Gröbner bases and noncommutative identities

(or: various elements of combinatorics in my work)

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I. Noncommutative identities

Conjecture. (M.Kontsevich) (around 1996)

(arXiv1109.2469, Section 3, Conjecture 1)

Let $(M_{ij})_{1\leqslant i,j\leqslant 3}$ be a matrix, whose entries are 9 independent *noncommutative* variables.

Consider the following three *birational transformations* on this 9 variables:

$I_1: \ M \to M^{-1}$

$I_2: \ M_{ij} \to M_{ij}^{-1}, \ \forall i, j$

$I_3: \ M \to M^t$

Let $\Phi = I_1 \circ I_2 \circ I_3$, then Φ^3 is equal to the identity,

modulo $\mathrm{Diag}_L \times \mathrm{Diag}_\mathrm{R}$ action by multiplication on diagonal matrices from the left and and from the right.

In other words, there exist two diagonal 3×3 matrices $D_L(M)$ and $D_R(M)$, whose entries are noncommutative rational functions in 9 variables M_{ij}, such that

$$(I_1 \circ I_2 \circ I_3)^3(M) = D_L(M)\, M\, D_R(M).$$

It was formulated in the Kontsevich 2011 Arbeitstagung talk 'Noncommutative identities'.

It was noticed at an early stage that Φ commutes with diagonal matrices.

Thus we can factorise our space by $\mathrm{Diag}_L(M)\times \mathrm{Diag}_R(M)$ action, and consider an induced transformation, which acts on orbits, and hence depends on 4 independent variables. We can set $M_{ij} = 1, \quad i, j = 1$, by appropriate choice of representatives in the orbits.

Actually one of the keys to the solution of this problem was a decision to work on the level of 4-parameter transformations (to reduce to 4-parameter representatives of orbits)

However it was not clear at the beginning which way is better to follow: to work with 3×3 or 2×2 matrices.

At the point of the Arbeitstagung talk there was an expression of $\Phi^3(a)$, which took 20 pages...

Other facts which was clear at that point:

1. The conjecture is true for commuting variables.

2. Φ commutes with right/left multiplication by diagonal or permutation matrices.

The transformation Φ in terms of 4-parameter matrices of the shape

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix}$$

encoded by the vector

$$\xi = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Closed formula for Φ.

$$\Phi\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} (d-1)^{-1}(c-d)(b(d-b)^{-1}(c-a)-a)^{-1} \\ (c-1)^{-1}(d-c)(c(c-a)^{-1}(d-b)-d)^{-1} \\ (b-1)^{-1}(a-b)(b(d-b)^{-1}(c-a)-a)^{-1} \\ (a-1)^{-1}(b-a)(c(c-a)^{-1}(d-b)-d)^{-1} \end{pmatrix}$$

Lemma 1. *(Symmetry).*

For any permutation σ *from the Klein four-group*

$$\Phi(\sigma(\xi)) = \sigma(\Phi(\xi)).$$

It reminds the fact that Φ commutes with any permutation of rows and columns of a 3×3 matrix, but it does not follow from that. Moreover it is true only for that particular formula for Φ, and there are other formulas (on the level of representatives of orbits for which this symmetry does not hold.

Since we always working up to representative in the orbit, it is necessary to know how elements of one orbit are related:

Lemma (Conjugacy)

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix} \text{and } M_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \gamma & \delta \end{pmatrix}$$

are in the same orbit w.r.t. to the $\mathrm{Diag}_L \times \mathrm{Diag}_R$ action if and only if there exists an element v, such that $\alpha = a^v, \beta = b^v, \gamma = c^v, \delta = d^v$, ($a^v$ denotes conjugation by v: vav^{-1}).

The main source of ('intrinsic') non-uniqueness of the presentation of the transformation Φ, which makes the problem difficult is that the transformation is presented by elements of the (skew)field of rational expressions on free variables. There is nothing like canonical form of these expressions.

The axiomatic construction of this object is developed by P.Cohn and C.Reutenauer in

[P.M. Cohn, C.Reutenauer On the construction of free field, Int. J. Alg. Comput., v.9, N3,4 (1999), 307-323]
but it does not provide a constructive way of working with these expressions.

The construction of such an object also discussed in [I.Gelfand, S.Gelfand, V.Retakh, R. Lee Wilson Quasideterminants,
arXiv math.QA0208146 (2004)]

We actually work with smaller object: we need only some elements to be invertible.

Lemma The certain set of elements in the ring of free polynomials need to be invertible, in order operators Φ and Φ^{-1} to be defined. The invertibility of the same set is sufficient in order to define any operator Φ^n.

Anyway, after we choose a presentation of Φ, we face the problem of comparing elements in $\mathcal{R}$. And this still remains a pure art.

The same element in $\mathcal{R}$ can look very differently. To derive ones from anothers, identities like the following two are used:

Id 1:

$$y - x = x(x^{-1} - y^{-1})y = y(x^{-1} - y^{-1})x$$

Id 2:

$$(x^{-1} - y^{-1})^{-1} = y(y - x)^{-1}x = x(y - x)^{-1}y$$

Hadamar product formulae

On the way we noticed a new identity which holds for the transformation Φ, and prowed it at least at some particular cases, including commutative case.

This identities is written in terms of Hadamard (componentwise) product of matrices $\star$:

Theorem In the commutative case

$$\Phi^0(\xi)\star\Phi^1(\xi)\star\Phi^2(\xi) = \Phi^0(\xi)\star\Phi^1(\xi)\star\Phi^{-1}(\xi) = \mathbf{1},$$

Another particular case, apart from matrices on commuting elements, could be considered. Namely, due to the symmetry lemma the subset of matrices $\mathcal{M}$ of the shape

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & b & a \end{pmatrix}$$

is closed under the operation Φ.

Thus on this subset we can derive the following identity.

Theorem On the subset

$$\mathcal{M} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & b & a \end{pmatrix} \right\}$$

the following identity holds:

$$\Phi^0(\xi) \star \Phi^{-1}(\xi) \star \Phi^1(\xi) = \mathbf{1}.$$

II. Gröbner bases for Sklyanin and other quadratic algebras

Over the last few years we found the way to use Gröbner bases techniques as the combinatorial tool to

- calculate Hilbert series, homologies,
- prove things like finiteness conditions,
- prove homological properties like Koszulity and Calabi-Yau; PBW, etc.

for various classes of quadratic algebras (algebras presented by generators and relations).

Examples

• **Sklyanin algebras** - algebras appearing in the inverse problem method for integrable models of quantum mechanics and statistical physics

• **homology of moduli spaces of pointed curves** given by Keel relations

• **contraction algebras** invariants associated to a curve contraction by Wemyss in frame of his work on minimal model program

• **potential algebras** (= vacualgebras, Jacobi algebras, etc.)

• W -**Witt algebra**: $A = U(W)/id(g)$,
$g \in U(W)$ has a polynomial growth.
(joint with S.Sierra, answer to conjecture in [Sierra, Petukhov, 2017]

• versions of $A_\infty-$ **structures** which model the TQFT for open strings

Gröbner bases and Hilbert series methods for quadratic potential algebras

I outline methods which we use for 3-dimensional Sklyanin algebras, for homology of moduli space of stable n-pointed curves of genus zero $\overline{M}_{0,n}$, $n = 6$, for proving finiteness results for potential algebras,

where the main ingredient is Gröbner bases technique.

It is usually a combination of arguments of various nature (like knowledge of the generic Hilbert series, passing to a finite field, etc.), with the Gröbner bases theory arguments per se.

First, we prove that the relations come from a potential.

Def *Potential algebra* (Jacobi, vacualgebra) given by cyclic invariant polynomial F is an algebra

$$A_F = k\langle x, y\rangle / \mathrm{id}(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$$

where noncommutative derivations $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} : k\langle x, y\rangle \to k\langle x, y\rangle$ are defined via action on monomials as:

$$\frac{\partial w}{\partial x} = \begin{cases} u & \text{if } w = xu, \\ 0 & \text{otherwise,} \end{cases} \qquad \frac{\partial w}{\partial y} = \begin{cases} u & \text{if } w = yu, \\ 0 & \text{otherwise.} \end{cases}$$

Polynomial F is *cyclic invariant* means $F = F \circlearrowleft$

where $u \circlearrowleft$ is a sum of all cyclic permutations of the monomial $u \in K\langle X\rangle$.

Sometimes it is not an easy task.

For example, $A = H_\bullet \overline{M}_{0,6}$ is given by the Keel relations.

Let $|M| = n$,

Generators:

correspond to the partitions P_A of M:

$M = A \sqcup C = P_A$, such that $|A|, |C| \geqslant 2$.

Relations:

$P_A P_B = P_B P_A$

$P_A P_B = 0$ if $A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c \neq \varnothing$

Plus linear relations:
for any distinct $i, j, k, m \in M$, $\sum\limits_{ijAkm} P_A = \sum\limits_{ikAjm} P_A$

We show that its dual $A^!$ is a potential algebra.

It becomes visible after a suitable change of variables:
P_A, with $|A| = 3$ stay the same
$V_p = \sum\limits_{|A|=2, p\in A} P_A$.

We use symmetries of the potential (which are preserved after the change of variables), this allows to actually calculate the potential.

1. not only cyclic, but any permutation of entries of the monomial contained in the polynomial with the same coefficient,

2. invariant under S_6 group action on variables, corresponding to permutations of points.

Then we consider the standard (potential) complex

$$0 \to A \xrightarrow{\delta} A^n \xrightarrow{\gamma} A^n \xrightarrow{\beta} A \xrightarrow{\alpha} \mathbb{K} \to 0,$$

where α is the augmentation map,

$$\beta(a_1, ..., a_n) = \sum x_i a_i,\ \delta(a) = (x_1 a, ..., x_n a)$$

and

$$\gamma(a_1, ..., a_n) = \begin{pmatrix} .. & .. & .. \\ .. & \partial_{x_i}\partial_{x_j}F & .. \\ .. & .. & .. \end{pmatrix} \begin{pmatrix} a_1 \\ .. \\ a_n \end{pmatrix}$$

As it is shown in [M.Wemyss, R.Bocklandt, T.Schedler, *Superpotentials and higher order derivations*] it is a subcomplex of the Koszul complex.

Since Koszul complex have meaning only for quadratic algebras, it is necessary to assume that potential have degree three.

$$\ldots \xrightarrow{d_{k+1}} (A^!_k)^* \otimes A \xrightarrow{d_k} (A^!_{k-1})^* \otimes A \xrightarrow{d_{k-1}} \ldots \xrightarrow{d_1} (A^!_0)^* \otimes A = A \longrightarrow \mathbb{K} \to 0,$$

each tensor product carries the natural structure of a free right A-module and

d_k are given by $d_k(\varphi \otimes u) = \sum_{j=1}^{n} \varphi_j \otimes x_j u$, where $\varphi_j \in (A^!_{k-1})^*$, $\varphi_j(v) = \varphi(v x_j)$.

For a degree three potential, potential complex frequently coincides with the Koszul complex.

More precisely, it turns out that they coincide if and only if $\sum\limits_x [\partial_x F, x] = 0$ is (up to a scalar multiple) the only degree 3 syzygy.

The next step is to prove that this complex is exact at all its entries except for perhaps one.

Due to the following lemma.

Lemma 2. *Numeric Koszulity together with exactness of the Koszul complex at all terms except for perhaps one implies Koszulity.*

Def. Numeric Koszulity

$H_A(t)H_{A^!}(-t) = 1$

To prove numeric Koszulity, we calculate the **Hilbert series** using Gröbner basis technique.

More precisely: 1) get a lower bound for the Hilbert series (lower bound is attained for generic algebras);

Prop. (Analogue of the Drinfeld dichotomy for Koszul algebras).

For any complex C defined in the variety $\mathcal{V}$, and for an open set $\mathcal{U} \subset \mathcal{V}$, where certain number of Hilbert series components are constant, the complex is either never exact in $\mathcal{U}$, or it is exact generically in $\mathcal{U}$.

So, we show first that there exist an example of algebra (corresponding to a point in the variety), for which the complex is exact.

Then it is generically exact due to Prop., and exactness gives a recurrence relation, which allows to calculate the series. This series is minimal.

2) Compute Gröbner basis and therefore show that the Hilbert series coincides with the lower bound, considering *only some* ambiguities.

To give a taste of how it works, I will show calculations of Hilbert series in the case of Sklyanin algebra, as it is done in:

[Iyudu, Shkarin, J.Algebra 2017],
[Iyudu, Shkarin, MPIM preprint 49.17].

Sklyanin algebra $S^{p,q,r}$ is given by quadratic relations on three variables:

$$
\begin{aligned}
pyz + qzy + rxx &= 0,\\
pzx + qxz + ryy &= 0,\\
pxy + qyx + rzz &= 0.
\end{aligned}
$$

Proof of: $H_A = (1-t)^{-3}$.

The problem of calculating Gröbner basis a priori is indeed highly non-trivial.

If we take just ordering $x > y > z$ (and DegLex on it) for initial Sklyanin relations, we can not understand the whole Gröbner basis.

We can calculate H_A till degree 3 (if we are very stubborn we can go till degree 4, but not further):

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & a_4 \\ 1 & 3 & 6 & 10 & 15 \end{array}$$

However, we can look for a change of variables, which would make the calculation of the Gröbner basis more convenient.

Empirical fact: for A with 3 quadratic relations on 3 generators, the Gröbner basis is simpler if the highest terms of the relations are xx, xy and yz (not xx, xy and xz as for the initial Sklyanin relations (w.r.t. $x > y > z$).

We look for a linear change of variables, which will ensure this.

and get the relations:

$$xx = -\frac{(a+b)^2+a^2b}{a^2-b^2}xz - \frac{(a+b)^2+a^3}{a^2-b^2}zx - yx - \frac{a}{a-b}yy;$$
$$xy = \frac{(a+b)^2+ab^2}{a^2-b^2}xz + \frac{(a+b)^2+a^2b}{a^2-b^2}zx + \frac{b}{a-b}yy;$$
$$yz = \frac{(a-b)}{a}zx - \frac{b}{a}zy + \frac{a+b}{a}zz.$$

What we do to find a change of variables.

The initial presentation of Sklyanin is a potential algebra with the potential

$$W = p\,xyz^{\circlearrowleft} + q\,xzy^{\circlearrowleft} + r(x^3 + y^3 + z^3).$$

$$\downarrow$$

$$W = a\,xyz^{\circlearrowleft} + b\,xzy^{\circlearrowleft} + (x + y + z)^3.$$

It is more convenient to get the shape of relations we require.

Now construct Gröbner basis in the ideal of relations.

Theorem If for some k, $F_k \subset G$, where

$$\left\{ \begin{array}{l} xz^k x = a_k xz^{k+1} \\ xz^k y = b_k xz^{k+1} \end{array} \right\} = F_k$$

(all mod $J = zA + yA$),

then either $F_{k+1} \subset G$ or $H_{k+1} \subset G$, where

$$H_{k+1} = xz^{k+1}y^m x = \frac{b}{a-b} xz^{k+1}y^{m+1}, \quad m = 0, 1, \dots$$

$$xz^{k+2} = 0.$$

Look at overlapping of F_k with initial defining relations.

These are obtained from the overlaps $(xz^ky)z = xz^k(yz)$, $(xz^kx)y = xz^k(xy)$ and $(xz^kx)x = xz^k(xx)$ respectively.

$$G_{k+1} = \begin{cases} -(a-b)xz^{k+1}x + bxz^{k+1}y + (ab_k - a - b)xz^{k+2} = 0, \\ -((a+b)^2 + a^2b)xz^{k+1}x + (a+b)((a-b)a_k - bb_k)xz^{k+1}y + ((a+b)^2 + ab^2)a_kxz^{k+2} = 0, \\ ((a-b)^2(a_k+b_k) + (a+b)^2 + a^3)xz^{k+1}x + a(a+b)b_kxz^{k+1}y + ((a+b)^2 + a^2b)a_kxz^{k+2} = 0. \end{cases}$$

If the 3×2 matrix of the $xz^{k+1}x$ and $xz^{k+1}y$ coefficients above has rank 2, these relations yield

$$\left\{ \begin{array}{l} xz^{k+1}x = a_{k+1}xz^{k+2} \\ xz^{k+1}y = b_{k+1}xz^{k+2} \end{array} \right\} = F_{k+1}$$

If this matrix has rank 1 (it can not be of rank 0 since $b - a \neq 0$), then the three relations in G_{k+1} simplify to the relations (mod J)

$$xz^{k+1}x = \tfrac{b}{a-b}xz^{k+1}y, \quad xz^{k+2} = 0.$$

Using induction and resolving the overlaps $(xz^{k+1}y^m x)y = xz^{k+1}y^m(xy)$, we get

$$xz^{k+1}y^m x = \frac{b}{a-b} xz^{k+1}y^{m+1}, \quad m = 0, 1, \dots$$

Normal words, generic case: $z^p y^m$, $z^p y^m x z^n$ with $p, m, n = 0, 1, \dots$

Normal words, branching off at k: $z^p y^m$, $z^p y^m x z^r$, $z^p y^m x z^{k+1} y^{q+1}$ with $p, m, q = 0, 1, \dots$, $0 \leqslant r \leqslant k+1$.

In both cases the number of normal words of degree n is $\frac{(n+1)(n+2)}{2}$.

Def We say that a quadratic algebra $A = A(V, R)$ is *PBW-algebra* if there are linear bases $x_1, \ldots, x_n$ and $g_1, \ldots, g_m$ in V and R respectively such that with respect to some compatible with multiplication well-ordering on the monomials in $x_1, \ldots, x_n$, $g_1, \ldots, g_m$ is a Gröbner basis of the ideal I_A generated by R.

In this case, $x_1, \ldots, x_n$ is called a *PBW-basis* of A, while $g_1, \ldots, g_m$ are called the *PBW-generators* of I_A.

Note, that this is the the definition used in [Polishchuk, Positselsky] book *Quadratic algebras*

For example, Odesski call A PBW if $H_A = (1-t)^{-n}$, and there are other versions of using the term PBW.

Theorem 3. *The algebra $A = S^{p,q,r}$ is PBW if and only if at least one of the following conditions is satisfied*:

(1) $pr = qr = 0$;

(2) $p^3 = q^3 = r^3$;

(3) $(p+q)^3 + r^3 = 0$ *and the equation* $t^2 + t + 1 = 0$ *is solvable in* $\mathbb{K}$.

III. Finiteness conditions for potential algebras

We consider finiteness conditions and questions of growth of noncommutative algebras, known as A_{con}s.

They appear in M.Wemyss work on minimal model program and noncommutative resolution of singularities. Namely, they serve as noncommutative invariants attached to a birational flopping contraction:

$f : X \to Y$

which contracts rational curve $C \simeq \mathrm{Pr}^1 \subset X$ to a point. X is a smooth quasi-projective 3-fold.

It is known due to [Van den Bergh], that A_{con}s are potential.

Finiteness questions are essential, because algebras with geometrical origin are finite dimensional or have a linear growth.

In [Iyudu, Smoktunowicz, IMRN 2017; IHES/M/16/19] we prove the following theorems on the finiteness conditions for 2-generated potential algebras.

It was shown by Michael Wemyss that the completion of a potential algebra can have dimension 8 and he conjectured that this is the minimal possible dimension. We show that his conjecture is true.

Theorem 4. *Let A_F be a potential algebra given by a potential F having only terms of degree 3 or higher. The minimal dimension of A_F is at least 8. Moreover, the minimal dimension of the completion of A_F is 8.*

Proof We use Golod-Shafarewich theorem, Gröbner bases arguments **plus** relation, which holds in any potential algebra:

$$[x, \frac{\partial F}{\partial x}] + [y, \frac{\partial F}{\partial y}] = 0$$

Non-Homogeneous case

Using the improved version of the Golod–Shafarevich theorem and involving the fact of potentiality we derive the following fact.

Theorem 5. *Let* A_F *be a potential algebra given by a not necessarily homogeneous potential* F *having only terms of degree* 5 *or higher. Then* A_F *is infinite dimensional.*

Homogeneous case

Theorem 6. *For the case of homogeneous potential of degree $\geqslant 3$, A_F is always infinite dimensional.*

Namely, we prove the following two theorems.

First, we deal with the case of homogeneous potentials of degree 3.

We classify all of them up to isomorphism.

From this we see that the corresponding algebras are infinite dimensional. We also compute the Hilbert series for each of them.

Classification of potential algebras, with homogeneous potential of degree 3.

Theorem 7. *There are three non isomorphic potential algebras with homogeneous potential of degree 3.*

1. $F = x^3$, $A = \mathbb{K}\langle x, y\rangle/\mathrm{Id}(x^2)$.

2. $F = x^2y + xyx + yx^2$, $A = \mathbb{K}\langle x, y\rangle/\mathrm{Id}(xy + yx, x^2)$.

3. $F = x^2y + xyx + yx^2 + xy^2 + yxy + y^2x$, $A = \mathbb{K}\langle x, y\rangle/\mathrm{Id}(xy + yx + y^2, x^2 + xy + yx) = \mathbb{K}\langle x, y\rangle/\mathrm{Id}(xy + yx + y^2, x^2 - y^2)$.

In each case:

*These relations form a Gröbner basis (w.r.t. degLex and $x > y$).

*A_F is infinite dimensional.

It has exponential growth for $F = x^3$ and the Hilbert series is $H_A = 1 + 2t + 2t^2 + 2t^3 + \dots$ in the other two cases (the normal words are y^n and $y^n x$).

Next, we consider the main case, when F is of degree $\geqslant 4$.

Theorem 8. *If $F \in \mathbb{K}\langle x, y\rangle$ is a homogeneous potential of degree $\geqslant 4$, then the potential algebra $= \mathbb{K}\langle x, y\rangle/\mathrm{Id}(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y})$ is infinite dimensional.*

Moreover, the minimal Hilbert series in the class $\mathcal{P}_n$ of potential algebras with homogeneous potential of degree $n+1 \geqslant 4$ is $H_n = \frac{1}{1-2t+2t^n-t^{n+1}}$.

Corollary 9. *Growth of a potential algebra with homogeneous potential of degree 4 can be polynomial (non-linear), but starting from degree 5 it is always exponential.*

Conjecture formulated in [Wemyss and Donovan, Duke 2015]

The conjecture says that the difference between the dimension of a potential algebra and its abelianization is a linear combination of squares of natural numbers starting from 2, with non-negative integer coefficients.

In [Toda, 2014] it is shown, that these integer coefficients do coincide with Gopakumar - Vafa invariants.

We give an example of solution of the conjecture using Gröbner bases arguments, for one particular type of potential, namely for the potential $F = x^2y + xyx + yx^2 + xy^2 + yxy + y^2 + a(y)$, where $a = \sum_{j=3}^{n} a_j y^j \in \mathbb{K}[y]$ is of degree $n > 3$ and has only terms of degree $\geqslant 3$.