Mirror symmetry and Feynman integrals

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Combinatorics and Arithmetic for Physics: special days

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based on 1309.5865, 1406.2664, 1601.08181 and work in progress

Spencer Bloch, Matt Kerr, Charles Doran, Andrey Novoseltsev

Feynman Integrals: parametric representation

Any Feynman integrals has the parametric representation in projective space $\nu = \nu_1 + \cdots + \nu_n$

$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{x_i \geqslant 0} \frac{\mathcal{U}_\Gamma^{\nu-(L+1)\frac{D}{2}}}{\Phi_\Gamma^{\nu - L\frac{D}{2}}} \delta(x_n = 1) \prod_{i=1}^n \frac{dx_i}{x_i^{1-\nu_i}}$$

The Symanzik polynomials $\mathcal{U}_\Gamma$ and Φ_Γ are homogeneous in the $x_1, \dots, x_n$

- $\mathcal{U}_\Gamma$ is of degree L in $\mathbb{P}^{n-1}$
- $\Phi_\Gamma = \mathcal{U}_\Gamma \sum_{i=1}^n m_i^2 x_i - \sum_{i,j} p_i \cdot p_j w_{ij}$ of degree $L+1$ in $\mathbb{P}^{n-1}$
- w_{ij} are homogeneous polynomials of degree $L+1$ in $\mathbb{P}^{n-1}$
- degree$(x_i) \leqslant 2$ for $1 \leqslant i \leqslant n$.

Feynman Integrals: motivic periods

In 2015 Francis Brown proved that when the integral converges one can construct a canonical family of motivic period that evaluates to the numerical period defined by the Feynman integral I_Γ

One can write the integral as a period of a variation of cohomology (variation of mixed Hodge structure) of a family of smooth varieties relative to a normal crossing divisor with a canonical local resolution of singularities

The motivic approach allows to pin down nice classes of functions:

- Multiple-polylogarithms for the mixed Tate case
- Elliptic polyogarithm non-mixed Tate

What kind of functions are generic Feynman integrals?

Feynman integral and Riemann-Hilbert problem

The central questions about amplitudes in QFT can be reformulated as Riemann-Hilbert problem for periods

- Compute period explicitly
  - *Numerically or by series expansion in the physical region*
- Derive the local monodromy
  - *unitarity of the S-matrix*
- Construct a complete system of differential equations
  - *Relate this to the integration-by-part method used in QCD*
- Understand the new class of special functions that are needed
  - *What is needed beyond beyond elliptic multiple polylogarithm?*

The geometry of a Feynman graph

The homogeneous polynomial of n variables and degree $L+1$ completely characterises the Feynman graph and its integral

$$\Phi_\Gamma = \mathcal{U}_\Gamma \times (\sum_{i=1}^{n} m_i^2 x_i) - \sum_{i,j} p_i \cdot p_j w_{ij}$$

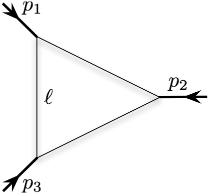
- We can recover both Symanzik polynomials
- Determines the graph topology
  - the number of propagators is the number of variables n
  - the loop order is the degree minus one $L = \deg(\Phi_\Gamma) - 1$
  - Number of vertices $v = 1 + n - L$ from Euler characteristic

From parametric representation to graph

The most general quadratic polynomial in $\mathbb{P}^2$

$$W_{2,3}(x_1, x_2, x_3) = \sum_{\substack{i_1+i_2+i_3=2\\ 0\leqslant i_r \leqslant 2}} w_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

The graph has $n = 3$ propagators, $L = 1$ loop, $v = 3$ vertices
This can only be a triangle graph



$$p_1 + p_2 + p_3 = 0; \qquad p_i^2 \neq 0$$

$$\Phi_\triangleright = (x_1 + x_2 + x_3)(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3) - (p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2)$$

- This is the most general quadric

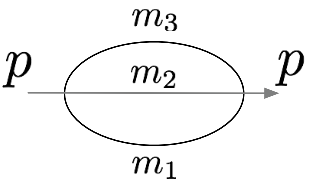
From parametric representation to graph

The most general cubic in $\mathbb{P}^2$ with $deg(x_i) \leqslant 2$

$$W_{3,3} = \sum_{\substack{i_1+i_2+i_3=3 \\ 0\leqslant i_r \leqslant 2}} w_{i_1,i_2,i_3} x_1^{i_1} x_2^{i_2} x_3^{i_3}$$

The graph has $n = 3$ propagators, $L = 2$ loops, $v = 2$ vertices

This can only be a sunset graph



$$\Phi_\ominus = (x_1x_2 + x_1x_3 + x_2x_3)(m_1^2x_1 + m_2^2x_2 + m_3^2x_3) - p^2x_1x_2x_3$$

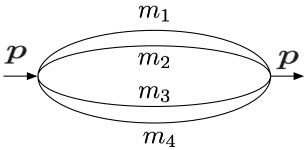
- Important restriction on the parameters of cubic $W_{3,3}$ from 7 parameters to 4 parameters

From parametric representation to graph

A quartic in $\mathbb{P}^3$ with $deg(x_i) \leqslant 2$

$$W_{4,4} = \sum_{\substack{i_1+i_2+i_3+i_4=4 \\ 0\leqslant i_r \leqslant 2}} w_{i_1,i_2,i_3,i_4} x_1^{i_1} x_2^{i_2} x_3^{i_3} x_4^{i_4}$$

The graph has $n = 4$ propagators, $L = 3$ loops, $v = 2$ vertices This can only be a three-loop sunset graph



$$\Phi_\ominus = (x_1x_2x_3 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2x_4)(m_1^2x_1 + m_2^2x_2 + m_3^2x_3 + m_4^2x_4) - p^2x_1x_2x_3x_4$$

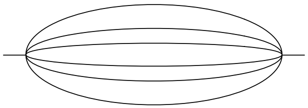
- Important restriction on the parameters of cubic $W_{4,4}$ from 19 parameters to 5 parameters

From parametric representation to graph

The most general polynomial of degree n in $\mathbb{P}^{n-1}$ with $deg(x_i) \leqslant 2$

$$W_{n,n} = \sum_{\substack{i_1+\cdots+i_n=n \\ 0\leqslant i_r\leqslant 2}} w_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n}$$

The graph has n propagators, $L = n-1$ loops, $v = 2$ vertices
This can only be a n-loop sunset graphs



$$\Phi_n = \prod_{i=1}^n x_i \left(\sum_{i=1}^n x_i^{-1}\right)\left(\sum_{i=1}^n m_i^2 x_i\right) - p^2 \prod_{i=1}^n x_i$$

Notice that the kinematics parameters always enter linearly

The sunset graphs Newton polytopes

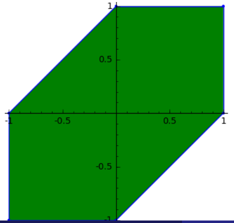
The graph polynomial Φ_n is given by the Laurent polynomial

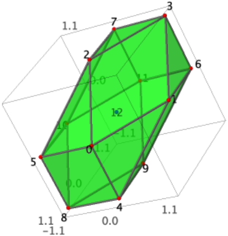
$$\Phi_n = 0 \Longleftrightarrow \sum_{1\leqslant i<j\leqslant n} x_i x_j^{-1} \xi_i^2 = p^2 - \sum_{i=1}^{n} \xi_i^2$$

It's Newton polytope Δ is reflexive with $p^2 - \sum_{i=1}^n \xi_i^2$ for single interior points

For $n = 3$ we have the hexagon (2 dim'l polyhedron in $\mathbb{Z}^3$ with 6 vertices)

For $n = 4$ we have the polytope 1529 in `Sagemath` classification





The toric approach

Consider the homogeneous polynomial of degree $L+1$ in $\mathbb{P}^{n-1}$

$$P(z_1,\dots,z_r) = \sum_{\substack{a_1,\dots,a_{n-1}\\ a_1+\cdots+a_{n-1}=L+1}} z_{a_1,\cdots,a_{n-1}} \prod_{i=1}^{n-1} x_i^{a_i}$$

with $\{z_r\} = \{z_{a_1,\cdots,a_{n-1}}\}$ and $\mathbf{a} = (a_1,\dots,a_{n-1})$ and $A = (\mathbf{a}_1,\cdots,\mathbf{a}_r)$ finite subset of $\mathbb{Z}^r$

For every vector $\ell \in \mathbb{L}$ such that

$$\mathbb{L} := \{(\ell_1,\dots,\ell_r) \in \mathbb{Z}^r, \ell_1+\cdots+\ell_r = 0, \ell_1\mathbf{a}_1+\cdots+\ell_r\mathbf{a}_r = 0\}$$

then there are the following differential operators

$$\Box_\ell := \prod_{l_i>0} \partial_{z_i}^{l_i} - \prod_{l_i<0} \partial_{z_i}^{-l_i}$$

and a system of n differential operator (including the Euler operator)

$$E_{\mathbf{a},\mathbf{c}} := \mathbf{a}_1 z_1 \frac{\partial}{\partial z_1} + \cdots + \mathbf{a}_r z_r \frac{\partial}{\partial z_r} - \mathbf{c} \qquad (\mathbf{c} \in \mathbb{C}^n)$$

The Gel'fand-Zelevinsky-Kapranov approach

GZK have shown that the functions

$$\Phi := \int_{|x_1|=\cdots=|x_{n-1}|=1} \prod_i P(z_1,\ldots,z_r)^{m_i} \prod_{i=1}^{n-1} x^{\beta_i} \frac{dx_i}{x_i}$$

satisfy $\Box_\ell \Phi = 0$ for all $\ell = (\ell_1,\ldots,\ell_r) \in \mathbb{L}$ and $E_{\mathbf{a},\mathbf{c}}\Phi = 0$

- The generic solution of GZK system are the hypergeometric series

$$\Phi_{\mathbb{L},\gamma}(z_1,\cdots,z_r) = \sum_{(\ell_1,\ldots,\ell_r)\in\mathbb{L}} \prod_{j=1}^{r} \frac{z_j^{\gamma_j+\ell_j}}{\Gamma(\gamma_j+\ell_j+1)}$$

with $(\gamma_1,\ldots,\gamma_r) \in \mathbb{C}^r$

- In general for a well choosen $\ell \in \mathbb{L}$ the differential operator factorizes a piece giving the Picard-Fuchs operator

The toric approach: consequences

- The following period integral

$$\pi_P = \int_{|x_1|=\cdots=|x_{n-1}|=1} \frac{1}{P(z_1,\ldots,z_r)} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

is the maximal cut integral

$$\text{Max-Cut}(I_\Gamma(1,\ldots,1)) := \int_{\mathbb{R}^{L(1,D-1)}} \prod_{i=1}^{V} \delta\Big(\sum_{j=1}^{n} \epsilon_{ij} q_j\Big) \prod_{i=1}^{n} \delta((q_i)^2) \prod_{i=1}^{L} d^D\ell_i$$

is actually

$$\pi_\Gamma = \Gamma\Big(\nu - \frac{LD}{2}\Big) \int_{|x_i|=1} \frac{\mathcal{U}_\Gamma^{\nu-(L+1)\frac{D}{2}}}{\Phi_\Gamma^{\nu-L\frac{D}{2}}} \delta(x_n=1) \prod_{i=1}^{n} \frac{dx_i}{x_i^{1-\nu_i}}$$

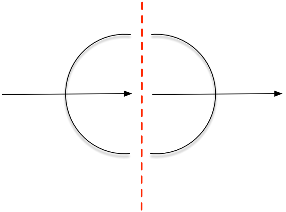
- One can derive the Picard-Fuchs operator from the graph polynomials

The differential operator: from the period

The analytic period of the elliptic curve around $p^2 \sim \infty$ has the same integrand as the Feynman integral but we have just changed the domain of integration

$$\pi_0(p^2) := \int_{|x|=|y|=1} \frac{1}{\Phi_3} \frac{dx}{x} \frac{dy}{y}$$

This is the imaginary part or the maximal cut of the amplitude



$$\Im\mathrm{m}(\mathcal{I}_{\ominus}(p^2)) = \oint_C \prod_{i=1}^{3} \delta(\ell_i^2 - m_i^2)\, \delta(\ell_1 + \ell_2 + p)\, d^2\ell_1 d^2\ell_2$$

The other period is $\pi_1(s) = \log(s)\,\pi_0(s) + \varpi_1(s)$ with $\varpi_1(s)$ analytic is obtained by looking at different unitarity cut cutting less lines

Sunset graphs and generalised Apéry numbers

The imaginary part (the maximal cut) of the multiple loop sunset integral is the following period integral

$$\pi_0(\underline{\xi}, p^2) = \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{p^2 - \left(\sum_{i=1}^n x_i \xi_1^2\right)\left(\sum_{i=1}^n x_i^{-1}\right)} \prod_{i=1}^n \frac{dx_i}{x_i}$$

which has the nice series expansion near $p^2 = \infty$

$$\pi_0(\underline{\xi}, p^2) = \sum_{k\geqslant 0} (p^2)^{-k-1} \sum_{r_1+\cdots+r_n=k} \left(\frac{k!}{r_1!\cdots r_n!}\right)^2 \prod_{i=1}^n (\xi_i^2)^{r_i}$$

The case $n = 3$ with $\xi_i = 1$ is a particular case of the random walk discussed by Alin Bostan yesterday $x_1 = x$, $x_2 = y$, $x_3 = 1$

$$\Phi_3 = p^2 - 3 - \left(x + \frac{1}{x} + y + \frac{1}{y} + \frac{x}{y} + \frac{y}{x}\right)$$

The differential operator: from the period

From the fundamental analytic period

$$\pi_0(\underline{\xi}, p^2) = \sum_{k\geqslant 0} (p^2)^{-k-1} \sum_{r_1+\cdots+r_n=k} \left(\frac{k!}{r_1!\cdots r_n!}\right)^2 \prod_{i=1}^{n} (\xi_i^2)^{r_i}$$

one can derive the Picard-Fuchs differential operator (the system has maximal unipotent monodromy around $p^2 = \infty$)

$$L_{\ominus}\pi_0(p^2) = \sum_{k\geqslant 0} c_k(p^2) \left(p^2 \frac{d}{dp^2}\right)^k \pi_0(p^2) = 0$$

- With this method one easily derives the PF at all loop order for the all equal mass sunset $\xi_i = 1$ and show the order(PF)$=$loop and the degree of the polynomial is $\lceil \frac{n}{2} \rceil$ [Verrill; Vanhove]

The differential operator: from the period

From the fundamental analytic period

$$\pi_0(\underline{\xi}, p^2) = \sum_{k\geqslant 0} (p^2)^{-k-1} \sum_{r_1+\cdots+r_n=k} \left(\frac{k!}{r_1!\cdots r_n!}\right)^2 \prod_{i=1}^{n} (\xi_i^2)^{r_i}$$

one can derive the Picard-Fuchs differential operator (the system has maximal unipotent monodromy around $p^2 = \infty$)

$$L_\ominus \pi_0(p^2) = \sum_{k\geqslant 0} c_k(p^2) \left(p^2 \frac{d}{dp^2}\right)^k \pi_0(p^2) = 0$$

- Gives for the 3-loop sunset the PF has order $2 + \#(\neq \text{masses}) = 3, 4, 5, 6$. [Doran, Novoseltsev, Vanhove; to appear]
- Coefficients are polynomials of degree 25 in the generic case with apparent singularities

Geometry of the sunset graph polynomial

The graph polynomial defines a geometry in $\mathbb{P}^{n-1}$

$$\left(x_1\xi_1^2+\cdots+x_n\xi_n^2\right)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)=p^2$$

- For $n=3$ we have a family of elliptic curve
- For $n=4$ we have a family of $K3$ surfaces with Picard rank $20-\#(\neq \text{masses})=19, 17, 18, 16$ [Doran, Novoseltsev, Vanhove; to appear]
- For $n=5$ and all equal mass this is a Barth-Nieto surface
- more Calabi-Yau varieties in Feynman graphs (train tracks, etc cf [Bourjaily et al.])

Calabi-Yau and sunset graph

The sunset graph polynomial has a reflexive Newton polytope and they define Fano toric varieties.
The Feynman integral is the integral of the natural holomorphic $(n, 0)$ of the Calabi-Yau defined by the graph polynomial

One can use maximal projective crepant partial resolution to define Calabi-Yau variety from the anticanonical surface $-K_{\mathbb{P}_\Delta}$ with $\mathbb{P}_\Delta = X_{\Sigma(\Delta^\circ)}$ is the toric manifold constructed from the complete fan $\Sigma(\Delta^\circ)$ where Δ° is the dual polytope.

[Verrill] had already remarked that and the fibers $\sum_{1\leqslant i\neq j\leqslant n} X_i X_j^{-1} = \lambda$ are smooth Calabi-Yau varieties
As she understood that the fan $\Sigma(\Delta^\circ)$ is the ones of the A_n root system

local mirror symmetry conjecture

For any reflexive polytope $\Delta \in \mathbb{R}^n$ the family of non-compact $n+1$-fold Calabi-Yau

$$Y_\ominus := \{u^2 + v^2 + \Phi_n = 0\} \in (\mathbb{C}^*)^n \times \mathbb{C}^2$$

is the mirror dual of $K_{\mathbb{P}_\Delta}$ The holomorphic form

$$\eta_\ominus = 2i\mathrm{Res}_{\Phi_n}\left(\frac{\wedge_{i=1}^n d\log x_i \wedge du \wedge dv}{u^2 + v^2 + \Phi_n}\right)$$

can be interpreted in terms of the (regulator) period of $X_\ominus = \{\Phi_n = 0\}$ and then the Feynman integral

The main conjecture by [Doran, Kerr] is that the local genus zero Gromov-Witten prepotential $\mathcal{F}_{\rm loc}$ determines the Feynman integral

This is proven for the elliptic curve case by [Bloch, Kerr, Vanhove]

Feynman integral and periods

$$I_\Gamma = \Gamma(\nu - \frac{LD}{2}) \int_{\Delta_n} \Omega_\Gamma; \qquad \Omega_\Gamma := \frac{\mathcal{U}^{\nu-(L+1)\frac{D}{2}}}{\Phi_\Gamma(\underline{x})^{\nu-L\frac{D}{2}}} \prod_{i=1}^{n-1} \frac{dx_i}{x_i^{1-\nu_i}}$$

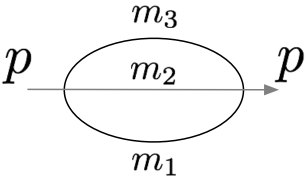
Ω_Γ algebraic differential form on the complement of the graph hypersurface

$$\Omega_\Gamma \in H^{n-1}(\mathbb{P}^{n-1} \backslash X_\Gamma) \qquad X_\Gamma := \{\Phi_\Gamma(\underline{x}) = 0, x_i \in \mathbb{P}^{n-1}\}$$

The domain of integration is the simplex Δ_n

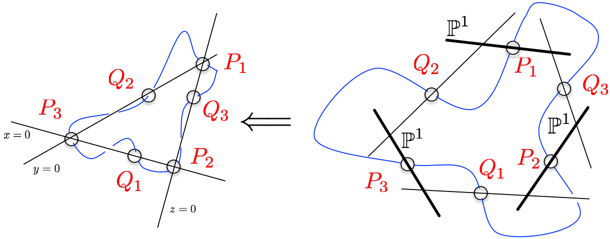
$$\Delta_n := \{x_1 \geqslant 0, \ldots, x_n \geqslant 0 | [x_1, \ldots, x_n] \in \mathbb{P}^{n-1}\}$$

Sunset as an elliptic dilogarithm



Feynman integral and periods

Δ_n and X_Γ are separated by performing a series of iterated blowups of the complement of the graph hypersurface [Bloch, Esnault, Kreimer]



The Feynman integral *are* periods of the relative cohomology after performing the appropriate blow-ups

$$H^{n-1}(\widetilde{\mathbb{P}^{n-1}}\backslash\widetilde{X_F}; \widetilde{\Delta_n}\backslash\widetilde{\Delta_n}\cap\widetilde{X_\Gamma})$$

Picard-Fuchs equation

$$\mathfrak{M}(s_{ij}, m_i) := H^\bullet(\widetilde{\mathbb{P}^{n-1}} \backslash \widetilde{X_F}; \widetilde{\Pi_n} \backslash \widetilde{\Pi_n} \cap \widetilde{X_\Gamma})$$

Since Ω_Γ varies when one changes the kinematic variables s_{ij} one needs to study a variation of (mixed) Hodge structure

Consequently the Feynman integral will satisfy a differential equation

$$L_{PF}\, I_\Gamma = S_\Gamma$$

The Picard-Fuchs operator will arise from the study of the variation of the differential in the cohomology when kinematic variables change

Generically there is an inhomogeneous term $S_\Gamma \neq 0$

The differential equation

By general consideration we know that since the integrand is a top form we have

$$L_\Gamma I_\Gamma = \int_{\Delta_n} d\beta_\Gamma = -\int_{\partial\Delta_n} \beta_\Gamma = \mathcal{S}_\Gamma \neq 0$$

Writing the differential equation as $\delta_s := s\frac{d}{ds}$ with $s = 1/p^2$

$$\left(\delta_s^2 + q_1(s)\delta_s + q_0(s)\right)\left(\frac{1}{s} I_\ominus(s)\right) = \mathcal{Y}_\ominus + \sum_{i=1}^{3} \log(m_i^2) c_i(s)$$

The differential equation

Using works from [del Angel,Müller-Stach] and [Doran, Kerr] we know that when rank of the D-module system of differential equations that $\mathcal{Y}_\ominus$ is the Yukawa coupling

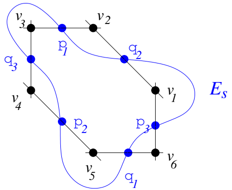
$$\mathcal{Y}_\ominus := \int_{\mathcal{E}(p^2)} \Omega_\ominus \wedge s\frac{d}{ds}\Omega_\ominus = \frac{2s^2\prod_{i=1}^4 \mu_i - 4s\sum_i m_i^2 + 6}{\prod_{i=1}^4(\mu_i^2 s - 1)}$$

The Yukawa coupling is the Wronskian of the Picard-Fuchs operator and only depends on the form of the Picard-Fuchs operator

$$\mathcal{Y}_\ominus = s\,\det\begin{pmatrix} \pi_0(s) & \pi_1(s) \\ \frac{d}{ds}\pi_0(s) & \frac{d}{ds}\pi_1(s)\end{pmatrix}$$

So far all we got can be deduced from the graph polynomial, and the associated Picard-Fuchs operator.

The differential equation



$$\left(\delta_s^2 + q_1(s)\delta_s + q_0(s)\right)\left(\frac{1}{s}\, I_\ominus(s)\right) = \mathcal{Y}_\ominus + \sum_{i=1}^{3} \log(m_i^2)\, c_i(s)$$

The mass dependent log-terms come from derivative of partial elliptic integrals on globally well-defined algebraic 0-cycles arising from the punctures on the elliptic curve [Bloch, Kerr,Vanhove]

$$c_1(s) = \frac{d}{ds}\int_{q_2}^{q_3} \Omega_\ominus$$

They are rational function by construction.

The two-loop sunset integral

We consider the sunset integral in two Euclidean dimensions

$$\mathcal{J}^2_\ominus = \int_{\Delta_3} \Omega_\ominus; \qquad \Delta_3 := \{[x:y:z] \in \mathbb{P}^2 | x \geqslant 0, y \geqslant 0, z \geqslant 0\}$$

- The sunset integral is the integration of the 2-form

$$\Omega_\ominus = \frac{zdx \wedge dy + xdy \wedge dz + ydz \wedge dx}{(m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - p^2 xyz} \in H^2(\mathbb{P}^2 - \mathcal{E}_{p^2})$$

- The sunset family of open elliptic curve

$$\mathcal{E}_{p^2} = \{(m_1^2 x + m_2^2 y + m_3^2 z)(xz + xy + yz) - p^2 xyz = 0\}$$

- For $m_1 = m_2 = m_3$ we have a modular curve $\mathcal{E}_{p^2} \simeq X_1(6)$

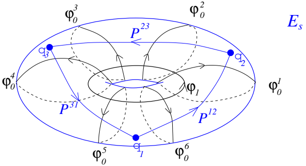
The 2-loop sunset integral as elliptic dilogarithm

The integral divided by a period of the elliptic curve is a function defined on the punctured torus [Bloch, Kerr,Vanhove]

$$\mathcal{J}_{\ominus} \equiv \frac{i\varpi_r}{\pi}\left(\mathcal{L}_2\left\{\frac{X}{Z},\frac{Y}{Z}\right\}+\mathcal{L}_2\left\{\frac{Z}{X},\frac{Y}{X}\right\}+\mathcal{L}_2\left\{\frac{X}{Y},\frac{Z}{Y}\right\}\right) \quad \text{mod period}$$

- ϖ_r is the elliptic curve period which is real on the line $0 < p^2 < (m_1+m_2+m_3)^2$
- The sunset integral is the regulator period (with tame Milnor symbol) in the K_2 of the elliptic curve [Bloch, Vanhove]

The 2-loop sunset integral as elliptic dilogarithm



$P_1 = [1,0,0]; \qquad Q_1 = [0,-m_3^2,m_2^2]; \qquad x(P_1)x(Q_1) = -1$

$P_2 = [0,1,0]; \qquad Q_2 = [-m_3^2,0,m_1^2]; \qquad x(P_2)x(Q_2) = -1$

$P_3 = [0,0,1]; \qquad Q_3 = [-m_2^2,m_1^2,0]; \qquad x(P_3)x(Q_3) = -1$

Representing the ratio of the coordinates on the sunset cubic curve as functions on $\mathcal{E}_\ominus \simeq \mathbb{C}^\times/q^{\mathbb{Z}}$

$$\frac{X}{Z}(x) = \frac{\theta_1(x/x(Q_1))\theta_1(x/x(P_3))}{\theta_1(x/x(P_1))\theta_1(x/x(Q_3))} \qquad \frac{Y}{Z}(x) = \frac{\theta_1(x/x(Q_2))\theta_1(x/x(P_3))}{\theta_1(x/x(P_2))\theta_1(x/x(Q_3))}$$

$\theta_1(x)$ is the Jacobi theta function

$$\theta_1(x) = q^{\frac{1}{8}}\frac{x^{1/2}-x^{-1/2}}{i}\prod_{n\geqslant 1}(1-q^n)(1-q^n x)(1-q^n/x)\,.$$

The 2-loop sunset integral as elliptic dilogarithm

$$\mathcal{L}_2\left\{\frac{X}{Z},\frac{Y}{Z}\right\} = -\int_{x_0}^{x} \log\left(\frac{X}{Z}(y)\right)\, d\log y$$

Since

$$\int \log(\theta_1(x))\, d\log x = \sum_{n\geqslant 1}\int (\mathrm{Li}_1(q^n x) + \mathrm{Li}_1(q^n/x) + \mathrm{cste})\, d\log(x)$$
$$= \sum_{n\geqslant 1}(\mathrm{Li}_2(q^n x) - \mathrm{Li}_2(q^n/x)) + \mathrm{cste}\,\log(x)$$

The 2-loop sunset integral as elliptic dilogarithm

We find

$$\mathcal{J}_{\ominus}(s) \equiv \frac{i\varpi_r}{\pi}\left(\hat{E}_2\left(\frac{x(P_1)}{x(P_2)}\right) + \hat{E}_2\left(\frac{x(P_2)}{x(P_3)}\right) + \hat{E}_2\left(\frac{x(P_3)}{x(P_1)}\right)\right) \mod \text{periods}$$

where

$$\hat{E}_2(x) = \sum_{n\geqslant 0}\left(\mathrm{Li}_2\left(q^n x\right) - \mathrm{Li}_2\left(-q^n x\right)\right) - \sum_{n\geqslant 1}\left(\mathrm{Li}_2\left(q^n/x\right) - \mathrm{Li}_2\left(-q^n/x\right)\right) .$$

Close to the form given by [Brown, Levin]. See as well [Adams, Bogner, Weinzeirl]

The sunset integral and the motive

- The integral is given by

$$\mathcal{J}^2_\ominus(t) = \int_0^\infty \int_0^\infty \frac{dxdy}{(x+y+1)(x+y+xy)-txy}$$

- The 2-form has only log-pole on $\mathcal{E}_t$ and there is a residue 1-form

$$\mathcal{J}^2_\ominus(t) = periods + \varpi_r \left\langle \epsilon_1\tau+\epsilon_2, \int d\tau \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)(\epsilon_1\tau+\epsilon_2)}{(m+n\tau)^3} \right\rangle$$

- Character $\psi : Lattice(\mathcal{E}_t) \to S^1$. Pairing $\langle \epsilon_1, \epsilon_2\rangle = -\langle \epsilon_2, \epsilon_1\rangle = 2i\pi$
- The amplitude integral is *not* the regulator map which involves a real projection $r : K_2(\mathcal{E}_t) \to H^1(\mathcal{E}_t, \mathbb{R})$
- The amplitude is multivalued in t whereas the regulator is single-valued

The sunset integral and the motive

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- The regulator is an Eichler integral

$$\mathcal{J}^2_\ominus(t) = periods + \varpi_r \int_\tau^{i\infty} \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)(\tau-x)}{(m+nx)^3}\, dx$$

The sunset integral and the motive

- The integral is given by

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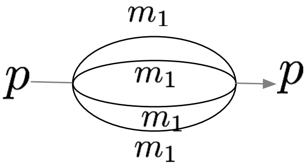
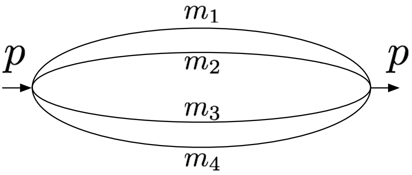
- The regulator is an Eichler integral

$$\mathcal{I}^2_{\ominus}(t) = periods + \varpi_1 \sum_{(m,n)\neq(0,0)} \frac{\psi_2(n)}{n^2(m+n\tau)}$$

The three-loop sunset as an elliptic trilogarithm



The three-loop sunset graph: integral



We look at the 3-loop sunset graph in $D = 2$ dimensions

- The Feynman parametrisation is given by

$$I^2_{\ominus}(m_i; K^2) = \int_{x_i \geqslant 0} \frac{1}{(m_4^2 + \sum_{i=1}^3 m_i^2 x_i)(1 + \sum_{i=1}^3 x_i^{-1}) - K^2} \prod_{i=1}^3 \frac{dx_i}{x_i}$$

three-loop sunset graph: differential equation

For the all equal mass case the geometry of the 3-loop sunset graph is a $K3$ surface (Shioda-Inose family for $\Gamma_1(6)^{+3}$) with Picard number 19 and discriminant of Picard lattice is 6

$$(m^2 + \sum_{i=1}^{3} m^2 x_i)(1 + \sum_{i=1}^{3} x_i^{-1}) \prod_{i=1}^{3} x_i - p^2 \prod_{i=1}^{3} x_i = 0$$

The $t = p^2/m^2$ Picard-Fuchs equation

$$\Big(t^2(t-4)(t-16)\frac{d^3}{dt^3} + 6t(t^2-15t+32)\frac{d^2}{dt^2} + (7t^2-68t+64)\frac{d}{dt} + t - 4\Big) J^2_{\ominus}(t) = -4!$$

- One miracle is that this picard-fuchs operator is the symmetric square of the picard-fuchs operator for the sunset graph [Verrill]

three-loop sunset graph: solution

- It is immediate to use the Wronskian method to solve the differential equation [Bloch, Kerr, Vanhove]

$$m^2\, I^2_{\circleddash}(t) = 40\pi^2 \log(q)\, \varpi_1(\tau)$$
$$-48\varpi_1(\tau)\left(24\mathcal{L}i_3(\tau,\zeta_6) + 21\,\mathcal{L}i_3(\tau,\zeta_6^2) + 8\mathcal{L}i_3(\tau,\zeta_6^3) + 7\mathcal{L}i_3(\tau,1)\right)$$

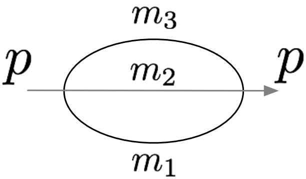
with $\mathcal{L}i_3(\tau,z)$ [Zagier; Beilinson, Levin]

$$\mathcal{L}i_3(\tau,z) := \mathrm{Li}_3(z) + \sum_{n\geqslant 1}\left(\mathrm{Li}_3(q^n z) + \mathrm{Li}_3\left(q^n z^{-1}\right)\right)$$
$$-\left(-\frac{1}{12}\log(z)^3 + \frac{1}{24}\log(q)\log(z)^2 - \frac{1}{720}(\log(q))^3\right).$$

- The 3-loop sunset integral is a regulator period of a motivic class of the K_3 of the the $K3$ surface [Bloch, Kerr, Vanhove]

Mirror Symmetry

sunset



sunrise

The sunset Gromov-Witten invariants

Around $1/s = p^2 = \infty$ the sunset Feynman has the expansion

$$\mathcal{J}_\ominus(s) = -\pi_0 \left(3R_0^3 + \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell>0 \\ (\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3\backslash(0,0,0)}} \ell(1-\ell R_0) N_{\ell_1,\ell_2,\ell_3} \prod_{i=1}^{3} Q_i^{\ell_i} \right).$$

where the Kähler parameters are $Q_i = m_i^2 e^{R_0}$ and R_0 is the logarithmic Mahler measure defined by

$$R_0 := i\pi - \int_{|x|=|y|=1} \log(\Phi_\ominus(x,y)/(xy)) \frac{d\log x d\log y}{(2\pi i)^2}.$$

This is related to the holomorphic $\pi_0(s)$ period near $s = 1/p^2 = 0$

$$\pi_0 = s\frac{dR_0(s)}{ds}$$

The sunset Gromov-Witten invariants

The numbers N_{ℓ_1,ℓ_2,ℓ_3} are local Gromov-Witten expressed in terms of the virtual integer number of degree ℓ rational curves by

$$N_{\ell_1,\ell_2,\ell_3} = \sum_{d|\ell_1,\ell_2,\ell_3} \frac{1}{d^3} n_{\frac{\ell_1}{d},\frac{\ell_2}{d},\frac{\ell_3}{d}} .$$

$\underline{\ell}$	(100)	$\overset{k>0}{(k00)}$	(110)	(210)	(111)	(310)	(220)	(211)	(221)
$N_{\underline{\ell}}$	2	$2/k^3$	-2	0	6	0	$-1/4$	-4	10
$n_{\underline{\ell}}$	2	0	-2	0	6	0	0	-4	10

$\underline{\ell}$	(410)	(320)	(311)	(510)	(420)	(411)	(330)	(321)	(222)
$N_{\underline{\ell}}$	0	0	0	0	0	0	$-2/27$	-1	$-189/4$
$n_{\underline{\ell}}$	0	0	0	0	0	0	0	-1	-48

The sunset Gromov-Witten invariants

For the all equal masses case $m_1 = m_2 = m_3 = 1$, the mirror map is

$$Q = e^{R_0} = -q \prod_{n \geqslant 1} (1 - q^n)^{n\delta(n)}; \qquad \delta(n) := (-1)^{n-1} \left(\frac{-3}{n}\right),$$

where $\left(\frac{-3}{n}\right) = 0, 1, -1$ for $n \equiv 0, 1, 2 \mod 3$.

The local Gromov-Witten numbers

$$\frac{N_\ell}{6} = 1, -\frac{7}{8}, \frac{28}{27}, -\frac{135}{64}, \frac{626}{125}, -\frac{751}{54}, \frac{14407}{343}, -\frac{69767}{512}, \frac{339013}{729}, -\frac{827191}{500}, \frac{8096474}{1331},$$
$$-\frac{367837}{16}, \frac{195328680}{2197}, -\frac{137447647}{392}, \frac{4746482528}{3375}, -\frac{23447146631}{4096}, \frac{115962310342}{4913},$$
$$-\frac{574107546859}{5832}, \frac{2844914597656}{6859}, -\frac{1410921149451}{800}, \frac{10003681368433}{1323}, \dots$$

The sunset mirror symmetry

- The sunset elliptic curve is embedded into a singular compactification X_0 of the local Hori-Vafa 3-fold

$$Y := \{1 - s(m_1^2 x + m_2^2 y + m_3^2)(1 + x^{-1} + y^{-1}) + uv = 0\} \subset (\mathbb{C}^*)^2 \times \mathbb{C}^2 ,$$

  limit of a family of elliptically-fibered CY 3-folds X_z
- The base given by $\Phi_\ominus$ is a toric del Pezzo surface of degree 6
- We have an isomorphism of A- and B-model $\mathbb{Z}$-variation of Hodge structure

$$H^3(\mathrm{X}_{z_0}) \cong H^{even}(\mathrm{X}^\circ_{Q_0}) ,$$

  and taking (the invariant part of) limiting mixed Hodge structure on both sides yields

the sunset Feynman integral given by the second regulator period of the motivic cohomology class is identified to the local Gromov-Witten prepotential for the 3-fold X

Mirror symmetry for elliptically fibered CY 3-fold

- In the degeneration limit the Yukawa coupling CY 3-fold X leads to the local Yukawa of the sunset elliptic curve

$$Y_{ijk} = \int_X \tilde{\Omega} \wedge \nabla_{\delta_i \delta_j \delta_k} \tilde{\Omega} \Longrightarrow Y^{\rm loc}_{0ij} \propto Y_\ominus = \int \Omega_\ominus \wedge \nabla_{\frac{d}{ds}} \Omega_\ominus$$

The holomorphic prepotential of [Huang, Klemm, Poretschkin]

$$F(Q_1, Q_2, Q_3, Q_4) = \frac{c_{ijk} t^i t^j t^k}{3!} + \frac{c_{ij}}{2!} t^i t^j + c_i t^i + c + \sum_{\beta \in H_2(M,\mathbb{Z})} n_0^\beta \, \mathrm{Li}_3(Q^\beta)$$

is mapped to the sunset integral with the identification of the Kähler parameter $Q_r = \exp(2\pi i t_r) = m_r^2 Q$ for $r = 1, 2, 3$ [Klemm private communication]

$$m_1^2 = \frac{(Q_1 Q_2 Q_4)^{\frac{1}{3}}}{Q_1^{\frac{2}{3}}}; \, m_2^2 = \frac{(Q_1 Q_2 Q_4)^{\frac{1}{3}}}{Q_2^{\frac{2}{3}}}; \, m_3^2 = \frac{(Q_1 Q_3 Q_4)^{\frac{1}{3}}}{Q_3^{\frac{2}{3}}}; \, Q = (Q_1 Q_2 Q_3 Q_4)^{\frac{1}{3}}$$

Mirror symmetry for higher sunset integrals

- The same construction applies to the 3-loop sunset graph where $\Phi_4 = 0$ defines a family of $K3$
- The same is conjectured to be true for the 4-loop sunset graph where $\Phi_5 = 0$ defines a family of CY 3-fold. [Doran, Kerr]
  - Not modular in general [Hulek, Verill].
  - Therefore (elliptic) polylogarithm not enough from 4-loop

At higher-loop loop the geometry is more intricate

- Need to go beyond the smoothness hypothesis for $K_{\mathbb{P}_\Delta}$ used in [Lian, Todorov, Yau]
- Need to extend the construction of the motivic cohomology classes and the regulator period of [Doran, Kerr]