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"Length functions on Coxeter groups with applicationes".

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PLAN. 1. Positive definite functions on Coxeter group (W,S).

(a) $\mathcal{B}_q(x) = q^{|x|}$, where for $x \in W$, $-1 \le q \le 1$

$|x| = \min\{k : x = s_1 s_2 .. s_k,\ s_j \in S\}$, „natural length"

(b) $Q_t(x) = t^{\|x\|}$, where

$\|x\| = \mathrm{card}\{s_1, s_2, \dots, s_k\}$, and $0 \le t \le 1$.

(c) $\mathcal{R}_t(x) = \prod^{*}_{s \in b(x)} (1 + t(s)\delta_s)$, where

$t : S \to [0,1]$, $b(x) = \{s_1, s_2, \dots, s_k\}$

R_t is called Coxeter-Riesz product.

2. Applications : (a) q- CCR relations $, -1 \leq q \leq 1$

$$\boxed{b_i b_j^* - q\, b_j^* b_i = \delta_{ij} I}\ , i,j=1,\dots N.$$

(b) New type $\underline{\text{II}}_1$ factors, - $VN_q(N)$

(c) Free ID of normal law $N(0,1)$, and also:

$$\frac{1}{\cosh(x)} = g_{1/2}(x)\ , \quad g_t(x) = c_t |\Gamma(t + i x)|^2, x \in \mathbb{R}$$

Meixner - Pollaczek laws

(d) Random matrices and generalized Brownian processes related to new positive definite functions on $S(\infty) = \bigcup_n S(n)$.

Ad 1(a). Coxeter group (W, S), $S = \{s_1, \dots, s_N\}$,

$W = gp(S)$, $s_i^2 = 1 = (s_i s_j)^{m_{ij}}$, where

$m_{ii} = 1$, $m_{ij} = m_{ji} \in \mathbb{N}$, $m_{ij} \geqslant 2$, $i \neq j$

Coxeter group $\supset$ Weyl groups = gps generated by orthogonal reflexions in $\mathbb{R}^k$ (by „root system").
(Cantor)

Examples: 1. Rademacher group

$$W^N = \underbrace{\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2}_{N} = gp\{r_1, r_2, \dots, r_N\},$$

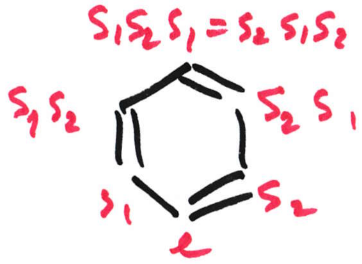
$r_i^2 = 1 = (r_i r_j)^2 \Rightarrow r_i r_j = r_j r_i$ (Rademacher system)

Ex. 2 Dyhedral group $D_N = gp(s_1, s_2)$: symetry gp of N-gon

$$s_i^2 = 1 = (s_1 s_2)^N$$

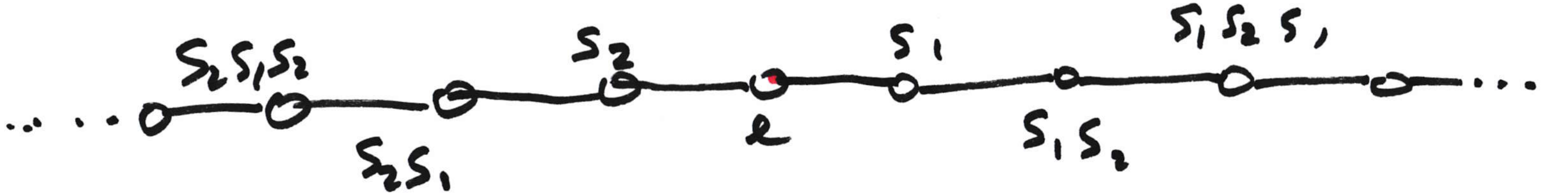
N=3.

$D_3 = S(3)$



$N=\infty$, $D_\infty = gp(s_1,s_2) = \mathbb{Z}_2 * \mathbb{Z}_2$, $s_i^2 = 1$, no other relations,

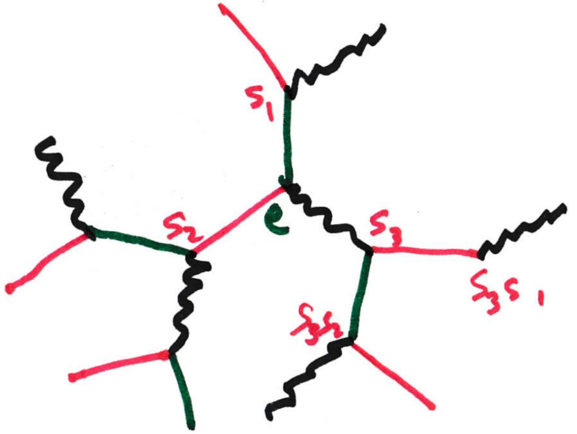
Cayley graph:



Free Coxeter group: $FC_N = \underbrace{\mathbb{Z}_2 * \mathbb{Z}_2 * \dots *}_{N} \mathbb{Z}_2 = gp\{s_1, s_2, \dots, s_N\}$

with only relations: $s_i^2 = 1$.

N=3 Cayley graph : $FC_3 = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$:



Main example of Coxeter (Weyl grp $A_N = S_{N+1}$ - the permutation gp:
= the symmetry group of (N+1)-simplex :

$S_{N+1} = gp\{\pi_1, \pi_2, \dots, \pi_N\}$: $\pi_i = (i, i+1) =$

Weyl group of type $B_N = S_N \ltimes \underbrace{\mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2}_{N} = gp\{\pi_0, \pi_1, \dots, \pi_{N-1}\}$

$= \{\sigma \in S_{2N}(\pm 1, \pm 2, \dots \pm N) : \sigma(-k) = -\sigma(k)\}$.

$\pi_0 =$

Important ex. : Affine Weyl groups → Demazure operators.

Th. (Bourbaki) For each $m_{ij} = m_{ji} \in \mathbb{N}$, $\underset{i \neq j}{m_{ij}} \geqslant 2$, $m_{ii} = 1$

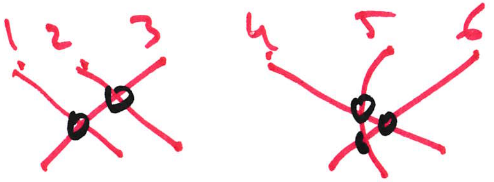
$\exists!$ Coxeter group (W, S) with relation

$$(s_i s_j)^{m_{ij}} = 1 .$$

Def: Natural length function : $W \ni x \rightarrow |x| = \min\{k : x = s_{i_1} .. s_{i_k} : s_j \in S\}$

i.e. $|x| = k$, if $x = s_{i_1} s_{i_2} .. s_{i_k}$, $S = \{s_1, \dots, s_n\}$

(minimal decomposition)

Ex. $W = S_N$, $x =$ 

, $|x| = 5 = cr(x)$

$$|x| = \text{card}\left\{ (i,j) : \begin{matrix} i < j \\ x(i) > x(j) \end{matrix} \right\} = cr(x) \text{, the number of crossings.}$$

Theorem 1. (MB + T. Januszkiewicz + R. Spatzier) For each $-1 \leq q \leq 1$

$$P_q(x) = q^{|x|}$$ is positive definite function on (W,S),

i.e the matrix: $$\left(q^{|y'x|}\right)_{x,y\in W}$$ is pos. define on $\ell^2(W)$

or $\exists$ unitary representation $\pi : W \to \mathcal{U}(\mathcal{H})$:

$$P_q(x) = <\pi_q(x)\xi_q|\xi_1>$$, for some $\xi_q \in \mathcal{H}$.

Corollary 1. If Coxeter group (W,S) is infinite, then W is not Kazhdan (property T) group.

For stronger applications we have :
(MB + R.Speicher + R. Szwarc)

Theorem 2. Let $\varphi : (W, S) \to B(\mathcal{H})$ - bounded oper. on a Hilbert space $\mathcal{H}$

s.t :

1. $\varphi(s) = \varphi(s)^*$, $s \in S$, $\|\varphi(s)\| \leq 1$.

2. $\varphi(xy) = \varphi(x)\varphi(y)$, if $|xy| = |x| + |y|$, $x, y \in W$

then φ is operator - pos. definite, i.e :

$$\sum_{x,y \in W} \langle \varphi(y^{-1}x)\, \alpha(x) \,|\, \alpha(y) \rangle \geq 0,$$ for each

$\alpha : W \to \mathcal{H}$, supp(α) is finite.

Main application for $(S(N), \pi_1, \pi_2, \dots, \pi_{N-1})$, $\pi_j = (j, j+1)$

Corollary: If $T : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is contraction $(\|T\| \leq 1)$ and also Yang-Baxter: $T_1 T_2 T_1 = T_2 T_1 T_2$, where

$$T_j : \mathcal{H}^{\otimes N} \to \mathcal{H}^{\otimes N}$$

, defined as follows:

$$T_1(\underbrace{x_1 \otimes x_2} \otimes x_3 \otimes \dots \otimes x_N) = T(\underbrace{x_1 \otimes x_2}) \otimes x_3 \otimes \dots \otimes x_N,$$

$$T_2(x_1 \otimes \underbrace{x_2 \otimes x_3} \otimes x_4 \otimes x_N) = x_1 \otimes T(\underbrace{x_2 \otimes x_3}) \otimes x_4 \otimes \dots \otimes x_N$$

then the function $P_T^{(N)} : S(N) \to B(\mathcal{H}^{\otimes N})$, defined as

$$P_T^{(N)}(\pi_j) = T_j$$

and $P_T^{(N)}(\pi_{i_1} \pi_{i_2} \dots \pi_{i_k}) = T_{i_1} T_{i_2} \dots T_{i_k}$, is (oper) positive defin.

Ex. Take $\mathcal{H} = \text{lin}\{e_1, e_2, \ldots, e_k\}$, and $q_{ij} = \overline{q_{ji}}$, $|q_{ij}| \leq 1$

and

$$T(e_i \otimes e_j) = q_{ij}(e_j \otimes e_i)$$, is Yang-Baxter op.

And positivity of $P_T^{(N)}$ implies existence of q-CCR relations:

$$\boxed{a_i a_j^* - q_{ij} a_j^* a_i = \delta_{ij} I}$$

(MB + R. Speicher 1991)

and existence of q-Brownian motion for $-1 \leq q \leq 1$

$q = 1$, classical BM

$q = -1$ Fermi BM

$q = 0$ Free BM (Voiculescu)

$\boxed{G_i = a_i + a_i^*}$ – q – Gaussian

r. variables

Def: Block (colored) length function on (W, S)

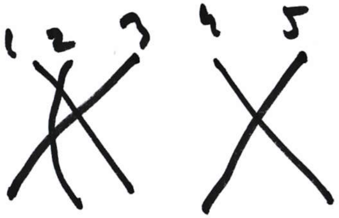
Let $x \in W$, $x = s_{i_1} \cdots s_{i_k}$ - minimal representation, $b(x) = \{s_{i_1}, \dots, s_{i_k}\}$ „colors"

The coloured length: $W \ni x \to \|x\| = \mathrm{card}\, b(x)$. (well defined!)

Ex. $W = (S_N, \pi_1, \pi_2, \dots \pi_{N-1})$, $\pi_j = (j, j+1)$ (simple transposition)

Then for $\sigma \in S_N$

$$\|\sigma\| = N - \#\text{ connected components of } \sigma :$$

$\sigma =$ 

$\pi_1 \pi_2 \pi_1 \pi_4$

, $b(\sigma) = \{\pi_1, \pi_2, \pi_4\}$

$\|\sigma\| = 5 - 2 = 3 = \mathrm{card}\, b(\sigma)$.

Theorem 3 (MB + R. Speicher). For $0 < t \leq 1$

$Q_t(x) = t^{\|x\|}$ is positive definite on (W, S).

Remark: Let $\gamma_1 = N(0,1)$ – the normal r.v $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$

then the prob. measure $\underbrace{\gamma_1 \boxplus \gamma_1 \boxplus \dots \boxplus \gamma_1}_{N} = \gamma_1^{\boxplus N}$

free convolution

has the following moments:

$$\int_{-\infty}^{\infty} u^{2k}\, d(\gamma_1^{\boxplus t}) = m_{2k}(\gamma_1^{\boxplus t}) = \sum_{V \in P_2(2k)} t^{k - \|V\|} = \sum_{V \in P_2(2k)} t^{cc(V)}$$

where $V = B_1 \cup B_2 \cup .. \cup B_k$ – 2-partition on $\{1, 2, .., 2k\}$

$\|V\| = k - cc(V)$, # connected components.

Def: Coxeter - Riesz product on (W, S).

For $\underline{t} : S \longrightarrow [0,1]$, we define for $x \in W$

$$\mathcal{R}_{\underline{t}}(x) = \prod_{s \in b(x)} t(s)$$

$$\mathcal{R}_{\underline{t}} = 1 + \sum_{s \in S} t(s)\delta_s + \sum_{x : b(x) = \{s_i, s_j\}} t(s_i)t(s_j)\delta_{\text{[illegible]}j} + \sum_{x : b(x) = \{s_i, s_j, s_k\}} t(s_i)t(s_j)t(s_k)\delta_x + \dots$$

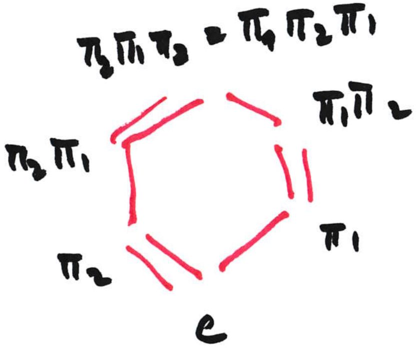
Theorem 4 (MB + Ś. Gal + W. Młotkowski) For each Coxeter group (W, S) and $t : S \longrightarrow [0,1]$, $\mathcal{R}_t$ is positive definite on W.

(19')

Theorem (MB + Gal + M Lotkowski) („Tokyo formula"

For arbitrary finite Coxeter group (W, S) and $q \in \mathbb{C}$

(1) $$\sum_{x \in W} (-1)^{|x|} q^{\|x\|} = (1-q)^{\mathrm{card}(S)} .$$



(2) $$\sum_{x \in W} (-1)^{|x|} R_{\underline{t}}(x) = \prod_{s \in S} (1 - t(s))$$

Remark: If $t(s) = t$, for $s \in S$, then for $x \in (W, S)$

$$\mathcal{R}_t(x) = Q_t(x) = t^{\|x\|} .$$

Def: **Sidon set** $E \subset W$ is by definition, such that for each $f \in \ell^\infty(E)$, bounded by 1, $|f(x)| \le 1$, there exists a function $\varphi \in \text{lin}\{\text{positive definite on } W\}$ such that

$$\varphi(x) = f(x), \quad \text{for } x \in E.$$

Ex 1. $W = (\mathbb{Z}, +)$, $E = \{3^k : k = 1, 2, 3, \dots\}$ is a **Sidon set**

2. $W = \bigoplus_{i=1}^{\infty} \mathbb{Z}_2^{(i)}$ (Cantor group), $\mathbb{Z}_2^{(i)} = \{e, r_i\}$, then the set $E = \{r_1, r_2, r_3, \dots\}$ is a **Sidon set**, since

$$\varphi = \boxed{\ast}_{i=1}^{\infty} (1 + f(r_i) r_i)$$ is pos. def. if $-1 \le f(r_i) \le 1$.

(this is so called **classical Riesz product function**. (measure $\mu = \hat{\varphi}$))

Since $R_t(s) = t(s)$, $s \in S$, therefore we get

Corollary: S is a Sidon set in (W, S).

and

Corollary: We have Haagerup-Pisier-Khinchine inequality: For arbitrary matrices $e_i \in M_N(\mathbb{C})$ let Schatten norm C_p:

$\|e_i\|_{C_p} = \{tr(|e_i|^p)\}^{1/p}$, $p \geq \cancel{1} 2$, Then the following inequality holds:

$$\max\{\|\sum e_i e_i^*\|^{1/2}_{C_p}, \|\sum e_i^* e_i\|^{1/2}_{C_p}\} \leq \Big\| \sum_{s_i \in S} e_i \otimes \lambda(s_i) \Big\|_{L^p(VN(W)) \otimes C_p} \leq K_p(W) \overset{\max}{\begin{matrix} \|\sum_i e_i e_i^*\|^{1/2}_{C_p}, \\ \|\sum_i e_i^* e_i\|^{1/2}_{C_p} \end{matrix}}$$

where $\lambda(x) : \ell^2(W) \to \ell^2(W)$, the left regular representation on W, $\lambda(x)\delta_g = \delta_{xg}$, $x, g \in W$

Corollary: For $a_j \in \mathbb{C}$, and Coxeter grp (W,S) we have for $p>2$:

$$\left(\sum_j |a_j|^2\right)^{1/2} \leq \Big\| \sum_{s_j \in S} a_j \lambda(s_j) \Big\|_{L^p(VN(W))} \leq K_p(W) \left(\sum_j |a_j|^2\right)^{1/2}$$

Open problem: We have proved that $K_p(W) \leq A \cdot \sqrt{p}$ for all Coxeter grp.

For $W =$ free Coxeter grp, $K_p(FC) \leq 2$,

even more
$$K_{2n}(FC) = \frac{1}{n+1}\binom{2n}{n},$$
M Bożejko 1975

This is equivalent with the Free Central Limit theorem

Problem: Find $K_3(FC) = ?$, in general $K_p(W)$ for large class of Coxeter groups?

Charact. Radial pos. def. functions on $S(\infty) = \bigcup S(N)$.

Th. Let $\varphi : S(\infty) \longrightarrow \mathbb{R}$, $\varphi(x) = \varphi(y)$, if $|x| = |y|$

$|\varphi(x)| \leq 1$, $\varphi(e) = 1$, φ is pos. definite on $S(\infty)$, iff

$$\boxed{\varphi(x) = \int_{-1}^{1} q^{|x|} \, d\mu(q)}$$

, for an unique prob. measure μ on $[-1, 1]$.

Problem: Show that for $q \neq 0$, $-1 \leq q \leq 1$

$\varphi_q(x) = \langle \pi_q(x) \xi | \xi \rangle$, π_q is irreducible unitary rep. of $S(\infty)$?

We knew that π_q is factorial rep.

Appl 2(b) · q-CCR relations } q-determinant

q-Fock spaces

q-Gaussian BM .

q-Von Neumann algebra $\Gamma_q(\mathbb{R}^N) = \Gamma_q(N)$.

App 2(c) : Meixner-Pollaczek (paper with T. Hasebe - 2013.)

Appl 2(e) : Random matrices and gen. Brownian proc

(paper with W. Bożejko, - 2013) .

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q-determinant :

$$\det_q[(a_{ij})_{i,j=1}^N] = \sum_{\sigma \in S(N)} q^{|\sigma|} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

Th. For $-1 \le q \le 1$, $\det_q(\langle \xi_i | \xi_j \rangle) \geq 0$

· q - Fock space: Free Fock space: $\mathcal{F}_0(\mathcal{H}) = \bigoplus_{N=0}^{\infty} \mathcal{H}^{\otimes N}$

If on $\mathcal{H}^{\otimes N}$, we introduce q - scalar product:

$$\langle \xi_1 \otimes \xi_2 \otimes \cdots \xi_N | \xi_1 \otimes \cdots \xi_N \rangle_q = \det_q(\langle \xi_i | \xi_j \rangle)$$

Then we get q - Fock space as the completion of $\mathcal{F}_0(\mathcal{H})$ with respect $\langle , \rangle_q$ - norm.

$q = 1$, we get symmetric Fock space

$q = -1$ antisymmetric Fock space