INTEGER RATIOS OF FACTORIALS AS HAUSDORFF MOMENTS VERSUS ALGEBRAICITY

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Plan of the talk

1. Glimpse over the Hausdorff moment problem: positivity, support of the measure, singularities...
2. Ratios of factorials as integer moments.
3. Algebraicity of ordinary generating functions.
4. Algebraicity of solutions of moment problem (inverse Mellin transform, Meijer G functions).
5. Perspective...

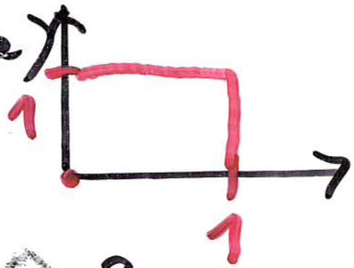
Hausdorff moment problem

Given an infinite sequence $\varrho(n)$, $n=0,1\ldots$ such that

$$\int_0^R x^n W(x)\,dx = \varrho(n), \quad n=0,1\ldots$$

Find $W(x)>0$, conditions: Hausdorff (1921)
These conditions are difficult to verify.
Support of the measure $(0,R)$.

Remarks:

a) 

$$\longrightarrow \int_0^1 x^n\cdot 1\cdot dx = \frac{1}{n+1}$$

b) $$\int_0^R x^n W(x) = \frac{1}{(n+1)!},$$ No solutions at all (?)

c) How do we know it is Hausdorff?

$$R = \lim_{n\to\infty}\left[\varrho(n)\right]^{\frac{1}{n}} \begin{cases} <\infty, & \text{Hausdorff} \\ \to\infty, & \text{Stieltjes} \end{cases}$$

Our approach is CONSTRUCTIVE

Examples:

Catalan(n): $\lim_{n\to\infty}\left[\frac{1}{n+1}\binom{2n}{n}\right]^{\frac{1}{n}} = 4$

Central Binomial(n): $\lim_{n\to\infty}\left[\binom{2n}{n}\right]^{\frac{1}{n}} = 4$

Ordinary generating function of Catalans(n)

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}z^n \equiv G(z) = \frac{1-\sqrt{1-4z}}{2z}$$

$z < \frac{1}{4}$

a = radius of convergence of ogf

Conjecture: $R(\text{support}) = \frac{1}{a}$

... always true

Main concern of this investigation

Positive integers, form vectors:

$$\left.\begin{array}{l}\vec{a} = (a_1, a_2, \dots a_K) \\ \vec{b} = (b_1, b_2, \dots b_K, b_{K+1})\end{array}\right\} K = 1, 2 \dots$$

Condition: $\sum_{i=1}^{K} a_i = \sum_{i=1}^{K+1} b_i$

* For what parameters $\vec{a}, \vec{b}$ is

$$\boxed{u_n(\vec{a}, \vec{b}) = \frac{(a_1 \cdot n)!(a_2 \cdot n)! \dots (a_K \cdot n)!}{(b_1 \cdot n)!(b_2 \cdot n)! \dots (b_K \cdot n)!(b_{K+1} \cdot n)!}}$$

integer?

Answer: for $\gcd(\vec{a}, \vec{b}) = 1$.

see conditions later

Example: Chebyshev (1852):

$$u_n = \frac{(30 \cdot n)!\, n!}{(15 \cdot n)!(10 \cdot n)!(6n)!}$$

$= 1, 77636318760, \dots$

Sloane: https://oeis.org/ → A211417

Methodology:

Mellin and inverse Mellin transform

$$\mathcal{M}[f(x); s] = \int_0^\infty x^{s-1} f(x)\, dx \equiv f^*(s)$$

$$\mathcal{M}^{-1}[f^*(s); x] = f(x)$$

Relation to the moment problem

$$n \longleftrightarrow s-1$$

In all applications treated here

$$\mathcal{M}^{-1}\left[\frac{\prod_j \Gamma(\beta_j + s)}{\prod_k \Gamma(\alpha_k + s)}; x\right] = W(x)$$

Ratios of Gamma functions

$\Rightarrow$ $W(x)$ are Meijer G-functions

convertible to generalized

hypergeometric functions

Arcsin Distribution
Unorthodox Approach

$$\binom{2n}{n} = 1, 2, 6, 20, 70, 252, 924 \ldots$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}} \equiv g(z)$$

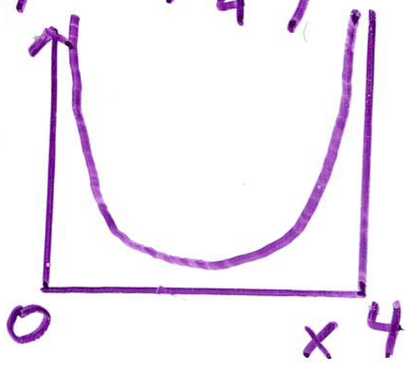
$g(z)$ is algebraic: $\quad 1-(1-4z)\,g^2(z)=0$

Moment problem via Mellin transform

$$\int_0^4 x^n W(x)\,dx = \int_0^4 x^{s-1} W(x)\,dx = \frac{4^s}{4\sqrt{\pi}} \, \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$

$$\frac{1}{4\sqrt{\pi}} \operatorname{MeijerG}\left([[\,],[0]],[[\tfrac{1}{2}],[\,]],\frac{x}{4}\right) = W(x)$$

$$= \frac{1}{\sqrt{(4-x)x}\,\pi}$$



$W(x)$ is algebraic:

$$\pi^2 x(x-4)\,W^2(x) + 1 = 0$$

M. Kontsevich (priv. communication) 2016

$$u_n = \frac{(6\cdot n)!\,n!}{(3\cdot n)!\,(2\cdot n)!\,(2\cdot n)!} =$$

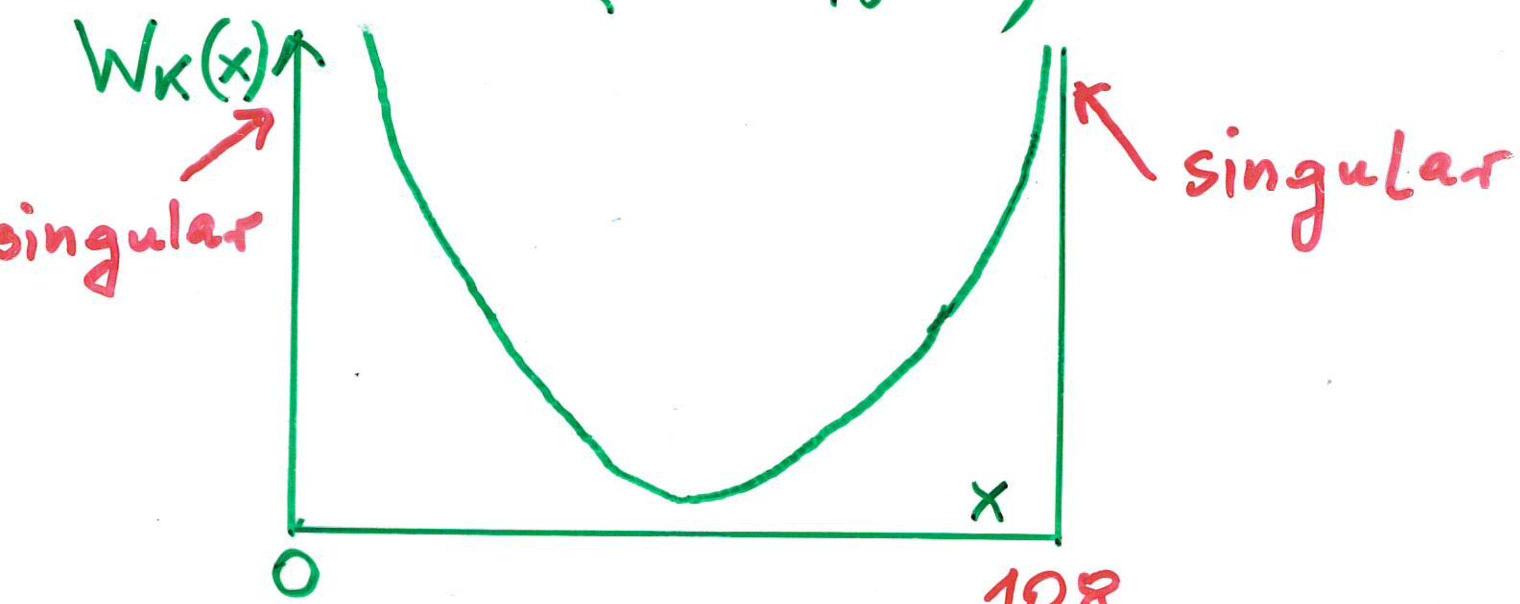
$$= 1, 30, 2310, 204204, 19122246\ldots$$

$$=\ \text{Sloane OEIS: A061162}$$

$$u_n = \int_0^{108} x^n W_K(x), \quad W_K(x) \text{ positive on } (0, 108)$$

$$W_K(x) = \frac{2^{2/3}\, 3\sqrt{3}\left(1+\frac{\sqrt{324-3x}}{18}\right)^{2/3}}{4\pi x^{5/6}\sqrt{324-3x}} +$$

$$\frac{2^{1/3}\sqrt{3}}{24\pi x^{1/6}\left(1+\frac{\sqrt{324-3x}}{18}\right)^{2/3}\sqrt{324-3x}}$$



Elementary function!

$$u_n = \frac{(6n)!\, n!}{(3n)!\,(2n)!\,(2n)!}$$ (Kontsevich)

Ordinary generating function (ogf) of these numbers:

$$h(z) = \sum_{n=0}^{\infty} u_n z^n = \sum_{n=0}^{\infty} \frac{(6n)!\, n!}{(3n)!\,(2n)!\,(2n)!} z^n =$$

$$= \frac{1}{\sqrt{1-108z}} \left[\left(1 - 216z + 12\sqrt{324z^2 - 3z}\right)^{\frac{1}{3}} + \left(1 - 216z + 12\sqrt{324z^2 - 3z}\right)^{-\frac{1}{3}} \right] \cdot \frac{1}{2}$$

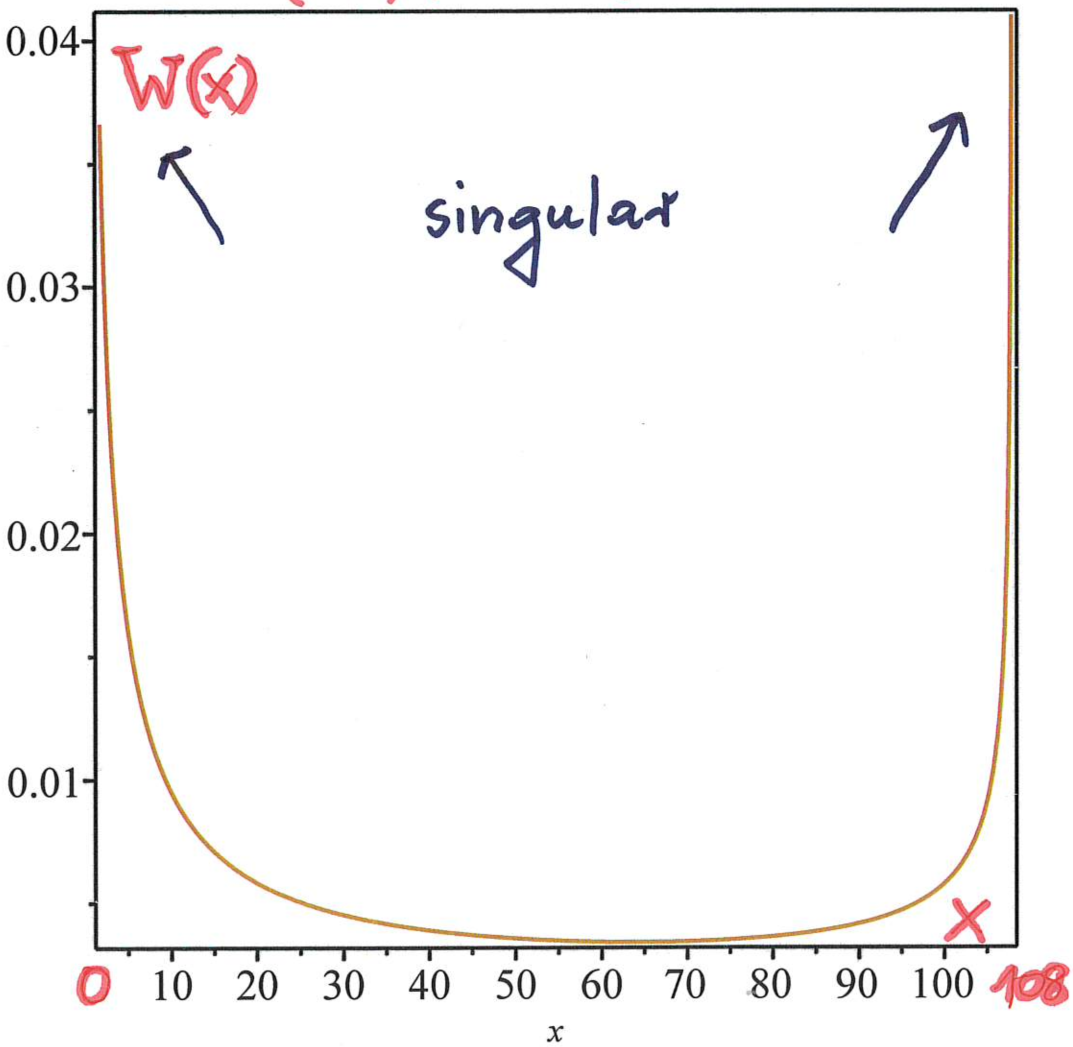
$\Longrightarrow$ Obviously, there is a connection with the weight function of the moment problem.

FURTHER SIMILARITY ???

ELBA

Weight $W(x)$ for moments

$$\frac{(6n)!\,n!}{(3n)!\,(2n)!\,(2n)!}$$

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> # Check for several initial moments:
> seq(round(evalf(Int(x^kk*W(x),x=0..108))),kk=0..4);
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1, 30, 2310, 204204, 19122246 (4)

> Everything agrees !

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> #############
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Two different functions in question:

a) Weight $W(x)$ of the moment problem

b) $h(z)$ ordinary g.f. of moments

Definition: (simplified)

Function is called algebraic if it satisfies algebraic equation of finite order with polynomial coefficients.

a)
$$1024\cdot\pi^6\cdot x^5(x-108)^3[W(x)]^6+ 384\cdot\pi^4x^4(x-108)^2[W(x)]^4+ 36\cdot\pi^2x^3(x-108)[W(x)]^2+ (x-216)^2=0$$

b)
$$16(108z-1)^3\cdot[h(z)]^6+ 24(108z-1)^2[h(z)]^4+ 9(108z-1)[h(z)]^2+ (1-216z)^2=0$$

$\Longrightarrow$ Both $W(x)$ and $h(z)$ satisfy bi-cubic equations !!!

Preliminary observation:

*: for some moments with ratios of factorials
ogf of moments are algebraic

**: at the same time the solution of the moment problem are algebraic

—————— * ——————

Questions:

1) Are these the rules or exceptions ???

2) Do both algebraicities synchronize ???

ELBA

Some elements of answers:

1. J. W. Bober, J. of the London Math. Society **79**, no. 2 (2009), p. 422

2. F. Rodriguez-Villegas
arXiv: math.NT/0701362 (unpublished)

3. Peter Bala, unpublished (2016) attached to: OEIS, attached to A262733
"Some integer ratios of factorials"

gcd(a, b) = 1, a, b – integers

*

It is essential that the moments are integers under following conditions:

I. $$u_n = \frac{[(a+b)\cdot n]!}{(a\cdot n)!(b\cdot n)!}$$

II. $$u_n = \frac{(2a\cdot n)!(b\cdot n)!}{(a\cdot n)!(2b\cdot n)![(a-b)n]!}, \quad a > b$$

III. $$u_n = \frac{(2an)!(2bn)!}{(a\cdot n)!(bn)![(a+b)n]!}$$

IV. 52 "sporadic cases" (no formula)
like: $$\frac{(9n)!\,n!}{(5n)!(3n)!(2n)!}$$

Theorem (Rodriguez-Villegas):
for I → IV the ordinary g.f.

$$h(z)=\sum_{n=0}^{\infty} u_n z^n \text{ are algebraic.}$$

$\left(\text{for } \frac{(30\cdot n)!\,n!}{(15n)!(10\cdot n)!(6n)!} \text{ is of order } 483\,840\right)$

Companion result (P. Bala, 2016):
Let $a \geq b$ be integers. Then

$$u_n = \frac{[(2a+1)\cdot n]! \cdot [(b+\frac{1}{2})n]!}{[(2b+1)\cdot n]! \cdot [(a+\frac{1}{2})\cdot n]!\,[(a-b)\cdot n]!}$$

are integers and its $h(z)$ is algebraic.

⟹ Existence of large sets of integer moments with algebraic generating functions.

⟹ Continue to investigate the (positive) solutions of the corresponding moment problems.

Example of complete exact solution of the moment problem (parameters $a \geq b$)

$$\frac{(2an)!\,(bn)!}{(2bn)!\,(an)!\,[(a-b)n]!} = \int_0^{R(a,b)} x^n\, W(a,b,x)\,dx$$

$$\text{Support} = R = \frac{a^a}{b^b}\,\frac{4^{a-b}}{(a-b)^{(a-b)}} = R(a,b)$$

$$W(a,b,x) =$$

$$\mathrm{Pref}(a,b)\cdot G^{2a+b,\,0}_{2a+b,\,2a+b}\left(\frac{x}{R(a,b)}\,\middle|\, \begin{matrix} \left\{\frac{j}{2b}-1\right\}_1^{2b},\ \left\{\frac{j}{a}-1\right\}_1^{a},\ \left\{\frac{j}{a-b}-1\right\}_1^{a-b} \\ \left\{\frac{j}{b}-1\right\}_1^{2b},\ \left\{\frac{j}{2a}-1\right\}_1^{2a} \end{matrix}\right)$$

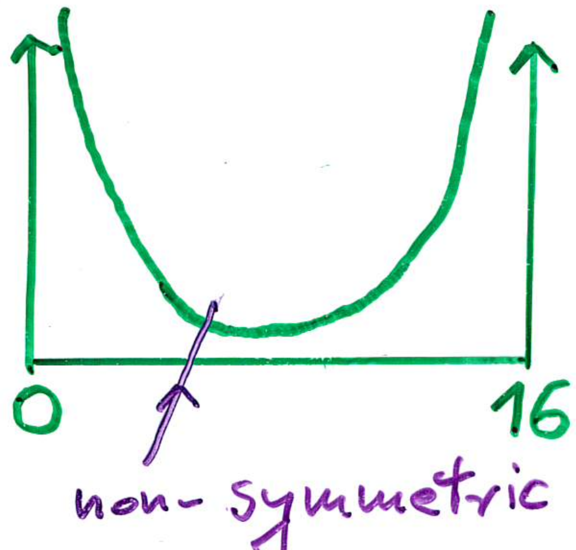
$\updownarrow$

Meijer G function

$\downarrow$

Finite sum of generalized hypergeom. functions

Example: $W(2,1,x) = \dfrac{1}{2\pi x^{3/4}\sqrt{4-\sqrt{x}}}$ on $x \in (0,16)$



Statement:

We have solved exactly the moment problem for all a and b

for I → IV