Combinatorics and Arithmetic for Physics (Jubilee, Tenth Anniversary Edition) IHES, 15-17 Nov. 2023

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Abstracts.–

Marek Bożejko

Speaker: Marek Bożejko (Institute of Mathematics Wroclaw University)
Title: Fock spaces associated with Coxeter groups of type B.
Abstract: In the talk we give the construction of Fock space related to the infinite hyperoctahedral group,which is related to the two-parameters function $F(q_+, q_-)$. We show that $F(q_+, q_-)$. is positive definite if and only if it is an extreme character of the infinite hyperoctahedral group and we classify the corresponding set of parameters q_+ and q_-. We apply our construction to a cyclic Fock space of type B, generalizing the results of Bozejko and Guta.

Ricardo Buring

Speaker: Ricardo Buring (Inria Saclay)
Title: Graph complex action on Poisson structures: from theory to computation.
Abstract: Poisson brackets on $\mathbb{R}^n$ are bi-linear skew-symmetric bi-derivations of $C^\infty(\mathbb{R}^n)$ satisfying the Jacobi identity, generalizing the canonical Poisson bracket from classical mechanics. Some interesting classes of examples are the Nambu–Poisson brackets defined via the Jacobian determinant with $n - 2$ arbitrary functions, the quadratic and cubic R-matrix Poisson brackets associated with Lie algebras, and all the bi-vector fields on $\mathbb{R}^2$.

While the flow of an arbitrary vector field on $\mathbb{R}^n$ induces a (cohomologically) trivial infinitesimal deformation of a Poisson bracket, some brackets admit nontrivial deformations. To infinitesimally deform any Poisson structure on $\mathbb{R}^n$, M. Kontsevich introduced an infinite family of formulas, depending nonlinearly and differential-polynomially on the Poisson bracket coefficients. Every such deformation formula is constructed from a linear combination of graphs that forms a cocycle in the graph complex, and graph coboundaries are naturally mapped to cohomologically trivial deformations. Nonzero graph cohomology classes (of

which the first example is the tetrahedron) are expected to deform some Poisson bracket nontrivially, but finding an example of this kind is an open problem since 1996.
We illustrate this story in a presentation of the newly developed software package gcaops (Graph Complex Action on Poisson Structures) for SageMath, which is presently used to expand the class of interesting non-examples.
After vigorous computation, the (classes of) vector fields associated with the respective Poisson coboundaries are produced explicitly, with short defining formulas and directed graph representations.

This talk is based on a part of my Ph.D. dissertation, which was supervised by A.V. Kiselev and D. van Straten.

Philippe Di Francesco

Speaker: Philippe Di Francesco (IPhT and UIUC)
Title: Macdonald dualities in genus one and two.
Abstract: We present dualities in the theory of Macdonald/Koornwinder polynomials in a unified way. Macdonald theory has a natural formulation in terms of the spherical Double Affine Hecke Algebra on a torus. We show that duality extends to the genus two Macdonald theory recently introduced by Arthamonov and Shakirov.
(Joint work with R. Kedem)

Gérard H. E. Duchamp

Speaker: Gérard H. E. Duchamp (LIPN UMR 7030 and IHP)
Title: Lazard Elimination on Arbitrary Alphabets, Lyndon Words and Iterated Smash-Products.
Abstract: Lazard elimination (LE) theorems provide uniform formulas for every alphabet (of arbitrary cardinality) and have similar schemes for groups, monoids, Lie algebras and unital associative algebras.
This tool gives rise to many implementable algorithms.
We will start from the most celebrated form of LE i.e. on the category of $\mathbf{k}$-Lie algebras ($\mathbf{k}$ being a unitary ring), concentrate on monoids and Lie algebras and provide examples on iterated smash-products, where the "rewriting on words" (string rewriting) plays a crucial rôle to understand the normal forms and how one converges to them.
If times permits we will give other applications of word indexing to hyperlogarithms and character theory.
Based on joint works with Vu Nguyen Dinh.

Harold Erbin

Speaker: Harold Erbin (Center for Theoretical Physics, MIT, USA)
Title: Introducing string field theory from a geometrical perspective.
Abstract: String field theory (SFT) is a second-quantized version of string theory: it provides an explicit regularization of all amplitudes and allows using all the standard techniques from QFT. In this talk, I will explain how SFT is constructed from the data of a 2d CFT (defining the spacetime background) and a decomposition of the moduli space of Riemann surfaces. The latter is background independent and determines a geometrical BV algebra, which implies that the SFT action is a solution of the BV master equation. It also induces an L-infinity algebra, which characterizes the form of the action and of its gauge symmetries. To conclude, I will exemplify this interplay between geometry and field theory by showing how neural networks can be used to construct data on the moduli spaces and compute the closed string tachyon potential.
`https://arxiv.org/abs/2006.16270`
`https://arxiv.org/abs/2211.09129`
`https://arxiv.org/abs/2301.01686`

Stephane Gaubert

Speaker: Stephane Gaubert (INRIA and CMAP, École polytechnique, IP Paris, CNRS)
Title: The Nullstellensatz and Positivstellensatz for Sparse Tropical Polynomial Systems, and Parametric Mean-Payoff Games.
Abstract: Grigoriev and Podolskii (2018) have established a tropical analogue of the effective Nullstellensatz, showing that a system of tropical polynomial equations is solvable if and only if a linearized system obtained from a truncated Macaulay matrix is solvable. They provided an upper bound of the minimal admissible truncation degree, as a function of the degrees of the tropical polynomials. We establish a tropical Nullstellensatz adapted to <u>sparse</u> tropical polynomial systems. Our approach is inspired by a polyhedral construction of Canny-Emiris (1993), refined by Sturmfels (1994). This leads to an improved bound of the truncation degree, which coincides with the classical Macaulay degree in the case of $n + 1$ equations in n unknowns. We also derive a tropical Positivstellensatz, allowing one to decide the inclusion of tropical basic semialgebraic sets. We finally show that solutions can be computed by a reduction to parametric mean-payoff games, providing a tropical analogue of eigenvalue methods to solve polynomial systems. This is a joint work with Marianne Akian and Antoine Béreau, based in particular on the following article:
The tropical nullstellensatz and positivstellensatz for sparse polynomial systems; In <u>Proceedings of the 2023 International Symposium on Symbolic and Algebraic</u>

Computation. ACM, July 2023;
doi:10.1145/3597066.3597089.

Volker Genz

Speaker: Volker Genz (Center for Geometry and Physics, IBS)
Title: Crystal operators on Cluster Algebras
Abstract: Crystal operators on canonical bases as introduced by Kashiwara/Lusztig provide in particular a toolbox to compute within the category of finite dimensional representations of finite dimensional simple Lie algebras. Motivated by this we previously introduced certain operators on the lattice of tropical points of mirror dual A- and X-cluster spaces. In this talk we give an update. In particular, the crystal structure gives rise to a binary operation on the canonical basis due to Gross-Hacking-Keel-Kontsevich. We expect this to have a wider range of applications in the theory of cluster algebras and in physics.

Darij Grinberg

Speaker: Darij Grinberg (Department of Mathematics, Drexel USA)
Title: The Redei–Berge symmetric function of a directed graph.
Abstract: In 1934, Laszlo Redei observed a peculiar property of tournaments (directed graphs that have an arc between every pair of distinct vertices): Each tournament has an odd number of Hamiltonian paths. In 1996, Chow introduced the "path-cycle symmetric function" of a directed graph, a symmetric function in two sets of arguments, which was later used in rook theory. We study Chow's symmetric function in the case when the y-variables are 0. In this case, we give new nontrivial expansions of the function in terms of the power-sum basis; in particular, we find that it is p-positive as long as the directed graph has no 2-cycles. We use our expansions to reprove Redei's theorem and refine it to a mod-4 congruence.
Joint work with with Richard P. Stanley.
A previous version of the same talk is available there:
https://www.cip.ifi.lmu.de/~grinberg/algebra/haverford2023.pdf

Dimitry Gurevich

Speaker: Dimitry Gurevich (IITP, Moscow)
Title: Combinatorics and quantum invariant differential operators on Reflection Equation algebras.
Abstract: Reflection Equation Algebras (without parameters) are very remarkable objects. They admit introducing q-analogs of certain symmetric functions

and quantum versions of some classical formulae of combinatorics: Capelli, Frobenius and others. Also, they admit introducing quantum analogs of differential operators. I plan to introduce some quantum invariant differential operators and to exhibit their properties.

Yuki Kanabuko

Speaker: Yuki Kanabuko (MPIM Bonn)
Title: Inequalities defining polyhedral realizations of affine types and extended Young diagrams.
Abstract: The crystal bases are powerful tools for studying the representation theory of Lie algebras or quantum groups.
By realizing crystal bases as combinatorial objects, one can reveal skeleton structures of representations.
Nakashima and Zelevinsky invented "polyhedral realizations", which are realizations of crystal bases as integer points in some polyhedral convex cones or polytopes. It is a natural problem to find explicit forms of inequalities that define the polyhedral convex cones and polytopes.
In this talk, we will briefly explain an outline of theory of Lie algebras and quantum groups and give explicit forms of inequalities in terms of combinatorial objects called extended Young diagrams when the associated Lie algebra is of a classical affine type.

Rinat Kedem

Speaker: Rinat Kedem (University of Illinois).
Title: Quantum relativistic Toda Hamiltonians associated with a family of cluster algebras.
Abstract: The quantum relativistic Toda Hamiltonians for classical root systems are obtained as the q-Whittaker limit of (dual) Macdonald operators or specialized Koornwinder operators. They are the conserved quantities of discrete evolutions in a family of quantum cluster algebras known as Q-systems. The polynomial eigenfunctions (q-Whittaker functions) can be constructed from the action of a subset of A-type cluster variables, which act as raising operators. This gives a uniform description for all classical root systems of such eigenfuctions. In some cases, the augmented cluster algebra quiver also gives a candidate for Baxter operators commuting with the quantum Hamiltonians.
Joint with Philippe Di Francesco.

Maxim Kontsevich

Speaker: Maxim Kontsevich (IHES)
Title: The miracle of integer eigenvalues.
Abstract: One can associate with any finite poset with N elements a square matrix depending on $2^(N-1)$ variables, such that each matrix coefficient is just one of variables. Remarkably, all eigenvalues of this matrix are integer linear combinations of variables. Even in the case of the trivial poset, the structure of eigenvalues gives a new result concerning representations of the symmetric group. This is a joint work with R.Kenyon, O.Ogievetsky, A.Pohoata, W.Sawin and S.Shlosman.

Gleb Koshevoy

Speaker: Gleb Koshevoy (ITTP, Moscow and IHES)
Title: On Manin–Schechtman orders related to directed graphs.
Abstract: Studying higher Zamolodchikov equations, in 1989 Manin and Schechtman introduced the notion of a higher Bruhat order on the d-element subsets of a set $[n] = \{1, 2, \ldots, n\}$. We consider a wider model, involving the so-called convex order on certain path systems in an acyclic directed graph, introduce local transformations, or flips, on such orders and establish a generalization of the Manin-Schechtman results. This is a joint work with V.Danilov and A. Karzanov.

Thomas Krajewski

Speaker: Thomas Krajewski (CPT, Marseille)
Title: A twisted version of Kitaev's quantum double model.
Abstract: The quantum double model has been introduced by Kitaev in order to propose a model of quantum computing based on anyonic excitations, thus stable against perturbations. We propose a deformation of this model, induced by an external 3 form. In this model, the role of the quantum double is replaced by its twisted version, a quasi Hopf algebra introduced in orbifold models by Dijkgraaf, Pasquier and Roche.

Marek Kuś

Speaker: Marek Kuś (Center for Theoretical Physics, Polish Academy of Sciences)
Title: Perception of visual art and topological data analysis.
Abstract: I will present applications of topological data analysis (TDA) to study differences in the perception of painting images. A psychological and neurophysiological experiment we conducted showed that physiological responses, in this case eye movements, show variation depending on how the image being perceived was produced. In our experiment, we compared responses for abstract

images painted by a professional artist (”real art”) and computer-generated using neural networks. The analysis of eye movements reveals, as expected, that observers focus on specific features within the image. It is reasonable to assume that our visual attention is primarily drawn to geometric objects In order to support these observations and reasoning we proposed to use methods of algebraic topology (“Topological Data Analysis”). We were able to show significant differences between the topological properties of “real” and generated images.

Hiroaki Nakamura

Speaker: Hiroaki Nakamura (Osaka University, Japan)
Title: Demi-shuffle duals of Magnus polynomials in a free associative algebra.
Abstract: We study two linear bases of the free associative algebra: one is formed by the Magnus type polynomials and the other is its dual basis (formed by what we call the ‘demi-shuffle’ polynomials) with respect to a standard pairing. As an application, we show a formula of Le-Murakami, Furusho type that expresses arbitrary coefficients of a group-like series in terms of its “regular” coefficients. This talk illustrates my recent paper published in Algebraic Combinatorics Volume 6 (2023) no. 4, pp. 929-939.

Toshiki Nakashima

Speaker: Toshiki Nakashima (Sophia University, Tokyo)
Title: Categorified Crystal Bases on Localized Quantum Coordinate Rings and Cellular Crystals.
Abstract: For a monoidal category $(\mathcal{T}, \circ)$, if there exists a“real commuting family $(C_i, R_{C_i}, \phi_i)_{i\in I}$”, we can define a localization $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ by $(C_i, R_{C_i}, \phi_i)_{i\in I}$. Let $R = R(\mathfrak{g})$ be the quiver Hecke algebra(=KLR algebra) associated with a simple Lie algebra $\mathfrak{g}$ and R-gmod the category of finite-dimensional graded R-modules, which is a monoidal category with a real commuting family $(C_i, R_{C_i}, \phi_i)_{i\in I}$. Thus, we get its localization $\widetilde{R}$-gmod. It has been shown that R-gmod categorifies the unipotent quantum coordinate ring $\mathcal{A}_q(\mathfrak{g})$, that is, the Grothendieck ring $\mathcal{K}(R\text{-gmod})$ is isomorphic to $\mathcal{A}_q(\mathfrak{g})$. For the localized category $\widetilde{R}$-gmod, its Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ defines the localized (unipotent) quantum coordinate ring $\widetilde{\mathcal{A}_q(\mathfrak{g})}$.

We shall give a certain crystal structure on the set of self-dual simple objects $\mathbb{B}(\widetilde{R}\text{-gmod})$ in $\widetilde{R}$-gmod. We also give the isomorphism of crystals from $\mathbb{B}(\widetilde{R}\text{-gmod})$ to the cellular crystal $\mathbb{B}_{\mathbf{i}} = B_{i_1} \otimes \cdots \otimes B_{i_N}$ for an arbitrary reduced word $\mathbf{i} = i_1 \cdots i_N$ of the longest Weyl group element. This result can be seen as a localized version for the categorification of the crystal base $B(\infty)$ for the subalgebra $U_q^-(\mathfrak{g})(\cong \mathcal{A}_q(\mathfrak{g}))$ of the quantum algebra $U_q(\mathfrak{g})$, given by Lauda-Vazirani.

Hadrien Notarantonio

Speaker: Hadrien Notarantonio (Inria Saclay), France)
Title: Solving equations from combinatorics via computer algebra.
Abstract: Enumerative combinatorics contains a vast landscape of problems that could hardly be solved without the consideration of special functional equations called "Discrete Differential Equations". Among these problems, the enumeration of walks, planar maps carrying hard particles, etc. These functional equations relate formal power series in n variables with specializations of them to some of the variables (the specializations being generating functions related to the enumeration of interest). When the involved variables are "nested", a celebrated result by Popescu (1986) implies algebraicity of the solutions. In 2006, Bousquet-Mélou and Jehanne provided an elementary proof of algebraicity of the solutions in the case $n = 2$. Their proof yields an algorithm, and it has been the state-of-the-art in enumerative combinatorics for solving these equations since then. In this talk, I will present a recent approach, based on the intensive use of effective algebraic geometry, in order to solve more efficiently such equations in the case n=2. Also, I will introduce and discuss recent advances in the case of systems of such equations.
The talk is based on joint works with Alin Bostan, Mohab Safey El Din and Sergey Yurkevich.

Karol A. Penson

Speaker: Karol A. Penson (LPTMC, Paris)
Title: Counting restricted set partitions with hypergeometrics.
Abstract: As it is known, the function $F(t) = exp(exp(t) - 1)$ is the exponential generating function (egf) of the (integer) Bell numbers $B(n)$, with $B(n) = 1,\ 1,\ 2,\ 5,\ 15,\ 52,\ 203, \cdots$, for $n = 0,\ 1 \cdots$. The $B(n)'s$ count the numbers of all partitions of a set of n distinguishable objects into indistinguishable "boxes". In this work, we investigate some properties of integers describing restricted partitions (RP), resulting from the deformations of the exponent in $F(t)$ in two following ways:
(A) - either we substract a finite number of terms from the exponent in $F(t)$, or
(B) - we retain only a finite number of terms in the exponent in $F(t)$. In case (A) the so obtained egfs generate the numbers of RP without "boxes" of certain sizes, whereas in case (B) the appropriate egfs generate the numbers of RP with only "boxes" of certain sizes. We treat many instances from (A) and (B), employing the multi-variable extensions of Hermite polynomials (also known as Gould-Hopper or Kampé de Fériet polynomials), developed by G. Dattoli et al. In this manner we obtain closed-form expressions for many enumerating sequences of RP, termed restricted Bell numbers, entirely in terms of finite operations on generalized hypergeometric functions. These formulas can be efficiently handled by

Computer Algebra Systems. Some drawbacks of this formalism are also pointed out.

Eric Pichon-Pharabod

Speaker: Eric Pichon-Pharabod (Université Paris-Saclay — Inria Saclay), France)
Title: Periods of hypersurfaces via effective homology.
Abstract: The period matrix of a smooth complex projective variety X encodes the isomorphism between the singular homology of X and its De Rham cohomology given by the De Rham theorem. Numerical approximations with sufficient precision of the entries of this matrix, called periods, allow to recover some algebraic invariants of the varieties, as shown for example by Torelli-type theorems. I will present a method relying on the computation of an effective description of the homology for obtaining such numerical approximations of the periods of hypersurfaces. This method is sufficiently efficient to allow computing the periods of dense quartic K3 surfaces in the three-dimensional projective space.

Sanjaye Ramgoolam

Speaker: Sanjaye Ramgoolam (Queen Mary University of London)
Title: Combinatorial topological quantum field theories and geometrical constructions of integers in finite group representation theory.
Abstract: Topological quantum field theories (TQFTs) which have a simple physical formulation as lattice gauge theory with finite gauge group G admit elegant expressions for partition functions on closed higher genus Riemann surfaces. There are expressions for the partition functions in terms of the combinatorial counting of flat G-bundles and in terms of dimensions of irreducible representations (irreps). Consideration of the partition functions of these G-Flat-TQFTs across different genuses gives finite algorithms which start from group multiplications and yield the spectrum of dimensions of irreps. The input into the algorithms is formed by identities which generalise the classic formula for the order of a group as a sum of squares of the dimensions of irreps. Considering the partition functions of the G-Flat-TQFTs for surfaces with boundaries leads to the derivation of integrality properties of certain partial sums along columns of the character table of G. Analogous considerations starting from a topological field theory based on the fusion ring of a finite group (denoted G-Fusion-TQFT) allows the proof of analogous integrality properties for partial sums along rows of the character table. These row-column relations between integrality properties of characters can be viewed as a mathematical reflection of a physical row-column duality between the G-flat TQFTs and the G-fusion TQFTs. The talk is based on the papers :
1. `https://arxiv.org/abs/2106.05598` : Integrality, duality and finiteness in combinatoric topological strings ;
2. `https://arxiv.org/abs/2304.10217` : Row-Column duality and

combinatorial topological strings ;
3. `https://arxiv.org/abs/2204.02266` : Combinatoric topological string theories and group theory algorithms.

Vasily Sazonov

Speaker: Vasily Sazonov (Paris-Saclay University, CEA, List)
Title: New Initial Approximation in Loop Vertex Expansion.
Abstract: Loop Vertex Expansion (LVE) was developed for the construction of QFT models with local and non-local interactions. Using LVE, one can prove the analyticity in the finite cardioid-like domain in the complex plain of the coupling constant of the free energies and cumulants of various vector, matrix, or tensor-type models. In this talk, applying the idea of choosing the initial approximation depending on the coupling constant, I construct the analytic continuation of the free energy of the quartic matrix model beyond the standard LVE cardioid.

Travis Scrimshaw

Speaker: Travis Scrimshaw (Hokkaido University, Japan)
Title: Limit shapes from skew Howe duality.
Abstract: The dual Cauchy identity is the character version of the $GL_n \times GL_k$ action on the exterior algebra of the natural representation. Additionally, (up to normalization) it is an example of a Schur measure on random partitions. By using other Lie groups (more precisely, dual reductive pairs), we can get analogous representation theoretic statements, which is known as skew Howe duality, and take the corresponding characters. In this talk, we will consider the measure by further specializing the characters to their dimensions to get a probability measure on partitions and describe their limit shapes for a number of dual reductive pairs. This is based on joint work with Anton Nazarov and Olga Postnova.

Andrea Sportiello

Speaker: Andrea Sportiello (Paris-North University, LIPN (UMR 7030), France)
Title: What does perturbative field theory teaches us on the statistical properties of Monge-Kantorovich Optimal Transportation in dimension 2?
Abstract: We all know that Feynman diagrams and integrals describe quantities in Particle Physics, and, sometimes, in Statistical Mechanics. We also know that, in these circumstances, we have to learn how to ”deal with infinities”, and regularise certain divergences. It may appear more suprising that the same ideas apply to this famous old problem in Functional Analysis, and also the same pathologies do arise. But at least, in this case, we see more clearly where do they come from. The precise content of the talk is still to be determined, but the message shall be twofold: on one side, we get results on the Optimal Transportation Problem in

itself; on the other side, we learn a lesson on why a more naive approach does fail, while a more accurate approach allows to cure the apparent divergences, which could be of general interest.

Adrian Tanasa

Speaker: Adrian Tanasa (Bordeaux University, France)
Title: Tensor models.
Abstract: Matrix models, seen as quantum field theoretical models, are known to represent a successful approach to 2D quantum gravity and to have many other interesting applications in Physics. Some of the main results of the study of matrix models in theoretical physics are the 't Hooft large N limit (the perturbative series can be reorganized in powers of 1/N (N being the matrix size)) and the double scaling limit mechanism (known to be related to the continuuos limit of the models). After a brief introduction dedicated to matrix models, I will focus in this talk on tensor models, which are a natural quantum field theoretical generalization of matrix models. In particular, I will present the implementation of the large N limit (N being now the size of the tensor) and the double scaling limit mechanisms for various tensor models. In the last part of the talk I will present how tensor models have been related (initially by Witten and then shortly after by Klebanov and Tarnopolsky) to the Sachdev-Ye-Kitaev model, which is known to be a particularly interesting toy model for holography.

Jean-Bernard Zuber

Speaker: Jean-Bernard Zuber (LPTHE, Sorbonne Université).
Title: Counting partitions by genus.
Abstract:
The counting of set partitions according to their genus is revisited. The case of genus 0 – non-crossing partitions – is well known. I will show how, using a functional equation between generating functions, all partitions may be reconstructed from the "(semi)-primitive" ones introduced by Cori and Hetyei. This will be carried out explicitly in genus 1 and 2. Time permitting, known results and conjectures about higher genus will be reviewed
(based on arxiv:2303.05875 and 2305.01100).

Karol Życzkowski

Speaker: Karol Życzkowski (Center for Theoretical Physics, PAS, Warsaw and Jagiellonian University, Cracow).
Title: Thirty-six entangled officers of Euler:
quantum solution of a classically impossible combinatorial problem.
Abstract:

A *quantum combinatorial designs* is composed of quantum states, arranged with a certain symmetry and balance. They determine distinguished quantum measurements and can be applied for quantum information processing. Negative solution to the famous problem of 36 officers of Euler implies that there are no two orthogonal Latin squares of order six.

We show that the problem has a solution, provided the officers are entangled, and construct orthogonal quantum Latin squares of this size [1,2]. The solution can be visualized on a chessboard of size six, which shows that 36 officers are splitted in nine groups, each containing of four entangled states [3]. It allows us to construct a pure nonadditive quhex quantum error detection code.

References:

1. S.A Rather, A.Burchardt, W. Bruzda, G. Rajchel-Mieldzioć, A. Lakshminarayan and K. Życzkowski, Thirty-six entangled officers of Euler, Phys. Rev. Lett. 128, 080507 (2022).

2. D. Garisto, Euler's 243-Year-Old 'Impossible' Puzzle Gets a Quantum Solution, Quanta Magazine, Jan. 10, 2022; https://www.quantamagazine.org/

3. K. Życzkowski, W. Bruzda, G. Rajchel-Mieldzioć, A. Burchardt, S. A. Rather, A. Lakshminarayan, $9 \times 4 = 6 \times 6$: Understanding the quantum solution to the Euler's problem of 36 officers, J. Phys.: Conf. Series 2448, 012003 (2023).