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HOPF ALGEBRAS OF DIAGRAMS

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We investigate several generalizations of the Hopf algebra **MQSym** whose constructions come from labelings of special diagrams in bijection with packed matrices. Their products come either from the Hopf algebras **WSym** or **WQSym**, respectively built on integer set partitions and set compositions. Realizations on bi-word are exhibited, and it is shown how these algebras fit into a commutative diagram. Hopf deformations and dendriform structures are also considered for some algebras in the picture.

Keywords: Hopf algebras; bipartite graphs; dendriform structures.

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1. Introduction

The purpose of the present paper is to tighten the links between a body of Hopf algebras related to physics and the realm of noncommutative symmetric functions. This paper can be seen as the continuation of [2], new ideas coming from diagram constructions of a special Field Theory introduced by Bender, Brody and Meister [1]. These diagrams arise in the expansion of

$$G(z) = \exp\left\{\sum_{n\geq 1} \frac{L_n}{n!}\left(z\frac{\partial}{\partial x}\right)^n\right\}\exp\left\{\sum_{m\geq 1} V_m \frac{x^m}{m!}\right\}\Bigg|_{x=0} \tag{1}$$

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and are bipartite finite graphs with no isolated vertex, and edges weighted with integers. These graphs are in bijective correspondence with packed matrices of integers up to a permutation of the columns and a permutation of the rows. From the algebraic point of view, a Hopf algebra named **LDIAG**(q_s, q_c, t) has been recently discovered [4], interpolating between the algebra **LDIAG** indexed by bipartite graphs and the algebra **MQSym** of matrix quasi-symmetric functions, indexed by packed matrices. The algorithm constructing the matrix from the associated diagram uses as an intermediate structure a particular packed matrix whose entries are sets. Such set matrices appear when one computes the internal product in **WSym** [10] and in **WQSym** [7, 9], an algebra isomorphic to the sum of Solomon–Tits. In this context, it becomes natural to investigate Hopf algebras on (set) packed matrices whose product comes from both **WSym** and **WQSym**, two well-known Hopf algebras respectively built on integer set partitions and set compositions.

The paper is organized as follows. In Sec. 2, the connection between the Quantum Field Theory of partitions and the Hopf algebra **LDIAG** of labeled diagrams is recalled. In Sec. 3, we investigate eight Hopf algebras of matrices related to labeled or unlabeled diagrams. In particular, we exhibit realizations on bi-words and show how some of these are bidendriform bialgebras, hence proving those algebras are in particular self-dual, free and cofree. We finally give (Sec. 7) a realization of the two-parameter deformation of **LDIAG**.

2. Algebras of Diagrams

Many computations carried out by physicists reduce to the "product formula", a bilinear coupling between two Taylor expandable functions, introduced by Bender, Brody, and Meister in their celebrated *Quantum field theory of partitions* (henceforth referred to as QFTP) [1]. For an example of such a computation derived from a partition function linked to the Free Boson Gas model, see [12].

The expansion of Formula 1 involves a summation over all diagrams of a certain type [1], a labeled version of which is described below. These diagrams are bipartite graphs with multiple edges. Bender, Brody and Meister [1] introduced QFTP as a toy model to show that every (combinatorial) sequence of integers can be represented by Feynman diagrams subject to suited rules.

The case where the expansions of the two functions occurring in their product formula have constant term 1 is of special interest. Indeed, these functions can be presented as exponentials which can be regarded as "free" through the classical Bell polynomials expansion [3]. Working out the formal case, one sees that the coupling results in a summation without multiplicity of a certain kind of labeled bipartite graphs which are equivalent, as a data structure, to pairs of unordered partitions of the set $\{1, 2, \ldots, n\}$. The sum reduces to a sum of topologically inequivalent diagrams (a monoidal basis of **DIAG**), at the cost of introducing multiplicities. These graphs, which can be considered as the Feynman diagrams of the QFTP, generate a Hopf algebra. Interpreting **DIAG** as the Hopf homomorphic image of its planar counterpart **LDIAG**, one gets access to the noncommutative world and to deformations: the product is deformed by taking into account, through two variables, the number of crossings of edges involved in the superposition or the transposition of two vertices. This gives the final picture of [12].

Labeled diagrams can be identified with their weight functions which are mappings $\omega : \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ such that the supporting subgraph

$$\Gamma_\omega = \{(i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid \omega(i, j) \neq 0\} \tag{2}$$

has projections, i.e., $\mathrm{pr}_1(\Gamma_\omega) = [1, p]$; $\mathrm{pr}_2(\Gamma_\omega) = [1, q]$ for some $p, q \in \mathbb{N}$.[a]

Let **ldiag** denote the set of labeled diagrams. With any element d of **ldiag**, one can associate the monomial $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$, called its multiplier, where $\alpha(d)$ (resp. $\beta(d)$) is the "white spot type" (resp. the "black spot type"), i.e., the multi-index $(\alpha_i)_{i\in\mathbb{N}^+}$ (resp. $(\beta_i)_{i\in\mathbb{N}^+}$) such that α_i (resp. β_i) is the number of white spots (resp. black spots) of degree i. For example, the multiplier of the labeled diagram of Fig. 1 is $\mathbb{L}^{(0,0,2,0,1)}\mathbb{V}^{(1,1,1,0,1)}$.

One can endow **ldiag** with an algebra structure denoted by **LDIAG** where the sum is the formal sum and the product is the shifted concatenation of diagrams, i.e., consists in juxtaposing the second diagram to the right of the first one and then adding to the labels of the black spots (resp. of the white spots) of the second diagram the number of black spots (resp. of white spots) of the first diagram. Then the application sending a diagram to its multiplier is an algebra homomorphism.

Moreover, the black spots (resp. white spots) of diagram d can be permuted without changing the monomial $\mathbb{L}^{\alpha(d)}\mathbb{V}^{\beta(d)}$. The classes of labeled diagrams up to this equivalence relation (permutations of white — or black — spots between

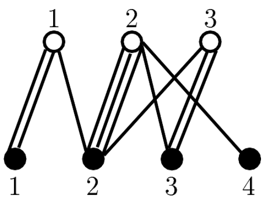

Fig. 1. A labeled diagram of shape 3×4.

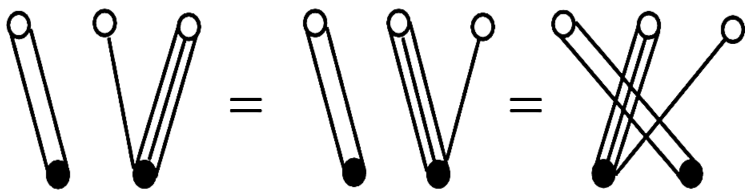

Fig. 2. Equivalent labeled diagrams.

[a]Remark that if p or q are zero, then both are and $\Gamma = \emptyset$.

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themselves, see Fig. 2) are naturally represented by unlabeled diagrams. The set of unlabeled diagrams will be henceforth denoted by **diag**.

The set **diag** can also be endowed with an algebra structure, denoted by **DIAG**, e.g., as the quotient of **LDIAG** by the equivalence classes of labeled diagrams. In **DIAG**, the product of d_1 by d_2 is basic concatenation, i.e., simply consists in juxtaposing d_2 to the right of d_1 [3].

3. Packed Matrices and Related HOPF Algebras

3.1. *The combinatorial objects*

In the sequel, we represent different kinds of diagrams using matrices to emphasize the parallel between this construction and the construction of **MQSym** ([5]).

3.1.1. *Set packed matrices*

Since the computations are similar in many cases, let us begin with the most general case and explain how one recovers the other cases by algebraic means. Let us consider the set **lldiag** of bipartite graphs with white and black vertices, and edges, all three labeled by initial intervals $[1, p]$ of $\mathbb{N}^+$. The diagrams **ldiag** are obtained by erasing the labels of the edges of such an element.

The set **lldiag** is in direct bijection with *set packed matrices*, that are two-row matrices containing disjoint subsets of $[1, n]$ for some $n \in \mathbb{N}$ with no line or column filled with empty sets, and such that the union of all subsets is $[1, n]$ itself. The bijection consists in putting k in the (i, j) entry of the matrix if the edge labeled k connects the white dot labeled i with the black dot labeled j. Figure 3 shows an example of such a matrix.

Note that set packed matrices are in bijection with pairs of *set compositions*, or, ordered set partitions of $[1, n]$: given a set packed matrix, compute the ordered sequence of the union of the elements in the same row (resp. column). For example, the set packed matrix of Fig. 3 gives rise to the set compositions $[\{2, 3, 6\}, \{1, 4, 5\}]$ and $[\{3\}, \{1, 5, 6\}, \{2, 4\}]$. Given two set compositions Π and Π', define $M_{ij} := \Pi_i \cap \Pi'_j$.

Hence the generating series counting set packed matrices by their maximum entry n is given by the square of the ordered Bell numbers, that is sequence A122725 in [11].

3.1.2. *Integer packed matrices*

As already said, if one forgets the labels of the edges of an element of **lldiag**, one recovers an element of **ldiag**. Its matrix representation is an *integer packed matrix*,

$$\begin{pmatrix} \{3\} & \{6\} & \{2\} \\ \emptyset & \{1, 5\} & \{4\} \end{pmatrix}$$

Fig. 3. A set packed matrix.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

Fig. 4. An integer packed matrix.

that is, a matrix with no line or column filled with zeros. The encoding is as follows: it consists in replacing the subsets by their cardinality. Figure 4 shows an example of such a matrix.

The generating series counting integer packed matrices by the sum of their entries is given by sequence A120733 of [11].

3.1.3. *Other packed matrices*

In the sequel, we shall also consider diagrams where one forgets about the labels of the white spots, or about the labels of the black spots, or about all labels. Those three classes of diagrams are respectively in bijection with matrices up to a permutation of the rows, a permutation of the columns, and simultaneous permutations of both.

3.2. *Word quasi-symmetric and symmetric functions*

Let us recall briefly the definition of two combinatorial Hopf algebras that will be useful in the sequel.

3.2.1. *The Hopf algebra* **WQSym**

We use the notations of [9]. The word quasi-symmetric functions are the noncommutative polynomial invariants of Hivert's quasi-symmetrizing action [7]

$$\mathbf{WQSym}(A) := \mathbb{C}\langle A\rangle^{\mathfrak{S}(A)_{QS}}. \tag{3}$$

When A is an infinite alphabet, $\mathbf{WQSym}(A)$ is a graded Hopf algebra whose basis is indexed by set compositions, or, equivalently, packed words. Recall that packed words are words w on the alphabet $[1,k]$ such that if $i \neq 1$ appears in w, then $i-1$ also appears in w. The bijection between both sets is that $w_i = j$ iff i is in the jth part of the set composition (see Fig. 5 for an example).

By definition, **WQSym** is generated by the polynomials

$$\mathbf{WQ}_u := \sum_{\mathrm{pack}(w)=u} w, \tag{4}$$

where $u = \mathrm{pack}(w)$ is the packed word having the same comparison relations between all elements as w.

$$32214132 \qquad \longleftrightarrow \qquad [\{4,6\},\{2,3,8\},\{1,7\},\{5\}]$$

Fig. 5. A packed word and its corresponding set composition.

The product in **WQSym** is given by

$$\mathbf{WQ}_u \mathbf{WQ}_v = \sum_{w \in u \star_W v} \mathbf{WQ}_w, \tag{5}$$

where the convolution $u \star_W v$ of two packed words is defined by

$$u \star_W v := \sum_{\substack{w_1, w_2; w = w_1 w_2 \text{ packed} \\ \mathrm{pack}(w_1) = u,\ \mathrm{pack}(w_2) = v}} w. \tag{6}$$

The coproduct is given by

$$\Delta \mathbf{WQ}_w(A) = \sum_{u, v; w \in u \Cup_W v} \mathbf{WQ}_u \otimes \mathbf{WQ}_v \tag{7}$$

where $u \Cup_W v$ denotes the *packed shifted shuffle* that is the shuffle of u and $v' = v[\max(u)]$, that is the word such that $v'_i = v_i + \max(u)$.

The dual algebra $\mathbf{WQSym}^*$ of **WQSym** is a subalgebra of the Parking quasi-symmetric functions **PQSym** [8]. This algebra has a multiplicative basis denoted by $\mathbf{F}^w$, where the product is the shifted concatenation, that is $u.v[\max(u)]$.

3.2.2. WSym

The algebra of word symmetric functions **WSym**, first defined by Rosas and Sagan in [10], where it is called the algebra of symmetric functions in noncommuting variables, is the Hopf subalgebra of **WQSym** generated by

$$\mathbf{W}_\pi := \sum_{\mathrm{sp}(u) = \pi} \mathbf{WQ}_u, \tag{8}$$

where $\mathrm{sp}(u)$ is the (unordered) set partition obtained by forgetting the order of the parts of its corresponding set composition.

Its dual $\mathbf{WSym}^*$ is the quotient of $\mathbf{WQSym}^*$

$$\mathbf{WSym}^* = \mathbf{WQSym}^* /_J \tag{9}$$

where J is the ideal generated by the polynomials $\mathbf{F}^u - \mathbf{F}^v$ with u and v corresponding to the same set partition. We denote by $\mathbf{F}^{\mathrm{sp}(u)}$ the image of $\mathbf{F}^u$ by the canonical surjection.

4. Hopf Algebras of Set Packed Matrices

4.1. *Set matrix quasi-symmetric functions*

The construction of the Hopf algebra **SMQSym** over set packed matrices is a direct adaptation of the construction of **MQSym** ([5, 7]). Consider the linear subspace spanned by the elements $\mathbf{SMQ}_M$, where M runs over the set of set packed matrices. We denote by $h(M)$ the number of rows of M. Then define

$$\mathbf{SMQ}_P\mathbf{SMQ}_Q := \sum_{R \in \text{Ш}(P,Q)} \mathbf{SMQ}_R \tag{10}$$

where the *augmented shuffle* of P and Q, $\text{Ш}(P,Q)$ is defined as follows: let Q' be obtained from Q by adding the greatest number inside P to all elements inside Q. Let r be an integer between $\max(p,q)$ and $p+q$, where $p = h(P)$ and $q = h(Q)$. Insert rows of zeros in the matrices P and Q' so as to form matrices $\tilde{P}$ and $\tilde{Q}'$ of height r. Let R be the matrix obtained by gluing $\tilde{Q}'$ to the right of $\tilde{P}$. The set $\text{Ш}(P,Q)$ is formed by all matrices with no row of 0's obtained this way.

For example,

$$\begin{aligned}
\mathbf{SMQ}_{\begin{pmatrix} \{3,4\} & \{1\} \\ \{2\} & \emptyset \end{pmatrix}} \mathbf{SMQ}_{\begin{pmatrix} \{2,3,4\} & \{1\} \end{pmatrix}} &= \mathbf{SMQ}_{\begin{pmatrix} \{3,4\} & \{1\} & \emptyset & \emptyset \\ \{2\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \{6,7,8\} & \{5\} \end{pmatrix}} \\
&\quad + \mathbf{SMQ}_{\begin{pmatrix} \{3,4\} & \{1\} & \emptyset & \emptyset \\ \{2\} & \emptyset & \{6,7,8\} & \{5\} \end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix} \{3,4\} & \{1\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{6,7,8\} & \{5\} \\ \{2\} & \emptyset & \emptyset & \emptyset \end{pmatrix}} \\
&\quad + \mathbf{SMQ}_{\begin{pmatrix} \{3,4\} & \{1\} & \{6,7,8\} & \{5\} \\ \{2\} & \emptyset & \emptyset & \emptyset \end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix} \emptyset & \emptyset & \{6,7,8\} & \{5\} \\ \{3,4\} & \{1\} & \emptyset & \emptyset \\ \{2\} & \emptyset & \emptyset & \emptyset \end{pmatrix}}.
\end{aligned} \tag{11}$$

The coproduct $\Delta\mathbf{SMQ}_M$ is defined by

$$\Delta\mathbf{SMQ}_A = \sum_{A = \binom{A_1}{A_2}} \mathbf{SMQ}_{\mathrm{std}(A_1)} \otimes \mathbf{SMQ}_{\mathrm{std}(A_2)}, \tag{12}$$

where $\mathrm{std}(A)$ denotes the standardized of the matrix A, that is the matrix obtained by the substitution $a_i \mapsto i$, where $a_1 < \cdots < a_n$ are the integers appearing in A. For example,

$$\begin{aligned}
\Delta\mathbf{SMQ}_{\begin{pmatrix} \{2,4\} & \{1\} \\ \{6\} & \{3,5\} \end{pmatrix}} &= \mathbf{SMQ}_{\begin{pmatrix} \{2,4\} & \{1\} \\ \{6\} & \{3,5\} \end{pmatrix}} \otimes 1 + \mathbf{SMQ}_{\begin{pmatrix} \{2,3\} & \{1\} \end{pmatrix}} \\
&\quad \otimes \mathbf{SMQ}_{\begin{pmatrix} \{3\} & \{1,2\} \end{pmatrix}} + 1 \otimes \mathbf{SMQ}_{\begin{pmatrix} \{2,4\} & \{1\} \\ \{6\} & \{3,5\} \end{pmatrix}}.
\end{aligned} \tag{13}$$

Rather than checking the compatibility between the product and the coproduct, one can look for a realization of **SMQSym** in terms of noncommutative bi-words, that will later give useful guidelines to understand homomorphisms between the different algebras.

4.2. *Realization of* SMQSym

A noncommutative bi-word is a word over an alphabet of bi-letters $\left\langle {a_i \atop \mathbf{b}_j} \right\rangle$ with $i, j \in \mathbb{N}^+$ where a_i and $\mathbf{b}_j$ are letters of two distinct ordered alphabets A and $\mathbf{B}$. We will denote by $\mathbb{C}\left\langle {A \atop \mathbf{B}} \right\rangle$ the algebra of the polynomials over the bi-letters $\left\langle {a \atop \mathbf{b}} \right\rangle$ for the product $\star$ defined by

$$\left\langle {u_1 \atop \mathbf{v}_1} \right\rangle \star \left\langle {u_2 \atop \mathbf{v}_2} \right\rangle = \left\langle {u_1 \cdot u_2 \atop \mathbf{v}_1 \cdot \mathbf{v}_2[\max(\mathbf{v}_1)]} \right\rangle, \tag{14}$$

where $\mathbf{v}_1 \cdot \mathbf{v}_2[k]$ denotes the concatenation of $\mathbf{v}_1$ with the word $\mathbf{v}_2$ whose letters are shifted by k and $\max(\mathbf{v}_1)$ denotes the maximum letter of $\mathbf{v}_1$.

In the sequel, we shall forget the letters a and $\mathbf{b}$ when there is no ambiguity about the alphabets A and $\mathbf{B}$, so that, for example,

$$\left\langle {1\,4\,2 \atop 2\,3\,6} \right\rangle \star \left\langle {2\,4 \atop 3\,1} \right\rangle = \left\langle {1\,4\,2\,2\,4 \atop 2\,3\,6\,9\,7} \right\rangle. \tag{15}$$

Lemma 4.1. *The product $\star$ is associative.*

Proof. Straightforward from its definition. □

With each set packed matrix, one associates the bi-word whose ith bi-letter is the coordinate in which the letter i appears in the matrix. For example,

$$\text{bi-word} \begin{pmatrix} \{2,7\} & \emptyset & \emptyset & \{3,5\} \\ \{8\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \{1\} & \{4,6\} & \emptyset \end{pmatrix} = \left\langle {3\,1\,1\,3\,1\,3\,1\,2 \atop 2\,1\,4\,3\,4\,3\,1\,1} \right\rangle. \tag{16}$$

A bi-word is said bi-packed if its two words are packed. The bi-packed of a bi-word is the bi-word obtained by packing its two words. The set packed matrices are obviously in bijection with the bi-packed bi-words.

Theorem 4.2. *Let $\left\langle {u \atop \mathbf{v}} \right\rangle$ be a bi-packed bi-word. Then*

- *The algebra* **SMQSym** *can be realized on bi-words by*

$$\mathbf{SMQ}_{\left\langle {u \atop \mathbf{v}} \right\rangle} := \sum_{\text{bi-packed}\left\langle {u' \atop \mathbf{v}'} \right\rangle = \left\langle {u \atop \mathbf{v}} \right\rangle} \left\langle {u' \atop \mathbf{v}'} \right\rangle. \tag{17}$$

- **SMQSym** *is a Hopf algebra.*
- **SMQSym** *is isomorphic as a Hopf algebra to the graded endomorphisms of* **WQSym**:

$$\mathrm{End}_{gr}\mathbf{WQSym} = \bigoplus_n \mathbf{WQSym}_n \otimes \mathbf{WQSym}_n^* \tag{18}$$

through the Hopf homomorphism

$$\phi\left(\mathbf{SMQ}_{\left\langle {u \atop \mathbf{v}} \right\rangle}\right) = \mathbf{WQ}_u \otimes \mathbf{F}^{\mathbf{v}}. \tag{19}$$

Proof. Since the map sending each set packed matrix to a bi-packed bi-word is a bijection, the first part of the theorem amounts to checking the compatibility of the product,

$$\mathbf{SMQ}_{\mathrm{biword}(P)}\mathbf{SMQ}_{\mathrm{biword}(Q)} = \sum_{R\in \underline{\sqcup\!\sqcup}(P,Q)} \mathbf{SMQ}_{\mathrm{biword}(R)}, \tag{20}$$

which is straightforward from the definition.

SMQSym being a connected graded algebra, it suffices to show that the coproduct Δ is a homomorphism of algebras. If one uses the representation of the basis elements by pairs of set compositions, the coproduct reads

$$\Delta\mathbf{SMQ}_{\Pi_1,\Pi_2} = \sum_{\Pi_1=[\Pi_1',\Pi_1'']} \mathbf{SMQ}_{\mathrm{std}(\Pi_1'),\mathrm{std}(\Pi_2|_{\Pi_1'})} \otimes \mathbf{SMQ}_{\mathrm{std}(\Pi_1''),\mathrm{std}(\Pi_2|_{\Pi_1''})}, \tag{21}$$

where $\Pi_1|_{\Pi_2}$ is the list of sets $[(\Pi_1)_i \cap \bigcup_j (\Pi_2)_j]_i$ from which one erases the empty sets. The proof then amounts to imitating the proof that **WQSym** is a Hopf algebra (see [7]).

One endows $\mathrm{End}_{gr}\mathbf{WQSym}$ with the coproduct Δ defined by

$$\Delta\mathbf{WQ}_{\Pi_1}\otimes\mathbf{F}^{\Pi_2} = \sum_{\Pi_1=[\Pi_1',\Pi_1'']} (\mathbf{WQ}_{\Pi_1'}\otimes\mathbf{F}^{\Pi_2|_{\Pi_1'}})\otimes(\mathbf{WQ}_{\Pi_1''}\otimes\mathbf{F}^{\Pi_2|_{\Pi_1''}}). \tag{22}$$

One then easily checks that ϕ is a surjective Hopf homomorphism and since the two spaces have same series of dimensions, we get the result. $\square$

For example,

$$\mathbf{SMQ}_{\begin{pmatrix} \{3,4\} & \{1\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{6\} & \emptyset \\ \{2\} & \emptyset & \emptyset & \{5,7\} \end{pmatrix}} = \mathbf{SMQ}_{\left\langle\begin{smallmatrix}1\,3\,1\,1\,3\,2\,3\\ 2\,1\,1\,1\,4\,3\,4\end{smallmatrix}\right\rangle}$$

$$= \sum_{\substack{j_1<j_2<j_3\\ k_1<k_2<k_3<k_4}} \left\langle \begin{matrix} j_1\, j_3\, j_1\, j_1\, j_3\, j_2\, j_3 \\ k_2\, k_1\, k_1\, k_1\, k_4\, k_3\, k_4 \end{matrix}\right\rangle. \tag{23}$$

Note that, from the point of view of the realization, the coproduct of **SMQSym** is given by the usual trick of noncommutative symmetric functions, considering an alphabet A of bi-letters ordered lexicographically as an ordered sum of two mutually commuting alphabets $A'\hat{+}A''$ of bi-letters such that if (x,y) is in A' then so is any bi-letter of the form (x,z). Then the coproduct is a homomorphism for the product.

4.3. *Set matrix half-symmetric functions*

4.3.1. *The Hopf algebra* **SMRSym**

Let **SMRSym** be the subalgebra of **SMQSym** generated by the polynomials $\mathbf{SMR}_{\pi_1,\Pi_2}$ indexed by a set partition π_1 and a set composition Π_2 and defined by

$$\mathbf{SMR}_{(\pi_1,\Pi_2)} := \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \mathbf{SMQ}_{\Pi_1,\Pi_2}. \tag{24}$$

For example,

$$\begin{aligned}\mathbf{SMR}_{\{\{14\},\{2\},\{3\}\},[\{134\},\{2\}]} &= \mathbf{SMQ}_{\begin{pmatrix}\{1,4\} & \emptyset\\ \emptyset & \{2\}\\ \{3\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{1,4\} & \emptyset\\ \{3\} & \emptyset\\ \emptyset & \{2\}\end{pmatrix}}\\ &\quad + \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \{2\}\\ \{1,4\} & \emptyset\\ \{3\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\emptyset & \{2\}\\ \{3\} & \emptyset\\ \{1,4\} & \emptyset\end{pmatrix}}\\ &\quad + \mathbf{SMQ}_{\begin{pmatrix}\{3\} & \emptyset\\ \emptyset & \{2\}\\ \{1,4\} & \emptyset\end{pmatrix}} + \mathbf{SMQ}_{\begin{pmatrix}\{3\} & \emptyset\\ \{1,4\} & \emptyset\\ \emptyset & \{2\}\end{pmatrix}}.\end{aligned} \tag{25}$$

Note that a pair constituted by a set partition and a set composition is equivalent to a set packed matrix up to a permutation of its rows. Hence, the realization on bi-words follows: for example,

$$\mathbf{SMR}_{\{\{14\},\{2\},\{3\}\},[\{134\},\{2\}]} = \sum_{\substack{j_1,j_2,j_3 \text{ distinct}\\ k_1<k_2}} \left\langle \begin{matrix} j_1\, j_2\, j_3\, j_1 \\ k_1\, k_2\, k_1\, k_1 \end{matrix} \right\rangle . \tag{26}$$

Proposition 4.3.

- **SMRSym** *is isomorphic to* $\oplus_n(\mathbf{WSym}_n \otimes \mathbf{WQSym}_n^*)$.
- **SMRSym** *is a co-commutative Hopf subalgebra of* **SMQSym**.

Proof. The first part of the proposition is a direct consequence of the following sequence of equalities:

$$\begin{aligned}\phi(\mathbf{SMR}_{\pi_1,\Pi_2}) &= \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \phi(\mathbf{SMQ}_{\Pi_1,\Pi_2})\\ &= \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \mathbf{WQ}_{\Pi_1} \otimes \mathbf{F}^{\Pi_2}\\ &= \mathbf{W}_{\pi_1} \otimes \mathbf{F}^{\Pi_2}.\end{aligned} \tag{27}$$

From its definition, **SMRSym** is stable for the product and Δ maps **SMRSym** to **SMRSym** $\otimes$ **SMRSym**. It follows that **SMRSym** is a Hopf subalgebra of **SMQSym**. One checks easily the co-commutativity by restricting Δ to **SMRSym**. □

4.3.2. *The Hopf algebra* **SMCSym**

Forgetting about the order of the columns instead of the rows leads to another Hopf algebra, **SMCSym**, a basis of which is indexed by pairs (Π_1, π_2) where Π_1 is a set composition and π_2 is a set partition. It is naturally the quotient (and not a subalgebra) of **SMQSym** by the ideal generated by the polynomials

$$\mathbf{SMQ}_{\Pi_1,\Pi_2} - \mathbf{SMQ}_{\Pi_1,\Pi_2'} \tag{28}$$

where $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$. Note that this quotient can be brought down to the bi-words. We denote by α the canonical surjection:

$$\alpha(\mathbf{SMQ}_{\Pi_1,\Pi_2}) =: \mathbf{SMC}_{\Pi_1,\mathrm{sp}(\Pi_2)}. \tag{29}$$

Proposition 4.4.

- **SMCSym** *is isomorphic to* $\oplus_n \mathbf{WQSym}_n \otimes \mathbf{WSym}_n^*$,
- **SMCSym** *is a Hopf algebra.*

Proof. The first property follows from the fact that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{SMQSym} & \xrightarrow{\phi} & \oplus_n \mathbf{WQSym}_n \otimes \mathbf{WQSym}_n^* \\ \downarrow \alpha & & Id \otimes \alpha' \downarrow \\ \mathbf{SMCSym} & \xrightarrow{\phi^C} & \oplus_n \mathbf{WQSym}_n \otimes \mathbf{WSym}_n^*, \end{array} \tag{30}$$

where α' denotes the canonical surjection $\alpha' : \mathbf{WQSym}_n^* \to \mathbf{WSym}_n^*$, and ϕ^C is the map sending $\mathbf{SMC}_{\Pi_1,\pi_2}$ to $\mathbf{W}_{\Pi_1} \otimes F^{\pi_2}$. Indeed, the image by ϕ of the ideal generated by the polynomials $\mathbf{SMQ}_{\Pi_1,\Pi_2} - \mathbf{SMQ}_{\Pi_1,\Pi_2'}$ for $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$ is the ideal $\tilde{J}$ of $\mathbf{WQSym}_n \otimes \mathbf{WQSym}_n^*$ generated by the polynomials $\mathbf{W}_{\Pi_1} \otimes (\mathbf{F}^{\Pi_2} - \mathbf{F}^{\Pi_2'})$. Since

$$\begin{aligned} (\mathbf{WQSym}_n \otimes \mathbf{WQSym}_n^*)/_{\tilde{J}} &= \mathbf{WQSym}_n \otimes \mathbf{WQSym}_n^*/_J \\ &= \mathbf{WQSym}_n \otimes \mathbf{WSym}_n^*, \end{aligned} \tag{31}$$

the result follows.

Using the representation of basis elements as pairs of set compositions, one obtains for two set compositions Π_2 and Π_2' satisfying $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$:

$$\begin{aligned} &\Delta(\mathbf{SMQ}_{\Pi_1,\Pi_2} - \mathbf{SMQ}_{\Pi_1,\Pi_2'}) \\ &\quad = \sum_{\Pi_1 = [\Pi_1', \Pi_1'']} (\mathbf{SMQ}_{\mathrm{std}(\Pi_1'),\mathrm{std}(\Pi_2|_{\Pi_1'})} \otimes \mathbf{SMQ}_{\mathrm{std}(\Pi_1''),\mathrm{std}(\Pi_2|_{\Pi_1''})} \\ &\qquad - \mathbf{SMQ}_{\mathrm{std}(\Pi_1'),\mathrm{std}(\Pi_2'|_{\Pi_1'})} \otimes \mathbf{SMQ}_{\mathrm{std}(\Pi_1''),\mathrm{std}(\Pi_2'|_{\Pi_1''})}). \end{aligned} \tag{32}$$

Since $\mathrm{sp}(\mathrm{std}(\Pi_2|_{\Pi_1''})) = \mathrm{sp}(\mathrm{std}(\Pi_2'|_{\Pi_1''}))$, we have

$$(\alpha \otimes \alpha) \circ \Delta(\mathbf{SMQ}_{\Pi_1,\Pi_2} - \mathbf{SMQ}_{\Pi_1,\Pi_2'}) = 0. \tag{33}$$

Hence, one defines the coproduct Δ in **SMCSym** by making the following diagram commute

$$\begin{array}{ccc} \mathbf{SMQSym} & \stackrel{\Delta}{\to} & \oplus_n \mathbf{SMQSym}_n \otimes \mathbf{WQSym}_n^* \\ \downarrow \alpha & & \alpha \otimes \alpha' \downarrow \\ \mathbf{SMCSym} & \stackrel{\Delta}{\to} & \oplus_n \mathbf{SMCSym}_n \otimes \mathbf{WSym}_n^*. \end{array} \tag{34}$$

More precisely, one has

$$\Delta\mathbf{SMC}_{\Pi_1,\pi_2} = \sum_{\Pi_1=[\Pi_1',\Pi_1'']} \mathbf{SMC}_{\mathrm{std}(\Pi_1'),\mathrm{std}(\pi_2|_{\Pi_1'})} \otimes \mathbf{SMC}_{\mathrm{std}(\Pi_1''),\mathrm{std}(\pi_2|_{\Pi_1''})},$$

where $\pi_2|_{\Pi_1}$ is the set of sets $\{(\pi_2)_i \cap \bigcup_j (\Pi_1)_j\}_i$ from which one erases the empty sets.

Since **SMQSym** is a Hopf algebra, we immediately deduce that Δ is an algebra homomorphism from **SMCSym** to **SMCSym** $\otimes$ **SMCSym**. $\square$

Note that **SMRSym** and **SMCSym** have the same Hilbert series, given by the product of ordered Bell numbers by unordered Bell numbers. This gives one new example of two different Hopf structures on the same combinatorial set since **SMCSym** is neither commutative nor cocommutative.

Note that the realization of **SMCSym** is obtained from the realization of **SMQSym** by quotienting bi-words by the ideal $\mathscr{J}$ generated by

$$\left\langle \begin{matrix} u \\ v \end{matrix} \right\rangle - \left\langle \begin{matrix} u \\ w \end{matrix} \right\rangle \tag{35}$$

where w is obtained from v by permuting its values.

4.4. *Set matrix symmetric functions*

The algebra **SMSym** of *set matrix symmetric functions* is the subalgebra of **SMCSym** generated by the polynomials

$$\mathbf{SM}_{\pi_1,\pi_2} = \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \mathbf{SMC}_{\Pi_1,\pi_2}. \tag{36}$$

For example,

$$\mathbf{SM}_{\{\{1,4\},\{2\},\{3\}\},\{\{1,3,4\},\{2\}\}} = \sum_{\substack{j_1,\,j_2,\,j_3 \text{ distinct} \\ k_1<k_2}} \left\langle \begin{matrix} j_1\, j_2\, j_3\, j_1 \\ k_1\, k_2\, k_1\, k_1 \end{matrix} \right\rangle_{\mathscr{J}}. \tag{37}$$

Theorem 4.5.

- **SMSym** *is a co-commutative Hopf subalgebra of* **SMCSym**.
- **SMSym** *is isomorphic as an algebra to* $\mathrm{End}_{gr}\mathbf{WSym} = \bigoplus_n \mathbf{WSym}_n \otimes \mathbf{WSym}_n^*$.
- **SMSym** *is isomorphic to the quotient of* **SMRSym** *by the ideal generated by the polynomials* $\mathbf{SMR}_{\pi_1,\Pi_2} - \mathbf{SMR}_{\pi_1,\Pi_2'}$ *with* $\mathrm{sp}(\Pi_2) = \mathrm{sp}(\Pi_2')$.

Proof. The space **SMSym** is stable for the product in **SMCSym**, as it can be checked from the realization. Furthermore, one has

$$\begin{aligned}
\Delta\mathbf{SM}_{\pi_1,\pi_2} &= \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \Delta\mathbf{SMC}_{\Pi_1,\pi_2} \\
&= \sum_{\mathrm{sp}(\Pi_1)=\pi_1} \sum_{\Pi_1=[\Pi_1',\Pi_1'']} \mathbf{SMC}_{\mathrm{std}(\Pi_1'),\mathrm{std}(\pi_2|_{\Pi_1'})} \otimes \mathbf{SMC}_{\mathrm{std}(\Pi_1''),\mathrm{std}(\pi_2|_{\Pi_1''})} \\
&= \sum_{\pi_1=\{\pi_1',\pi_1''\}} \mathbf{SM}_{\mathrm{std}(\pi_1'),\mathrm{std}(\pi_2|_{\pi_1'})} \otimes \mathbf{SM}_{\mathrm{std}(\pi_1''),\mathrm{std}(\pi_2|_{\pi_1''})},
\end{aligned} \tag{38}$$

where $\pi_2|_{\pi_1}$ is the set of sets $\{(\pi_2)_i \cap \bigcup_j (\pi_1)_j\}_i$ from which one erases the empty sets. The co-commutativity of Δ is obvious from (38), thus proving the first part of the theorem.

The second part of the theorem is a consequence of the following fact: $\phi^C(\mathbf{SM}_{\pi_1,\pi_2}) = \mathbf{W}_{\pi_1} \otimes F^{\pi_2}$.

The proof of the third part is the same as in Proposition 4.4. □

5. Hopf Algebras of Packed Integer Matrices

5.1. *Matrix quasi-symmetric functions*

Let SA_n be the set of set packed matrices such that if one reads the entries by columns from top to bottom and from left to right, then one obtains the numbers 1 to n in the usual order (see Fig. 6).

$$\begin{pmatrix} \{1,2\} & \emptyset \\ \emptyset & \{4\} \\ \{3\} & \emptyset \end{pmatrix} \qquad \begin{pmatrix} \{1,3\} & \emptyset \\ \emptyset & \{4\} \\ \{2\} & \emptyset \end{pmatrix} \tag{39}$$

Fig. 6. An element of SA and an element not in SA.

Denote by SA the set $SA := \bigcup_{n\geq 0} SA_n$. One easily sees that SA is in bijection with the packed integer matrices. Indeed, the bijection $\daleth$ consists in substituting each set of a matrix by its cardinality. The reverse bijection exists since each integer is the cardinality of a set, fixed by the reading order of the matrix. For example,

$$\daleth\begin{pmatrix} \{1,2\} & \emptyset & \{6\} \\ \emptyset & \{3,4,5\} & \{7,8,9,10\} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \end{pmatrix}. \tag{40}$$

Let us consider the subspace $\mathbf{MQSym}'$ of $\mathbf{SMQSym}$ spanned by the elements of SA. For example,

$$\begin{aligned}\mathbf{MQ}_{\begin{pmatrix}2&0&1\\0&3&4\end{pmatrix}} &:= \mathbf{SMQ}_{\begin{pmatrix}\{1,2\}&\emptyset&\{6\}\\\emptyset&\{3,4,5\}&\{7,8,9,10\}\end{pmatrix}}\\ &= \sum_{\substack{j_1<j_2\\k_1<k_2<k_3}}\left\langle\begin{matrix}j_1\,j_1\,j_2\,j_2\,j_2\,j_1\,j_2\,j_2\,j_2\,j_2\\k_1\,k_1\,k_2\,k_2\,k_2\,k_3\,k_3\,k_3\,k_3\,k_3\end{matrix}\right\rangle.\end{aligned} \tag{41}$$

By definition of the reading order, the product of two elements of SA is a linear combination of elements of SA and the coproduct of an element of SA is a linear combination of tensor products of two elements of SA. So

Theorem 5.1. $\mathbf{MQSym}'$ *is a Hopf subalgebra of* $\mathbf{SMQSym}$ *and it is isomorphic as a Hopf algebra to* $\mathbf{MQSym}$.

Proof. As $\mathbf{MQSym}'$ is generated by a set indexed by packed integer matrices, it is sufficient to check that the product and the coproduct have the same decompositions as in $\mathbf{MQSym}$. This can be obtained by a straightforward computation. □

Note that this last theorem gives a realization on bi-words different from the realization given in [7].

5.2. *Matrix half-symmetric functions*

We reproduce the same construction as for set packed matrices. We define three algebras $\mathbf{MRSym}$ (resp. $\mathbf{MCSym}$, $\mathbf{MSym}$) of packed matrices up to permutation of rows (resp. of columns, resp. of rows and columns).

5.2.1. *The Hopf algebra* MRSym

Let $\mathbf{MRSym}$ be the subalgebra of $\mathbf{MQSym}$ generated by the polynomials

$$\mathbf{MR}_A := \sum_B \mathbf{MQ}_B \tag{42}$$

where B is obtained from A by any permutation of its rows. As for $\mathbf{SMRSym}$, the realization of $\mathbf{MRSym}$ on bi-words is automatic. For example,

$$\begin{aligned}\mathbf{MR}_{\begin{pmatrix}2&1&0\\0&3&4\end{pmatrix}} &= \mathbf{MQ}_{\begin{pmatrix}2&1&0\\0&3&4\end{pmatrix}} + \mathbf{MQ}_{\begin{pmatrix}0&3&4\\2&1&0\end{pmatrix}}\\ &= \sum_{\substack{j_1\neq j_2\\k_1<k_2<k_3}}\left\langle\begin{matrix}j_1\,j_1\,j_2\,j_2\,j_2\,j_1\,j_2\,j_2\,j_2\,j_2\\k_1\,k_1\,k_2\,k_2\,k_2\,k_2\,k_3\,k_3\,k_3\,k_3\end{matrix}\right\rangle.\end{aligned} \tag{43}$$

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Theorem 5.2.

(1) **MRSym** *is a co-commutative Hopf subalgebra of* **MQSym**,
(2) **MRSym** *is also the subalgebra of* **SMRSym** *generated by the elements* $\mathbf{SMR}_A$ *where* A *is any matrix such that each element of the set composition of its columns is an interval of* $[1, n]$.

Proof. From its definition, **MRSym** is stable for the product and Δ maps **MRSym** to $\mathbf{MRSym} \otimes \mathbf{MRSym}$. It follows that **MRSym** is a Hopf subalgebra of **MQSym**. One easily checks the co-commutativity of the restriction of Δ to **MRSym**.

The second part of the theorem amounts to observing that $\mathbf{MR}_A = \mathbf{SMR}_{\daleth^{-1}A}$. □

5.2.2. *The Hopf algebra* **MCSym**

We construct the algebra **MCSym** as the quotient of **MQSym** by the ideal generated by the polynomials $\mathbf{MQ}_A - \mathbf{MQ}_B$ where B can be obtained from A by a permutation of its columns.

Theorem 5.3.

(1) **MCSym** *is a commutative Hopf algebra*,
(2) **MCSym** *is isomorphic as a Hopf algebra to the subalgebra of* **SMCSym** *generated by the elements* $\mathbf{SMC}_A$, *such that each element of the set partition of its columns is an interval of* $[1, n]$.

Proof. The proof of the first part of the theorem is almost the same as Proposition 4.4(1).

The dimensions are the same, so it is sufficient to check that the product and the coproduct have the same decomposition in both algebras. □

5.2.3. *Dimensions of* **MRSym** *and* **MCSym**

The dimension of the homogeneous component of degree n of **MRSym** or **MCSym** is equal to the number of packed matrices with sum of entries equal to n, up to a permutation of their rows.

Let us denote by $\mathrm{PMuR}(p, q, n)$ the number of such $p \times q$ matrices. One has obviously

$$\dim \mathbf{MRSym}_n = \sum_{1 \le p, q \le n} \mathrm{PMuR}(p, q, n). \tag{44}$$

The integers $\mathrm{PMuR}(p, q, n)$ can be computed through the induction

$$\mathrm{PMuR}(p, q, n) = \mathrm{MuR}(p, q, n) - \sum_{1 \le k, l \le p, q} \binom{q}{l} \mathrm{PMuR}(k, l, n), \tag{45}$$

where $\mathrm{MuR}(p,q,n)$ is the number of $p \times q$ possibly unpacked matrices with sum of entries equal to n, up to a permutation of their rows.

Solving this induction and substituting it in Eq. (44), one gets

$$\dim \mathbf{MRSym}_n = \sum_{i=1}^{n+1} (-1)^{n-i} T_{n+1,i+1} \mathrm{MuR}(n,i,n), \tag{46}$$

where

$$T_{n,k} = \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{j+k-1}{j}, \tag{47}$$

that is the number of minimum covers of an unlabeled n-set that cover k points of that set uniquely (sequence A056885 of [11]). The generating series of the $T_{n,k}$ is

$$\sum_{i,j} T_{i,j} x^i y^j = \frac{1-x}{(1+x)(1-x-xy)}. \tag{48}$$

The integer $\mathrm{MuR}(n,i,n)$, computed via the Pólya enumeration theorem, is the coefficient of x^n in the cycle index $Z(G_{n,i})$, evaluated over the alphabet $1 + x + \cdots + x^n + \cdots$, of the subgroup $G_{n,i}$ of $\S_{in}$ generated by the permutations $\sigma \cdot \sigma[n] \cdot \sigma[2n]. \cdots .\sigma[in]$ for $\sigma \in \S_n$ (here $\cdot$ denotes the concatenation).

This coefficient is also the number of partitions $N_{n,i}$ of n objects with i colors whose generating series is

$$\sum_n N_{n,i} x^n = \prod_k \left(\frac{1}{1-x^k} \right)^{\binom{i+k}{i}}. \tag{49}$$

Hence

Proposition 5.4.

$$\dim \mathbf{MRSym}_n = \sum_{i=1}^{n+1} (-1)^{n-i} T_{n+1,i+1} N_{n,i}. \tag{50}$$

The first values are

$$\begin{aligned} \mathrm{Hilb}(\mathbf{MRSym}) &= \mathrm{Hilb}(\mathbf{MCSym}) \\ &= 1 + t + 4\,t^2 + 16\,t^3 + 76\,t^4 + 400\,t^5 + 2356\,t^6 + 15200\,t^7 \\ &\quad + 106644\,t^8 + 806320\,t^9 + 6526580\,t^{10} + \cdots \end{aligned} \tag{51}$$

5.3. *Matrix symmetric functions*

The algebra **MSym** of *matrix symmetric functions* is the subalgebra of **MCSym** generated by

$$\mathbf{M}_A := \sum_B \mathbf{MR}_B, \tag{52}$$

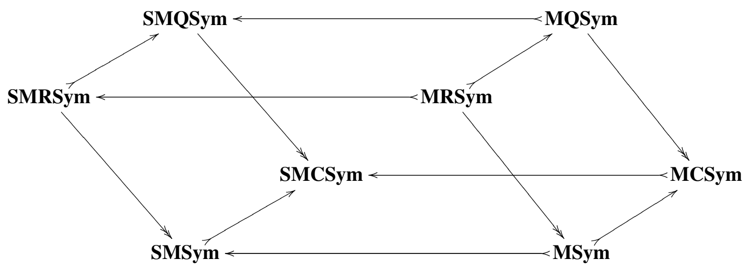
where B is obtained from A by any permutation of its rows.

Theorem 5.5.

(1) **MSym** *is a commutative and co-commutative Hopf subalgebra of* **MCSym**.
(2) **MSym** *is isomorphic as a Hopf algebra to the subalgebra of* **SMSym** *generated by the elements* $\mathbf{SM}_A$, *such that each element of the set partition of its columns is an interval of* $[1, n]$.
(3) **MSym** *is isomorphic to the quotient of* **MRSym** *by the ideal generated by the polynomials* $\mathbf{MR}_A - \mathbf{MR}_B$ *where* B *can be obtained from* A *by a permutation of its columns.*

Proof. The proof follows the same lines as the proof of Theorem 4.5. $\square$

From all the previous results, we deduce that the following figure commutes.



6. Dendriform Structures Over SMQSym

6.1. *Tridendriform structure*

A *tridendriform algebra* is an associative algebra whose multiplication can be split into three operations

$$x \cdot y = x \prec y + x \circ y + x \succ y\,, \tag{53}$$

where $\circ$ is associative, and such that

$$(x \prec y) \prec z = x \prec (y \cdot z),\ (x \succ y) \prec z = x \succ (y \prec z),\ (x \cdot y) \succ z = x \succ (y \succ z), \tag{54}$$

$$(x \succ y) \circ z = x \succ (y \circ z),\ (x \prec y) \circ z = x \circ (y \succ z),\ (x \circ y) \prec z = x \circ (y \prec z). \tag{55}$$

6.2. *Tridendriform structure on bi-words*

One defines three product rules over bi-words as follows:

(1) $\left\langle {u_1 \atop v_1} \right\rangle \prec \left\langle {u_2 \atop v_2} \right\rangle = \left\langle {u_1 \atop v_1} \right\rangle \star \left\langle {u_2 \atop v_2} \right\rangle$ if $\max(u_1) > \max(u_2)$, and 0 otherwise.
(2) $\left\langle {u_1 \atop v_1} \right\rangle \circ \left\langle {u_2 \atop v_2} \right\rangle = \left\langle {u_1 \atop v_1} \right\rangle \star \left\langle {u_2 \atop v_2} \right\rangle$ if $\max(u_1) = \max(u_2)$, and 0 otherwise.
(3) $\left\langle {u_1 \atop v_1} \right\rangle \succ \left\langle {u_2 \atop v_2} \right\rangle = \left\langle {u_1 \atop v_1} \right\rangle \star \left\langle {u_2 \atop v_2} \right\rangle$ if $\max(u_1) < \max(u_2)$, and 0 otherwise.

Proposition 6.1. *The algebra of bi-words endowed with the three product rules* $\prec, \circ$, *and* $\succ$ *is a tridendriform algebra.*

Moreover, **SMQSym** *is stable by those three rules. More precisely, one has*:

$$\mathbf{SMQ}_{\left\langle {u \atop \mathbf{v}} \right\rangle} \prec \mathbf{SMQ}_{\left\langle {u' \atop \mathbf{v}'} \right\rangle} = \sum_{\substack{w=x.y\in u\star_W u' \\ |x|=|u|;\ \max(y)<\max(x)}} \mathbf{SMQ}_{\left\langle {w \atop \mathbf{v}\mathbf{v}'[\max(v)]} \right\rangle}, \tag{56}$$

$$\mathbf{SMQ}_{\left\langle {u \atop \mathbf{v}} \right\rangle} \circ \mathbf{SMQ}_{\left\langle {u' \atop \mathbf{v}'} \right\rangle} = \sum_{\substack{w=x.y\in u\star_W u' \\ |x|=|u|;\ \max(y)=\max(x)}} \mathbf{SMQ}_{\left\langle {w \atop \mathbf{v}\mathbf{v}'[\max(v)]} \right\rangle}, \tag{57}$$

$$\mathbf{SMQ}_{\left\langle {u \atop \mathbf{v}} \right\rangle} \succ \mathbf{SMQ}_{\left\langle {u' \atop \mathbf{v}'} \right\rangle} = \sum_{\substack{w=x.y\in u\star_W u' \\ |x|=|u|;\ \max(y)>\max(x)}} \mathbf{SMQ}_{\left\langle {w \atop \mathbf{v}\mathbf{v}'[\max(v)]} \right\rangle}. \tag{58}$$

So **SMQSym** *is a tridendriform algebra.*

Proof. The first part of the proposition amounts to checking the compatibility relations between the three rules. It is immediate.

The stability of **SMQSym** by any of the three rules and the product relations are also immediate: the bottom row can be any word whose packed word is $\mathbf{v}\mathbf{v}'[\max(v)]$, so that the formula reduces to a formula on the top row which is equivalent to the same computation in **WQSym** (see [9]). The compatibility relations automatically follow from their compatibility at the level of bi-words. □

Corollary 6.2. **SMCSym**, **MQSym** *and* **MCSym** *are tridendriform.*

Proof. As in the case of **SMQSym**, one only has to check that the algebras are stable by the three product rules since the compatibility relations automatically follow.

The case of **SMCSym** is direct since Formulas (56)–(58) have only the word $\mathbf{v}\mathbf{v}'[\max(v)]$ in the bottom row of their basis elements. For the same reason, the case of **MCSym** directly follows from the case of **MQSym**. The case of **MQSym** is the same as the case of **SMQSym** itself. □

6.3. *Bidendriform structures*

Let us define two product rules $\ll = \prec$ and $\gg = \circ + \succ$ on bi-words. We now split the nontrivial parts of the coproduct of the **SMQ** of **SMQSym**, as

$$\Delta_{\ll}(\mathbf{SMQ}_A) = \sum_{\substack{A=\binom{B}{C},\ A\neq B,C \\ \max(B)=\max(A)}} \mathbf{SMQ}_{\mathrm{std}(B)} \otimes \mathbf{SMQ}_{\mathrm{std}(C)}, \tag{59}$$

$$\Delta_{\gg}(\mathbf{SMQ}_A) = \sum_{\substack{A=\binom{B}{C},\ A\neq B,C \\ \max(C)=\max(A)}} \mathbf{SMQ}_{\mathrm{std}(B)} \otimes \mathbf{SMQ}_{\mathrm{std}(C)}. \tag{60}$$

Let us recall that under certain compatibility relations between the two parts of the coproduct and other compatibility relations between the two product rules $\ll := \prec$ and $\gg := \circ + \succ$ defined by Foissy [6], we get bidendriform bialgebras.

Theorem 6.3. **SMQSym** *is a bidendriform bialgebra.*

Proof. The co-dendriform relations, the one concerning the two parts of the coproduct, are easy to check since they only amount to knowing which part of a matrix cut in three contains its maximum letter.

The bi-dendriform relations are more complicated but reduce to a careful check that any part of the coproduct applied to any part of the product only brings a limited amount of mutually disjoint cases. Let us for example check the relation

$$\Delta_{\gg}(a \ll b) = a' b'_{\gg} \otimes a'' \ll b''_{\gg} + a' \otimes a'' \ll b + b'_{\gg} \otimes a \ll b''_{\gg}, \tag{61}$$

where the pairs (x', x'') (resp. $(x'_{\ll}, x''_{\ll})$ and $(x'_{\gg}, x''_{\gg})$) correspond to all possible elements occurring in $\overline{\Delta} x$ (resp. $\Delta_{\ll} x$ and $\Delta_{\gg} x$), summation signs being understood (Sweedler's notation).

First, the last row of all elements in $(a \ll b)$ only contain elements of a. Since by application of $\Delta_{\gg}$, the maximum of b has to go in the right part of the tensor product, this means that there has to be also elements coming from a in this part of the tensor product. Now, the elements of $\Delta_{\gg}(a \ll b)$ where all elements of b are in the right part of the tensor product, are obtained, for the left part by elements coming from the top rows of a and for the right part by elements coming from the other rows of a multiplied by b in such a way that the last row only contains elements of a, hence justifying the middle term $a' \otimes a'' \ll b$.

If both components of $\Delta_{\gg}(a \ll b)$ contain elements coming from b, then the left part cannot contain the maximum element of b (hence justifying the $b'_{\gg}$ and $b''_{\gg}$, the left part being multiplied by elements coming from a if any (this is the difference between the first and the third term of the expansion of $\Delta_{\gg}(a \ll b)$), the right part being multiplied by $\ll$ with elements coming from a since the last row must contain elements coming from a. □

Recall that **SMCSym** is the quotient of **SMQSym** by the ideal generated by $\mathbf{SMQ}_A$–$\mathbf{SMQ}_B$ where A and B are the same matrices up to a permutation of their columns, the row containing the maximum element is the same for any element of a given class, so that the left coproduct and the right coproduct are compatible with the quotient. Moreover, the left and right coproduct are internal within **MQSym**, so that

Corollary 6.4. **SMCSym** *and* **MQSym** *are bidendriform sub-bialgebras of* **SMQSym**.

Proof. We already know that **SMCSym** and **MQSym** are dendriform subalgebras of **SMQSym** since they are tridendriform subalgebras of this algebra. The compatibility relations come from the compatibility relations on **SMQSym**, so that there only remains to check that the coproduct goes from X to $X \otimes X$, where X is either **SMCSym** or **MQSym**. This is an easy computation. □

Corollary 6.5. **SMQSym**, **SMCSym**, *and* **MQSym** *are free, cofree, self-dual Hopf algebras and their primitive Lie algebras are free.*

Proof. This follows from the characterization of bidendriform bialgebras given by Foissy [6]. □

7. A Realization of $L\mathrm{Diag}(q_c, q_s)$ on Bi-Words

Let us define a two-parameter generalization of the algebra **MQSym**. For this purpose, consider bi-words with parameter-commuting bi-letters depending on the bi-letters as follows:

$$\begin{aligned} \left\langle \begin{matrix} yx \\ zt \end{matrix} \right\rangle &= q_c \left\langle \begin{matrix} xy \\ tz \end{matrix} \right\rangle \quad \text{if } y > x, \\ \left\langle \begin{matrix} xx \\ zt \end{matrix} \right\rangle &= q_s \left\langle \begin{matrix} xx \\ tz \end{matrix} \right\rangle \quad \text{if } z < t. \end{aligned} \tag{62}$$

Let us now define the realization as a sum of bi-words of a packed integer matrix with p rows and q columns:

$$\mathbf{LD}_M := \sum_{\substack{j_1 < \cdots < j_p \\ k_1 < \cdots < k_q}} \prod_{a=1}^{p} \prod_{b=q}^{1} \left\langle \begin{matrix} j_a \\ k_b \end{matrix} \right\rangle^{M_{ab}}. \tag{63}$$

For example,

$$\mathbf{LD}_{\begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix}} = \sum_{\substack{j_1 < j_2 \\ k_1 < k_2}} \left\langle \begin{matrix} j_1^5 \, j_1^3 \, j_2^3 \, j_2 \\ k_2^5 \, k_1^3 \, k_2^3 \, k_1 \end{matrix} \right\rangle. \tag{64}$$

We then have

Theorem 7.1.

- *The subspace spanned by the* **LD** *has a structure of associative algebra. Moreover, the matrices indexing the product* $\mathbf{LD}_A \mathbf{LD}_B$ *are equal to the matrices appearing in* $\mathbf{M}_A \mathbf{M}_B$ *in* **MQSym**, *and the coefficient of* $\mathbf{LD}_C$ *in this product is a monomial* $q_s^{x(A,B,C)} q_c^{y(A,B,C)}$ *computed as follows: let us call* left *the part of* C *coming from*

A and right *the part of C coming from B. Then*

$$x(A,B,C) = \sum_{r \text{ row of } C} \left(\sum_{i \in \text{ left } (r)} i \right) \left(\sum_{j \in \text{ right } (r)} j \right). \tag{65}$$

$$y(A,B,C) = \sum_{r<r' \text{rows of } C} \left(\sum_{i \in \text{ left } (r)} i \right) \left(\sum_{j \in \text{ right } (r')} j \right). \tag{66}$$

- *The specialization $q_s = q_c = 1$ gives back* **MQSym**.

Proof. Since each partially commuting bi-word has only one expression such that the top row is weakly increasing and the bottom row is weakly decreasing at the spots where the top row is constant, we can define without ambiguity the *canonical* element of a bi-word. The set of canonical elements is in bijection with integer matrices.

Now, if two bi-words appearing in a product of two **LD** have canonical elements whose corresponding matrices have the same packed matrix, they follow exactly the same rewriting steps to get to their canonical element. So in particular, the product of two **LD** decomposes as a linear combination of **LD**. Moreover, since the product on bi-words is associative and compatible with the partial commutations, then so is the product of the **LD**, hence proving that they span an algebra.

By definition of the realization of **MQSym** on bi-words, **MQSym** is obtained from this algebra by specifying $q_c = q_s = 1$, that is, replacing partially parameter-commuting bi-letters by partially commuting bi-letters, so that the matrices appearing in a product of two **LD** are the same as the matrices appearing in the product of the same packed matrices in **MQSym**. Finally, the coefficient of a given matrix is obviously a monomial in q_s and q_c and the powers of q_s and q_c are straightforward from the definition of the commutations: a bi-letter of the right has to exchange with any bi-letter of the left whose top value is greater than or equal to its top value. Each exchange amounts either to multiplying by q_c if those values differ, or to multiplying by q_s if they are equal. This is equivalent to the formulas of the statement. □

For example, one has:

$$\begin{aligned} \mathbf{LD}_{\begin{pmatrix} 2 & 0 \\ 1 & 4 \end{pmatrix}} \star \mathbf{LD}_{(1)} = \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}} + q_s^5\, \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 1 \end{pmatrix}} + q_c^5\, \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 0 \end{pmatrix}} \\ + q_c^5 q_s^2\, \mathbf{LD}_{\begin{pmatrix} 2 & 0 & 1 \\ 1 & 4 & 0 \end{pmatrix}} + q_c^7\, \mathbf{LD}_{\begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 4 & 0 \end{pmatrix}} \end{aligned} \tag{67}$$

since

$$\sum_{\substack{j_1<j_2\\k_1<k_2}} \left\langle \begin{matrix} j_1\, j_1\, j_2\, j_2\, j_2\, j_2\, j_2 \\ k_1 k_1 k_2 k_2 k_2 k_2 k_1 \end{matrix} \right\rangle \star \sum \left\langle \begin{matrix} j \\ k \end{matrix} \right\rangle$$

$$\begin{aligned} &= \sum_{\substack{j_1<j_2<j_3\\k_1<k_2<k_3}} \left\langle \begin{matrix} j_1^2\, j_2^4\, j_2\, j_3 \\ k_1^2\, k_2^4\, k_1\, k_3 \end{matrix} \right\rangle + \sum_{\substack{j_1<j_2(=j_3)\\k_1<k_2<k_3}} q_s^5 \left\langle \begin{matrix} j_1^2\, j_2\, j_2^4 j_2 \\ k_1^2\, k_3\, k_2^4\, k_1 \end{matrix} \right\rangle \\ &\quad + \sum_{\substack{j_1<j_3<j_2\\k_1<k_2<k_3}} q_c^5 \left\langle \begin{matrix} j_1^2\, j_3\, j_2^4\, j_2 \\ k_1^2\, k_3\, k_2^4\, k_1 \end{matrix} \right\rangle + \sum_{\substack{j_1(=j_3)<j_2\\k_1<k_2<k_3}} q_s^2 q_c^5 \left\langle \begin{matrix} j_1\, j_1^2\, j_2^4\, j_2 \\ k_3\, k_1^2\, k_2^4\, k_1 \end{matrix} \right\rangle \\ &\quad + \sum_{\substack{j_3<j_1<j_2\\k_1<k_2<k_3}} q_c^7 \left\langle \begin{matrix} j_3\, j_1^2\, j_2^4\, j_2 \\ k_3\, k_1^2\, k_2^4\, k_1 \end{matrix} \right\rangle. \end{aligned} \tag{68}$$

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