Tropical Combinatorial Nullstellensatz and Fewnomials

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Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.

If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \max$, $\otimes := +$.

If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.

Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_{-\infty} := \mathbb{Z}^+ \cup \{-\infty\}$ are commutative tropical semi-rings. $-\infty$ plays a role of 0, in its turn 0 plays a role of 1;

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• $n \times n$ matrices over $\mathbb{Z}_{-\infty}$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

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Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.

Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \max_j\{Q_j\}$;

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(Classical) Weak Combinatorial Nullstellensatz

Polynomial $h \in F[X_1, \dots, X_n]$ over any field F (not necessarily algebraically closed unlike Hilbert's Nullstellensatz) with $deg_{X_i}(h) \leq d_i$, $1 \leq i \leq n$. Consider subsets $D_i \subset F$, $|D_i| > d_i$, $1 \leq i \leq n$.

Theorem

*(**N. Alon, 1999**). h can't vanish on the grid $D_1 \times \cdots \times D_n$.*

It provides a deterministic test, whether a polynomial with given degrees in each variable, vanishes identically.

Support $Supp(h) \subset \mathbb{Z}^n$ is the set of multiindices $I = (i_1, \dots, i_n)$ such that monomial $X^I = X_1^{i_1} \cdots X_n^{i_n}$ occurs in h.

Conjecture (J.-J. Risler - F. Ronga, 1990). *h* can't vanish on $Supp(h)$.

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We call a pair of sets $S, D \subset \mathbb{R}^n$ *iso-ordered* if there exists a bijection $g : S \to D$ such that for any two points $(x_1, \dots, x_n), (y_1, \dots, y_n) \in S$ we have $x_i \le y_i \Leftrightarrow g(x_i) \le g(y_i)$, $1 \le i \le n$.

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A tropical polynomial f in n variables can't vanish on any set $D \subset \mathbb{R}^n$ iso-ordered to $Supp(f)$.

For $deg_{X_i}(f) \le d_i$, $1 \le i \le n$ and a grid $D = D_1 \times \cdots \times D_n$, $|D_i| = d_i + 1$ we get a tropical analog of Alon's weak combinatorial Nullstellensatz.

For $D = Supp(f)$ we get a tropical analog of Risler-Ronga conjecture.

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We call a pair of sets $S, D \subset \mathbb{R}^n$ *iso-ordered* if there exists a bijection $g : S \to D$ such that for any two points $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in S$ we have $x_i \leq y_i \Leftrightarrow g(x_i) \leq g(y_i)$, $1 \leq i \leq n$.

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A tropical polynomial f in n variables can't vanish on any set $D \subset \mathbb{R}^n$ iso-ordered to $Supp(f)$.

For $deg_{X_i}(f) \leq d_i$, $1 \leq i \leq n$ and a grid $D = D_1 \times \cdots \times D_n$, $|D_i| = d_i + 1$ we get a tropical analog of Alon's weak combinatorial Nullstellensatz.

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(**Schwartz-Zippel, 1979**) *For a set $S \subset \mathbb{C}$ a polynomial $h \in \mathbb{C}[X_1, \dots, X_n]$ can vanish at most at*

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The bound is sharp and provides a probabilistic test, whether a polynomial of a given degree vanishes identically.

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Let $D_1, \dots, D_n \subset \mathbb{R}$, $d < \min_{1 \le i \le n}\{|D_i|\}$. A tropical polynomial f either of degree d or $deg_{X_i}(f) \le d$, $1 \le i \le n$ can vanish at most at

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Let a polynomial $h = \sum_{1 \leq j \leq k} a_{I_j} X^{I_j} \in \mathbb{C}[X_1, \dots, X_n]$ be k-sparse. For pairwise distinct primes $p_1, \dots, p_n$ denote a point $s_i := (p_1^i, \dots, p_n^i) \in \mathbb{Z}^n$.

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Tropical Universal Testing Set over $\mathbb{R}$

Theorem

The minimal size of a universal testing set over $\mathbb{R}$ for k-sparse tropical polynomials equals k.

Tropical Universal Testing Sets over $\mathbb{Q}$

Denote by $s(k, n)$ the minimal size of universal testing sets over $\mathbb{Q}$ for k-sparse tropical polynomials in n variables.

Theorem

- *(Exact Bound for 2 variables)* $s(k, 2) = 2k - 1$;
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Denote by $k(s, n)$ the function inverse to $s(k, n)$, i. e. the minimal number k such that for any set S with s points in $\mathbb{Q}^n$ there exists a k-sparse tropical polynomial in n variables vanishing on S.

Geometrically, a tropical polynomial is a convex piece-wise linear function. Its domains of linearity are convex polyhedra in $\mathbb{R}^n$, and its tropical roots are the boundaries of these polyhedra, thus each root belongs to at least two of these polyhedra.

For a set $S \subset \mathbb{Q}^n$ its *single* (respectively, *double*) *covering* is a family of pairwise disjoint (in their interiors) polytopes with the vertices in S not containing points of S in their interiors such that every point of S belongs to at least one (respectively, two) of the polytopes. Denote by $k_1(s, n)$ (respectively, $k_2(s, n)$) the minimal number of polytopes that suffices to single (respectively, double) cover any s points in $\mathbb{Q}^n$.

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For a set $S \subset \mathbb{Q}^n$ its *single* (respectively, *double*) *covering* is a family of pairwise disjoint (in their interiors) polytopes with the vertices in S not containing points of S in their interiors such that every point of S belongs to at least one (respectively, two) of the polytopes. Denote by $k_1(s, n)$ (respectively, $k_2(s, n)$) the minimal number of polytopes that suffices to single (respectively, double) cover any s points in $\mathbb{Q}^n$.

Lemma

$k(s, n) \geq k_2(s, n)$.

Constructive Upper Bound

Denote by $E(n)$ the maximal number such that any large enough set of points in $\mathbb{R}^n$ contains a subset of $E(n)$ points being vertices of a convex polytope without points of this set in the interior of the polytope.

Theorem

*(**P. Valtr, 1992**) • $E(n)$ is bounded from above by an explicit function of a factorial growth;*
• $7 \leq E(3) \leq 22$.

Lemma

• $k_1(s, n) \geq s/E(n)$ for large enough s;
• $k_2(s, n) \geq k_1(s, n)$.

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Small complexity tropical formula has small number of roots

Consider a tropical formula admitting gates max, $+$ and multiplying by a positive integer (positive reals would be also allowed if to extend the concept of tropical polynomials). The latter operation is the tropical taking a power. The size of a formula is the number of max, $+$ in it (so, the tropical powering is for free).

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Tropical circuit admits $\oplus := \max$, $\otimes := +$ (defines a convex piece-wise linear function).

Rational tropical circuit admits $\oplus$, $\otimes$, $\oslash$ (defines a piece-wise linear function).

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Consider a rational tropical function
$t := 1 \oslash x^{\otimes 2} \oplus (-1) \otimes x^{\otimes 2} = \max\{-2x + 1, 2x - 1\}$.

Lemma

(borrowed from the deep learning).
k-th iteration t_k of t has 2^k intervals of linearity (on the interval $[0, 1]$).

Transform the rational tropical circuit computing $t_1, t_2, \ldots$ into a tropical circuit computing $r_1, s_1, r_2, s_2, \ldots$ being tropical numerators and denominators of $t_k = r_k \oslash s_k$ according to the usual rules of adding, multiplying, dividing tropical fractions. Then one of r_k, s_k has at least 2^{k-1} tropical roots due to the Lemma.

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Transform the rational tropical circuit computing $t_1, t_2, \dots$ into a tropical circuit computing $r_1, s_1, r_2, s_2, \dots$ being tropical numerators and denominators of $t_k = r_k \oslash s_k$ according to the usual rules of adding, multiplying, dividing tropical fractions. Then one of r_k, s_k has at least 2^{k-1} tropical roots due to the Lemma.

Tropical Linear Variety

$K = \mathbb{C}((t^{1/\infty})) = \{c = c_0 t^{i_0/q} + c_1 t^{(i_0+1)/q} + \cdots\}$

is a field of Puiseux series where $i_0 \in \mathbb{Z}$, $1 \leq q \in \mathbb{Z}$.

Consider a linear system $A \cdot X = b$ with $m \times n$ matrix $A = (a_{i,j})$ and vector $b = (b_i)$ over K. Denote by $P \subset K^n$ the linear plane determined by this system.

Tropicalization $Trop(c) = i_0/q$, $Trop(0) = \infty$.

The closure in the Euclidean topology $V := \overline{Trop(P)} \subset \mathbb{R}^n$ is called a **tropical linear variety**.

We study the complexity of the problem of recognizing a tropical linear variety, i.e., for a given real algebraic vector $v = (v_1, \ldots, v_n) \in (\mathbb{R} \cap \overline{\mathbb{Q}})^n$ to test whether $v \in V$.

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Algorithmical Representation of Puiseux Series

Primitive element $z \in K$ is given as a root of a polynomial equation $h(t, z) = 0$ where $h \in \mathbb{Z}[t, Z]$, and by means of a further specifying a beginning of the expansion of z as a Puiseux series over the field $\overline{\mathbb{Q}}$ of algebraic numbers (to make a root of h to be unique with this beginning of the expansion). One can produce such an expansion within the polynomial complexity (Chistov).

Also we are supplied with rational functions $h_{i,j}, h_i \in \mathbb{Q}(t)[Z]$ such that $a_{i,j} = h_{i,j}(z)$, $b_i = h_i(z)$, $1 \leq i \leq m$, $1 \leq j \leq n$.

We suppose that $\deg(h), \deg(h_{i,j}), \deg(h_i) \leq d$.

In addition, we assume that each rational coefficient of the polynomials $h, h_{i,j}, h_i$ is given as a quotient of a pair of integers with absolute values less than 2^M. The latter means that the bit-size of this rational number is bounded by $2M$.

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Reduction: from Puiseux to Power Series

First, the algorithm cleans the denominator in the exponents of the Puiseux series of z replacing $t^{1/q}$ by t for a suitable $q \leq d$ to make z to be a Laurent series with integer exponents (and keeping the same notation for z, h, $h_{i,j}$, h_i). The coordinates of the vector v we also multiply by s and keep the same notation for v.

We say that two coordinates v_{j_1}, v_{j_2} of v are *congruent* if $v_{j_1} - v_{j_2} \in \mathbb{Z}$. Consider any solution x of $A \cdot X = b$. For each congruence class $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}$ of v select from x all the monomials with the exponents which belong to α, denote by $x^{(\alpha)} := (x_1^{(\alpha)}, \dots, x_n^{(\alpha)})$ the resulting vector consisting of these selected subsums of $x_1, \dots, x_n$.

Then $x^{(0)}$ which corresponds to the congruence class of the integers satisfies $A \cdot X = b$, and any other $x^{(\alpha)}$ with $\alpha \notin \mathbb{Z}$ satisfies the homogeneous linear system $A \cdot x^{(\alpha)} = 0$, hence $A \cdot (t^{-\alpha} \cdot x^{(\alpha)}) = 0$ and $t^{-\alpha} \cdot x^{(\alpha)}$ is a Laurent series.

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Partitioning the Coordinates of a Vector into Congruence Classes

Lemma

Vector $v \in \overline{Trop(P)}$ iff the conjunction of the following statements for all the congruence classes $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}$ holds. System $A \cdot X = b$ when $\alpha \in \mathbb{Z}$ (or respectively, the homogeneous system $A \cdot X = 0$ when $\alpha \notin \mathbb{Z}$) has a solution $x = (x_1, \dots, x_n)$ in Laurent series $x_1, \dots, x_n$ satisfying the conditions either $Trop(x_j) + \alpha = v_j$ when v_j belongs to the congruence class of α, or $Trop(x_j) + \alpha > v_j$ otherwise, $1 \le j \le n$.

We assume that the vector v is provided in the following way. A primitive real algebraic element $u \in \overline{\mathbb{Q}} \cap \mathbb{R}$ is given as a root of a polynomial $g \in \mathbb{Z}[Y]$ together with specifying a rational interval $[e_1, e_2]$ which contains the unique root u of g. In addition, certain polynomials $g_j \in \mathbb{Q}[Y]$, $1 \le j \le n$ are given such that $v_j = g_j(u)$. We suppose that $\deg(g), \deg(g_j) \le d$ and that the absolute values of the numerators and denominators of the (rational) coefficients of g, g_j and of e_1, e_2 do not exceed 2^M.

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We assume that the vector v is provided in the following way. A primitive real algebraic element $u \in \overline{\mathbb{Q}} \cap \mathbb{R}$ is given as a root of a polynomial $g \in \mathbb{Z}[Y]$ together with specifying a rational interval $[e_1, e_2]$ which contains the unique root u of g. In addition, certain polynomials $g_j \in \mathbb{Q}[Y]$, $1 \leq j \leq n$ are given such that $v_j = g_j(u)$. We suppose that $\deg(g), \deg(g_j) \leq d$ and that the absolute values of the numerators and denominators of the (rational) coefficients of g, g_j and of e_1, e_2 do not exceed 2^M.

Partitioning the Coordinates of a Vector into Congruence Classes

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Shifting Vector to Zero

To detect whether for a pair of the coordinates the congruence $v_{j_1} - v_{j_2} \in \mathbb{Z}$ holds, the algorithm computes an integer approximation $e \in \mathbb{Z}$ of $|v_{j_1} - v_{j_2} - e| < 1/2$ (provided that it does exist) and then verifies whether $v_{j_1} - v_{j_2} = e$. This supplies us with the partition of the coordinates $v_1, \dots, v_n$ into the classes of congruence.

Thus, for the time being we fix a congruence class α. The algorithm searches for vectors $x = (x_1, \dots, x_n)$ satisfying the conditions in Lemma.

Denote by $J \subset \{1, \dots, n\}$ the set of j such that v_j belongs to the fixed congruence class. For every $j \in J$ we replace $a_{i,j}$ by $t^{v_j - \alpha} \cdot a_{i,j}$, $1 \le i \le m$. For every $j \notin J$ let $\alpha + s - 1 < v_j < \alpha + s$ for a suitable (unique) integer s, then we replace $a_{i,j}$ by $t^s \cdot a_{i,j}$, $1 \le i \le m$.

After this replacement the algorithm searches for vectors $x = (x_1, \dots, x_n)$ satisfying the properties $Trop(x_j) = 0$, $j \in J$ and $Trop(x_j) \ge 0$, $j \notin J$.

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Truncation of Puiseux/Power Series

Then by elementary transformations with the rows of matrix A over the quotient-ring $\mathbb{Q}(t)[Z]/h$ and an appropriate permutation of columns, the algorithm brings A to the form $a_{i,i} = 1,\ a_{i,j} = 0,\ 1 \le i \ne j \le m$ (one can assume w.l.o.g. that $rk(A) = m$).

For $m < j \le n$ denote $r_j := -\min_{1 \le i \le m}\{Trop(a_{i,j})\}$. If $r_j < 0$ we put the coordinate $x_j = 1$. Else if $r_j \ge 0$ we put $x_j = y_{j,0} + y_{j,1} \cdot t + \cdots + y_{j,r_j} \cdot t^{r_j}$ with the indeterminates $y_{j,0}, \dots, y_{j,r_j}$ over $\overline{\mathbb{Q}}$.

Below w.l.o.g. we carry out the calculations for the case of the congruence class of integers $\alpha \in \mathbb{Z}$. When $\alpha \notin \mathbb{Z}$ one should put below $b_i = 0,\ 1 \le i \le m$ (cf. Lemma).

For $1 \le i \le m$ denote $s_i = \min_{m < j \le n}\{Trop(a_{i,j}),\ Trop(b_i)\}$. The i-th equation of system $A \cdot X = b$ one can rewrite as $x_i + \sum_{m < j \le n} a_{i,j} \cdot x_j = b_i$.

For every $s_i \le k \le 0$ one can express the coefficient of $\sum_{m < j \le n} a_{i,j} \cdot x_j - b_i$ at the power t^k as a linear function $L_{i,k}$ over $\overline{\mathbb{Q}}$ in the indeterminates $Y := \{y_{j,l},\ m < j \le n,\ 0 \le l \le r_j\}$.

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Reduction to a System of Linear Equations and Inequations

Consider the linear system

$L_{i,k} = 0,\ 1 \leq i \leq m,\ s_i \leq k < 0$

in the indeterminates Y. The algorithm solves this system and tests whether each of n linear functions from the family

$L := \{L_{i,0},\ i \in J, 1 \leq i \leq m;\ y_{j,0},\ j \in J, m < j \leq n\}$

does not vanish identically on the space of solutions of the system.

If all of them do not vanish identically then take any values of Y which fulfil the system with non-zero values of all the linear functions from the family L. Then equation $x_i + \sum_{m<j\leq n} a_{i,j} \cdot x_j = b_i$ determines uniquely x_i with $Trop(x_i) = 0$ when $i \in J$ and $Trop(x_i) \geq 0$ when $i \notin J$. This provides a solution x of the input system $A \cdot X = b$ satisfying $Trop(x_j) = 0$ when $j \in J$ and $Trop(x_j) \geq 0$ when $j \notin J$ (cf. Lemma).

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If all of them do not vanish identically then take any values of Y which fulfil the system with non-zero values of all the linear functions from the family L. Then equation $x_i + \sum_{m<j\le n} a_{i,j} \cdot x_j = b_i$ determines uniquely x_i with $Trop(x_i) = 0$ when $i \in J$ and $Trop(x_i) \ge 0$ when $i \notin J$. This provides a solution x of the input system $A \cdot X = b$ satisfying $Trop(x_j) = 0$ when $j \in J$ and $Trop(x_j) \ge 0$ when $j \notin J$ (cf. Lemma).

Otherwise, if some of the linear functions from the family L vanishes identically on the space of solutions of the system then the input system $A \cdot X = b$ has no solutions satisfying the conditions of Lemma.

Reduction to a System of Linear Equations and Inequations

Consider the linear system

$L_{i,k} = 0,\ 1 \leq i \leq m,\ s_i \leq k < 0$

in the indeterminates Y. The algorithm solves this system and tests whether each of n linear functions from the family

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Employing Vandermonde Matrix to Solve a System of Linear Equations/Inequations

To test the above requirement of identically non-vanishing of the linear functions from the family L the algorithm finds a basis $w_1, \dots, w_r \in (\overline{\mathbb{Q}})^N$ and a vector $w \in (\overline{\mathbb{Q}})^N$ where $N = |Y|$ such that the r-dimensional space of solutions of the linear system is the linear hull of the vectors $w_1, \dots, w_r$ shifted by the vector w.

If each linear function from the family L does not vanish identically on this space then all of them do not vanish on at least one of the vectors from the family

$F := \{w + \sum_{1 \le l \le r} p^l \cdot w_l,\ 1 \le p \le |J|r + 1 \le nr + 1\}$

because any linear function can vanish on at most of r vectors from F due to the non-singularity of the Vandermonde matrices.

So, the algorithm substitutes each of the vectors of F into the linear functions from L and either finds a required one Y or discovers that the input system $A \cdot X = b$ has no solution satisfying the conditions of Lemma.

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Complexity

To estimate the complexity of the designed algorithm observe that it solves the linear system of the size bounded by a polynomial in n, d with the coefficients from a finite extension of $\mathbb{Q}$ having the bit-size less than linear in M and polynomial in n, d. Thus, the algorithm solves this system within the complexity polynomial in M, n, d , by a similar magnitude one can bound the complexity of the executed substitutions

Theorem

There is an algorithm which for a tropical linear variety $V := \overline{Trop(P)}$ defined by a linear system $A \cdot X = b$ over the field K of Puiseux series, recognizes whether a given real algebraic vector $v \in ((\mathbb{R} \cap \overline{\mathbb{Q}}) \cup \{\infty\})^n$ belongs to V. If yes then the algorithm yields a solution $x \in K^n$ of $A \cdot X = b$ with $Trop(x) = v$. The complexity of the algorithm is polynomial in the bit-sizes of the system $A \cdot X = b$ and of the vector v.

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