

Let $\mathfrak{S}$ be one of the 74 non-singular models of small-step walks in $\mathbb{N}^2$. The following assertions are equivalent:

(1) the full generating function $F_{\mathfrak{S}}(t;x,y)$ is D-finite

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(3) the excursions sequence $[t^n]\,F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

(5) $\exists I \in \mathbb{Q}(x,t), J \in \mathbb{Q}(y,t)$ s.t. $I(x) = J(y)$ on the curve $\chi_{\mathfrak{S}}(x,y) = \frac{1}{t}$.

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▷ Many contributors (2010–2018): Bernardi, B., Bousquet-Mélou, Chyzak, Denisov, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Salvy, Wachtel

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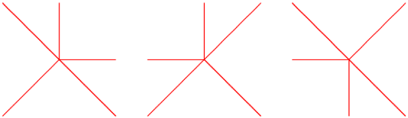
▷ Proof uses various tools: algebra, complex analysis, probability theory, computer algebra, etc.

[B., Raschel, Salvy, 2014]: $F_{\mathfrak{S}}(t;0,0)$ is not D-finite for the models



▷ For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

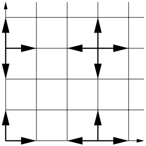
is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

[Denisov, Wachtel, 2015]

▷ The irrationality of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being D-finite.

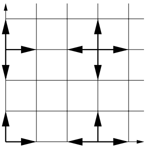
[Katz, 1970; Chudnovsky, 1985; André, 1989]

D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



The kernel $\mathcal{K} = 1 - t \cdot \sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of (x, y) into, respectively:

$$\left(\tfrac{1}{x}, y\right), \left(\tfrac{1}{x}, \tfrac{1}{y}\right), \left(x, \tfrac{1}{y}\right).$$



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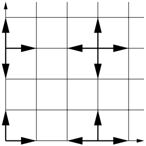
$$\mathcal{K}(t; x, y)xyF(t; x, y) = xy - txF(t; x, 0) - tyF(t; 0, y)$$



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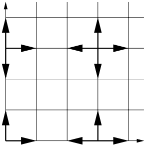

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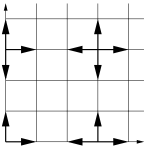

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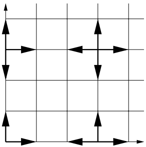

The kernel $\mathcal{K} = 1 - t \cdot \sum_{(i,j)\in\mathfrak{S}} x^i y^j = 1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)$ is invariant under the change of (x, y) into, respectively:

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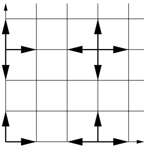
Summing up yields the orbit equation:

$$\sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xy\,F(t;x,y)\big) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{\mathcal{K}(t;x,y)}$$



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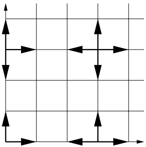
\end{aligned}$$

Taking positive parts yields:

$$[x^> y^>] \sum_{\theta\in\mathcal{G}} (-1)^\theta \theta\big(xy\,F(t;x,y)\big) = [x^> y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{\mathcal{K}(t;x,y)}$$



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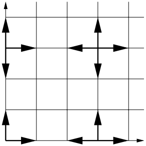
D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



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$$\mathsf{GF} = \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{kernel}}\right)$$

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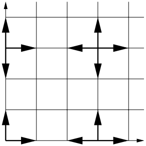

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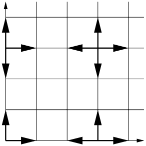
$$\text{GF} = \text{PosPart}\left(\frac{\text{OS}}{\text{ker}}\right) \text{ is D-finite [Lipshitz, 1988]}$$

D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



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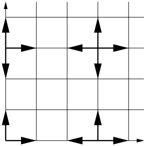
$$\mathsf{GF} = \mathsf{PosPart}\left(\frac{\mathsf{OS}}{\mathsf{ker}}\right)$$ is D-finite [Lipshitz, 1988]

▷ Argument works if $\mathsf{OS} \neq 0$: algebraic version of the reflection principle

D-Finiteness via the finite group [Bousquet-Mélou, Mishna, 2010]



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▷ Creative Telescoping finds a differential equation for PosPart(OS/ker)

Models with D-Finite $F(t;1,1)$

	OEIS	$\mathfrak{S}$	Pol size	LDE size	Rec size		OEIS	$\mathfrak{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: DF confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- ▷ Explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Models with D-Finite $F(t;1,1)$

	OEIS	$\mathfrak{S}$	algebraic?	asymptotics		OEIS	$\mathfrak{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$		$A=1+\sqrt{2},\ B=1+\sqrt{3},\ C=1+\sqrt{6},\ \lambda=7+3\sqrt{6},$			$\mu=\sqrt{\frac{4\sqrt{6}-1}{19}}$
9	A151302		N	$\frac{1}{3}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3}\sqrt{\frac{7}{3\pi}}\frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)}\frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi}\frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)}\frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

- ▷ Computerized discovery: conv. acc. + LLL/PSLQ [B., Kauers, 2009]
- ▷ Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]
- ▷ Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Explicit expressions and transcendence for models 1–19

Theorem [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let $\mathfrak{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t;x,y)$ at $(x,y) \in \{0,1\}^2$, only 4 are algebraic: for $\mathfrak{S} =$  at $(1,1)$, and $\mathfrak{S} =$  at $(1,0),(0,1),(1,1)$

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Example (King walks in the quarter plane, A025595)

F 
$$(t;1,1) = \frac{1}{t}\int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix} \,\middle|\, \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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▷ Computer-driven discovery and proof; no human proof yet.

▷ Proof uses creative telescoping, ODE factorization, ODE solving.

Bonus: hypergeometric functions occurring in $F(t;x,y)$

	$\mathfrak{S}$	occurring ${}_2F_1$	w		$\mathfrak{S}$	occurring ${}_2F_1$	w
1		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$16t^2$	11		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$
2		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$16t^2$	12		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
3		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2}{(12t^2+1)^2}$	13		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
4		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t(t+1)}{(4t+1)^2}$	14		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$
5		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$64t^4$	15		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$64t^4$
6		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$	16		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(t+1)}{(1-4t^2)^2}$
7		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{16t^2}{4t^2+1}$	17		${}_2F_1\left(\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\middle\vert\, w\right)$	$27t^3$
8		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$	18		${}_2F_1\left(\begin{matrix}\frac{1}{3}\ \frac{2}{3}\\ 1\end{matrix}\middle\vert\, w\right)$	$27t^2(2t+1)$
9		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19		${}_2F_1\left(\begin{matrix}\frac{1}{2}\ \frac{1}{2}\\ 1\end{matrix}\middle\vert\, w\right)$	$16t^2$
10		${}_2F_1\left(\begin{matrix}\frac{1}{4}\ \frac{3}{4}\\ 1\end{matrix}\middle\vert\, w\right)$	$\frac{64t^2(t^2+t+1)}{(12t^2+1)^2}$				

▷ All related to the complete elliptic integrals $\int_0^{\pi/2}(1-k^2\sin^2\theta)^{\pm\frac{1}{2}}\, d\theta$

Summary: walks with small steps in $\mathbb{N}^2$

quadrant models $\mathfrak{S}$: 79

$|\mathcal{G}_{\mathfrak{S}}| < \infty$: 23

$|\mathcal{G}_{\mathfrak{S}}| = \infty$: 56

orbit sum $\neq 0$: 19

orbit sum $= 0$: 4

asymptotics + GB

kernel method + CT

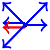
Guess'n'Prove

non-D-finite

D-finite

algebraic

Theorem: differential finiteness $\Longleftrightarrow$ finiteness of the group !

- 2D quadrant models: 527
  - $|\mathcal{G}_{\mathfrak{S}}| < \infty$: 118
    - orbit sum $\neq 0$: 95
      - kernel method: 94
        - **D-finite**
      - CA: 
        - **D-finite**
    - orbit sum $= 0$: 23
      - 22: reducible to Kreweras/Gessel
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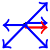
      - CA: 
        - **algebraic**
  - $|\mathcal{G}_{\mathfrak{S}}| = \infty$: 409
    - **non-D-finite?**

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]

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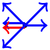
    - CA:  → **D-finite**
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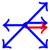
    - CA:  → **algebraic**
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[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017]

Question: differential finiteness $\iff$ finiteness of the group?

Answer: probably yes

Extensions: walks with small steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ million models, of which ≈ 11 million inherently 3D

3D octant models $\mathfrak{S}$ with ≤ 6 steps: 20804

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kernel method

D-finite

orbit sum $= 0$: 62

2D-reducible: 43

D-finite

not 2D-reducible: 19

non-D-finite?

$|\mathcal{G}_{\mathfrak{S}}| = \infty$: 20634

non-D-finite?

[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];
completed by [Bacher, Kauers, Yatchak, 2016]

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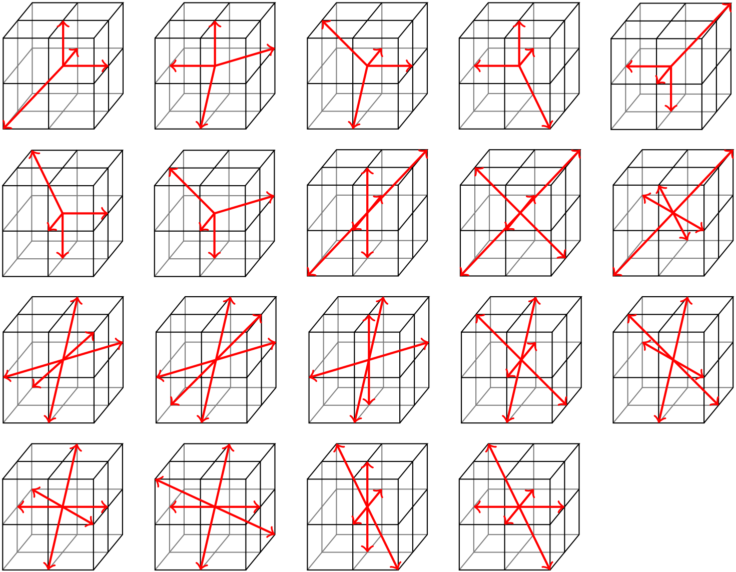
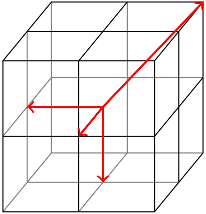




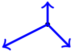
Two different computations suggest:

$$k_{4n} \approx C \cdot 256^n / n^{3.3257570041744...},$$

so excursions are very probably transcendental (and even non-D-finite) quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

- $|\text{orbit}| < \infty$: 240
  - $\text{OS} \neq 0$: 431 → **D-finite**
  - $\text{OS} = 0$: 9 → **D-finite?**
- $|\text{orbit}| = \infty$: 12 870
  - α rational: 16 → **non-D-finite?**
  - α irrational: 12 854 → **non-D-finite**

[B., Bousquet-Mélou, Melczer, 2018]

- Example: For the model 

$$xyF(t;x,y) = [x^{>0}y^{>0}]\frac{(x-2x^{-2})(y-(x-x^{-2})y^{-1})}{1-t(xy^{-1}+y+x^{-2}y^{-1})}$$

quadrant models with steps in $\{-2,-1,0,1\}^2$: 13 110

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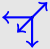
[B., Bousquet-Mélou, Melczer, 2018]

Question: differential finiteness $\iff$ finiteness of the orbit?

Answer: ?

Conjecture 1 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $F(t^{1/2};0,0)$ equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left(\frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1+36t)^2}\right) + \sqrt{1-12t} \cdot {}_2F_1\left(\begin{matrix} -\frac{1}{6} & \frac{2}{3} \\ & 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1-12t)^2}\right)\right).$$

Conjecture 2 [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function $F(t;0,0)$ equals

$$\frac{(1-24\,U+120\,U^2-144\,U^3)\,(1-4\,U)}{(1-3\,U)\,(1-2\,U)^{3/2}\,(1-6\,U)^{9/2}},$$

where $U = t^4 + 53\,t^8 + 4363\,t^{12} + \cdots$ is the unique series in $\mathbb{Q}[[t]]$ satisfying

$$U\,(1-2\,U)^3\,(1-3\,U)^3\,(1-6\,U)^9 = t^4\,(1-4\,U)^4.$$

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Thanks for your attention!