ELSEVIER Theoretical Computer Science 267 (2001) 105–120

Theoretical Computer Science

www.elsevier.com/locate/tcs

Direct and dual laws for automata with multiplicities ☆

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Abstract

We present here theoretical results coming from the implementation of the package called AMULT (automata with multiplicities). We show that classical formulas are optimal for the bounds. Especially they are almost everywhere optimal for the fields $\mathbb{R}$ and $\mathbb{C}$. We characterize the dual laws preserving rationality and examine compatibility between the geometry of the K-automata and these laws. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Automata with multiplicities; Rational laws; Dual laws; Geometry of K-automata; Shuffle compatibility

1. Introduction

Noncommutative formal series (i.e. functions on the free monoid, with values in a – commutative or not – semiring) encode an infinity of data. Rational series can be represented by linear recurrences, corresponding to automata with multiplicities [6], and therefore they can be generated by finite state processes. Literature can be found on these "weighted automata" (e.g. [15, 24]) and their theoretical (e.g. [12]) and practical (e.g. [4, 21]) applications (recently one of us solved a conjecture in operator theory using these tools [5]). The theory was founded by Schützenberger in 1961 [25] where the link between recognizable and rational series is shown (see also [26]), extending to rings (and to semirings [2]) Kleene's result for languages [14] (corresponding to boolean coefficients). In 1974, for the case of fields, Fliess extended the proof of the equivalence of minimal linear representations, using Hankel matrices [7]. All these results allow us to construct an algorithmic processing for these series and their associated operations. In fact, classical constructions of language theory have multiplicity analogues which can be used in every domain where linear recurrences between words

☆ Partially supported by the Scientific Research Program of MENRT.

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PII: S0304-3975(00)00298-X

are handled (for example these automata have been extensively used in theory of control [20]).

All these operations can be found in the AMULT package over automata with multiplicities. This package is a component of the environment SEA (Symbolic Environment for Automata [1]) under development at the University of Rouen. Rational series can be represented on the computer by three ways (formula, rational expressions, K-automata).

The structure of this paper is the following: in Section 3 (the first section after introductory paragraphs), we recall the classical construction for simple rational laws $(+, \cdot, *, \times)$. The compositions are based on polynomial formulas which has an important consequence on composition of K-automata choosen at random ($K=\mathbb{R}$ or $\mathbb{C}$). In fact, this first result says that the classical formulas are almost everywhere optimal (which is clear from experimental tests at random). In the general case, the bounds are reached.

In Section 4, we show that the three laws known to preserve rationality (shuffle, Hadamard and infiltration products) are of the same nature: they arise by dualizing alphabetic morphisms. Moreover, they are, up to a deformation, the only ones of this kind, which of course, shows immediately in the implemented formulas. The second and third parts of this section are devoted to show the sharpness of the classical bounds and to begin the study of the compatibility of these laws with the geometry of K-automata.

2. Definitions, notations and basic results

Following Eilenberg ([6, p. 136]), we recall that a K-subset S of E is a mapping $E \to K$ (where K is a semiring). Then, if A is an alphabet, a K-automaton is a triplet $\mathscr{A}=(Q,I,T)$ given by a finite set Q (the states of $\mathscr{A}$) with K-subsets I (the initial states) and T (the final states), and a K-subset E of $Q \times A \times Q$ (the transitions and their multiplicities).

The school of Schützenberger ([6, p. 158]) has introduced the concept of a linear representation (λ,μ,γ) with $\lambda \in K^{1\times n}$, $\mu : A \to K^{n\times n}$ and $\gamma \in K^{n\times 1}$ (see [2]). The correspondence between these two data structures is then, up to a relabelling, one to one and goes as follows. For each letter $a \in A$, one can construct a $Q\times Q$-matrix $\mu(a)$ by $\mu_{i,j}(a) = E(i,a,j)$ for each pair $(i,j) \in Q^2$. The mapping μ is extended as a morphism from A^* to $K^{Q\times Q}$. If we consider the mapping I as a row matrix belonging to $K^{Q\times 1}$ and the mapping T as a column matrix belonging to $K^{1\times Q}$, the behaviour of $\mathscr{A}$ is given by

$$\mathscr{B}(\mathscr{A}) = \sum_{w\in A^*} IE(w)Tw,$$

that is the series recognized by $\mathscr{A}$. If we define a labelling $Q=\{q_1,\dots,q_n\}$, we receive through this correspondence a triplet (λ,μ,γ) defined by $\lambda \in K^{1\times n}$, $\mu : A \to K^{n\times n}$ and $\gamma \in K^{n\times 1}$, which is called a linear representation of $\mathscr{B}(\mathscr{A})$. We will use the terms of *K-automaton* and *linear representation* to denote the same object.

In this paper, we describe a general theory of the laws of K-automata which can be applied for any semiring K. It uses matrix operations which are implied by the definitions of the linear representations, such as the computation of the coefficients of a recognized word or the computation of $\mathscr{B}(\mathscr{A})$ for example. Since a lot of zeros appear in the matrices of transition, we take dynamic structures (such as the tables in Maple) for implementation. When we choose a special semiring (boolean semiring, field, etc.), some algorithms can certainly be made more efficient by specialization.

Let $K\langle\langle A\rangle\rangle$ be the set of noncommutative formal series with A a finite alphabet and K a semiring (commutative or not). A series denoted $S = \sum_{w\in A^*} \langle S|w\rangle w$ is recognizable if and only if there exists a linear representation (λ, μ, γ), such that for all $w \in A^*$, one has $\langle S|w\rangle = \lambda\mu(w)\gamma$. Throughout the paper, we will denote by $S:(\lambda,\mu,\gamma)$ this property. The integer n is called the *dimension* of the linear representation (λ,μ,γ) [7].

Let $K^{\mathrm{rat}}\langle\langle A\rangle\rangle$ be the set of rational noncommutative formal series, that is the set generated from the letters and the laws $\cdot$ (Cauchy product), $*$ (star operation, partially defined), $\times$ (external product) and $+$ (union or sum). The preceding four laws will be called rational laws of the first kind. The following important theorem for series [25] is the analogue of Kleene's theorem for languages. It is remarkable that it holds whatever K be.

Theorem 1 (Schützenberger [25]). *A formal series is recognizable if and only if it is rational.*

Notice that, in the boolean case, $\times$ (the external product) is trivial, but it permits to take for granted that $L=\emptyset$ and then $\emptyset^* = 1$ are regular (see [14, 13]).

A reduced K-automaton (λ,μ,γ) is a K-automaton of minimal dimension among all the K-automata with behaviour S.[1] This minimum is called the *rank* of the series S [25]. In case K is a field, the rank of S is the dimension of the linear span of the shifts of S. It is the smallest number of states of a K-automaton with behaviour S. Here, minimization (up to an equivalence) is possible [25] (see also [2]). An explicit algorithm is given in full details in [10] as well as the construction of intertwining matrices (Figs. 1and 2).

3. Constructing usual laws

3.1. Operations on linear representations

We expound here universal formulas for constructing linear representations. They can be applied to any semiring K. Let us recall some classical facts. Classical operations

[1] Existence is assumed by definition, unicity is proved in case K is $\mathbb{B}$ (for deterministic automata) or a (commutative or not) field [10] and indeed is the automaton of quotients but is problematic in general.

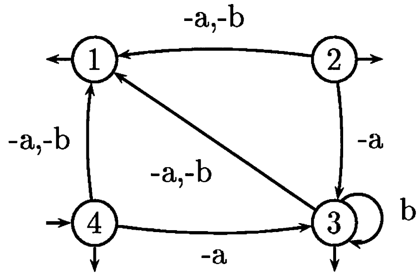


Fig. 1. A $\mathbb{Z}$-automaton for the rational series $1 - b + (a + 2b - 2)b^*a$.

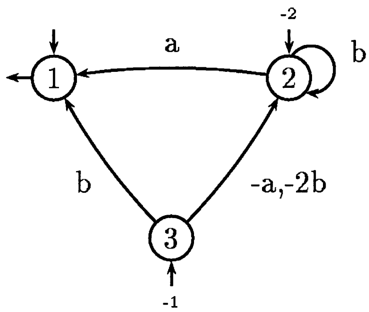


Fig. 2. Minimal $\mathbb{Z}$-automaton for the automaton of Fig. 1.

on series are sum, Cauchy product and star (unary and partially defined). By definition, the sum of two series R and S is

$$R + S = \sum_{w \in A^*} (\langle R|w\rangle + \langle S|w\rangle)w,$$

their concatenation (or Cauchy product) is

$$R.S = \sum_{w \in A^*} \left(\sum_{uv=w} \langle R|u\rangle\langle S|v\rangle \right) w$$

and the star of a series S is

$$S^* = \sum_{n \geqslant 0} S^n = 1 + SS^*$$

if its constant term is zero (such a series is said to be proper). The preceding operations have polynomial counterparts in terms of linear representations. They are well known [4, 9] and we gather them in the following proposition.

Proposition 2. *Let R (resp. S) be a rational series and $\mathscr{A}_r = (\lambda^r, \mu^r, \gamma^r)$ (resp. $S : \mathscr{A}_s = (\lambda^s, \mu^s, \gamma^s)$) be a K-automaton which recognizes R (resp. S). Let n (resp. m) be the*

dimension of $\mathscr{A}_r$ (resp. $\mathscr{A}_r$). The linear representations of the sum, the concatenation and the star are, respectively,
$R+S$

$$\mathscr{A}_r \boxplus \mathscr{A}_s = \left(\begin{pmatrix} \lambda^r & \lambda^s \end{pmatrix}, \left(\begin{array}{c|c} \mu^r(a) & 0_{n\times m} \\ \hline 0_{m\times n} & \mu^s(a) \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^r \\ \gamma^s \end{pmatrix} \right), \tag{1}$$

$R.S$

$$\mathscr{A}_r \boxdot \mathscr{A}_s = \left(\begin{pmatrix} \lambda^r & 0_{1\times m} \end{pmatrix}, \left(\begin{array}{c|c} \mu^r(a) & \gamma^r\lambda^s\mu^s(a) \\ \hline 0_{m\times n} & \mu^s(a) \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^r\lambda^s\gamma^s \\ \gamma^s \end{pmatrix} \right), \tag{2}$$

If $\lambda^s\gamma^s = 0$, S^*:

$$\mathscr{A}_s^{\boxed{*}} = \left(\begin{pmatrix} 0_{1\times m} & 1 \end{pmatrix}, \left(\begin{array}{c|c} \mu^s(a) + \gamma^s\lambda^s\mu^s(a) & 0_{m\times 1} \\ \hline \lambda^s\mu^s(a) & 0 \end{array} \right)_{a\in A}, \begin{pmatrix} \gamma^s \\ 1 \end{pmatrix} \right). \tag{3}$$

Proof. Formula (1) is straightforward.

To prove formula (2), let $(\lambda,\mu,\gamma) = \mathscr{A}_r \boxdot \mathscr{A}_s$. One proves by induction that

$$\mu(w) = \begin{pmatrix} \mu^r(w) & \sum\limits_{\substack{uv=w \\ v\neq 1}} \mu^r(u)\gamma^r\lambda^s\mu^s(v) \\ 0_{m\times n} & \mu^s(w) \end{pmatrix}$$

and then $\lambda\mu(w)\gamma = \sum_{uv=w} \lambda^r\mu^r(u)\gamma^r\lambda^s\mu^s(v)\gamma^s = \sum_{uv=w} \langle R|u\rangle\langle S|v\rangle$.

Concerning the formula (3), let $(\lambda^*,\mu^*,\gamma^*) = \mathscr{A}_s^{\boxed{*}}$. Again,

$$\mu^*(w) = \begin{pmatrix} M & 0_{m\times 1} \\ \sum\limits_{n=1}^{|w|} \sum\limits_{u_1\cdots u_n = w, u_i\neq 1} (\lambda_s\mu_s(u_1)\gamma_s)\cdots(\lambda_s\mu_s(u_{n-1})\gamma_s)(\lambda_s\mu_s(u_n)) & 0 \end{pmatrix},$$

where $M \in K^{m\times n}$. We then have

$$\begin{aligned} \lambda^*\mu^*(w)\gamma^* &= \sum_{n=1}^{|w|} \sum_{u_1\cdots u_n=w, u_i\neq 1} (\lambda_s\mu_s(u_1)\gamma_s)\cdots(\lambda_s\mu_s(u_n)\gamma_s) \\ &= \sum_{n=1}^{|w|} \langle S^n|w|\rangle = \sum_{n\geqslant 0} \langle S^n|w|\rangle = \langle S^*|w\rangle. \qquad \square \end{aligned}$$

Remark 3. (1) Formulas (1) and (2) provide associative laws on triplets. They can be found explicitly in [4].

(2) Formula (3) makes sense even when $\lambda^s\gamma^s \neq 0$ (this fact will be used in the density result of Section 3.3.

(3) Of course if $S:(\lambda,\mu,\gamma)$ and $\alpha \in K$ then $\alpha\times S:(\alpha\lambda,\mu,\gamma)$ and $S\times\alpha:(\lambda,\mu,\gamma\alpha)$.

(4) For the sum $(\mathscr{A}_r \boxplus \mathscr{A}_s)$, $\mathscr{A}_r$ and $\mathscr{A}_s$ are just placed side by side.

The product $\mathscr{A}_r \boxdot \mathscr{A}_s$ has the following components:

- *States:* The union of the sets of states of $\mathscr{A}_r$ and $\mathscr{A}_s$.
- *Inputs:* Inputs of $\mathscr{A}_r$.
- *Transitions:* Transitions of $\mathscr{A}_r$ and $\mathscr{A}_s$ and, for each letter a, each state r_i of $\mathscr{A}_r$ and each state s_j of $\mathscr{A}_s$, a new arc $r_i \xrightarrow{a} s_j$ is added with the coefficient $(\gamma_r)_i(\lambda_s\mu_s(a))_j$.
- *Outputs:* The scalar product $\lambda_s\gamma_s$ is computed once for all and there is an output on each q_i with the coefficient $(\gamma_r)_i\lambda_s\gamma_s$, the outputs of $\mathscr{A}_S$ being unchanged.

For $\mathscr{A}^{\boxed{*}}$, one adds a new state q_{n+1} with an input and an output bearing coefficient 1, every coefficient $\mu_{i,j}(a)$ is multiplied by $(1+\gamma_i\lambda_j)$ and new transitions $q_{n+1} \xrightarrow{a} q_i$ with coefficient $\sum_k \lambda_k\mu_{k,i}(a)$ (i.e. the "charge" of the state q_i after reading a) are added.

In case $K=\mathbb{B}$, one recovers the classical nondeterministic boolean constructions implemented in softwares such as Automate [3], AMoRE [18], Grail [23] and SEA [1].

3.2. Sharpness

Here, we study the efficiency of the formulas of Section 3.1 with respect to the dimension of the output automaton. We prove that, if we consider the set of algorithms such that the dimension of the output is only function of the dimension of the input automata, there does not exist such an algorithm more efficient than the classical constructions.

Let $\mathscr{B}=(S_i)_{1\leqslant i\leqslant n}$ be a finite sequence of series generating a stable module (i.e. the linear span of $\mathscr{B}$ is stable under all the transition matrices) and $S=\sum_{i=1}^n \lambda_i S_i$. It is well known that the triplet

$$\left(\sum_{i=1}^n \lambda_i e_i, ([\mu_{i,j}(a)]_{1\leqslant i,j\leqslant n})_{a\in A}, \sum_{i=1}^n \langle S_i|1\rangle e_i^*\right)$$

(where $e_i=(0,\dots,1,\dots,0)$ with the entry 1 at place i, e_i^* the transpose of e_i, and $a^{-1}S_i=\sum_{j=1}^n (\mu(a))_{ij}S_j$ for any letter $a\in A$) is a linear representation of S. Here, to each series of one variable, $S=\sum_{p\geqslant 0}\alpha_p a^p$, of rank n, over a field K, we associate the triplet $\tau(S)$ given by $\mathscr{B}=(a^{-p}S)_{0\leqslant p\leqslant n-1}$.

Remark 4. Of course, if $a\in A$, we have $S\in K\langle\langle a\rangle\rangle \subset K\langle\langle A\rangle\rangle$ and this will neither affect the rank nor the following constructions.

First, in the case of one variable, Lemma 5 proves that the classical formulas (of first kind) are optimal (with respect to the dimension of the output automaton).

Lemma 5. *Let $S_{\alpha,n}=1/(1-\alpha a)^n$ and $T_n=a^{n-1}/(1-a^n)$ be $\mathbb{Q}$-series.*

(1) *The rank of $S_{\alpha,n}$, $S_{\alpha,n}+S_{\beta,m}$ $(\alpha\neq\beta)$, and $S_{\alpha,n}.S_{\alpha,m}$ are, respectively, n, $n+m$ and $n+m$.*

(2) *The rank of T_n is n and that of T_n^* is $n+1$.*

Proof. Straightforward. □

3.3. Density

Indeed, testing our package showed us that almost everytime the compound K-automaton was minimal when the data were chosen at random. The crucial point in the proof of Theorem 6 is the fact that certain polynomial indicators are not trivial. To be – a bit (complete technicality is to be found below) – more precise, for a K-automaton (K a field), the condition "to be minimal" can be shown to be polynomial in the sense that there is a polynomial $P(\lambda_i, \mu_{j,k}(a), \gamma_l)_{1 \leqslant i,j,k,l \leqslant n; a \in A}$ in the data which vanishes *only* if $\mathscr{A} = (\lambda, \mu, \gamma)$ is not minimal. Now we suppose that $K = \mathbb{R}$ or $\mathbb{C}$ and, in order to show that, given a – say binary – law $\diamondsuit$, the lifted K-automaton $\mathscr{A} \boxed{\diamond} \mathscr{B}$ is almost every time minimal when $\mathscr{A}$ and $\mathscr{B}$ are chosen at random, it suffices to show that the big polynomial P(data of $\mathscr{A} \boxed{\diamond} \mathscr{B}$) (which is a polynomial as all the laws $\boxed{\diamond}$ under consideration here are) is not trivial.

The following theorem proves that if the data are chosen at random in bounded domains, the compound K-automaton is almost surely minimal. More precisely:

Theorem 6. *Let A be a finite alphabet and $\mathscr{A}_i = (\lambda_i, \mu_i, \gamma_i)$ two K-automata of dimension n_i $(i = 1, 2)$, chosen at random within bounded nontrivial disks of K $(K = \mathbb{R}$ or $\mathbb{C})$. Then the probability that the K-automaton $\mathscr{A}_1 \boxplus \mathscr{A}_2$ (resp. $\mathscr{A}_1 \boxdot \mathscr{A}_2$, $\mathscr{A}_1^{\boxed{*}}$) be minimal is* 1.

Proof. The proof lays on the following lemma.

Lemma 7. *There is a polynomial mapping $P : K^{|A| \times n^2 + 2n} \to K^s$ such that $P(\lambda, \mu, \gamma) = 0$ if and only if (λ, μ, γ) (a K-automaton of dimension n) is not minimal.*

Proof. By a theorem of Schützenberger [25], the representation (λ, μ, γ) is minimal if and only if $\lambda\mu(K\langle A \rangle) = K^{1 \times n}$ (resp. $\mu(K\langle A \rangle)\gamma = K^{n \times 1}$). As there is a prefix (resp. suffix) subset $U \subset A^*$ (resp. $V \subset A^*$) such that $\lambda\mu(U)$ (resp. $\mu(V)\gamma$) is a basis, we have $U \subset A^{<n}$ (resp. $V \subset A^{<n}$). Let $A^{<n} = \{w_1 = 1, w_2, \ldots, w_m\}$ $(m = (|A|^n - 1)/(|A| - 1))$. One constructs the $m \times n$ (resp. $n \times m$) matrix

$$L = \begin{pmatrix} \lambda\mu(w_1) \\ \lambda\mu(w_2) \\ \vdots \\ \lambda\mu(w_m) \end{pmatrix} \quad (\text{resp. } M = (\mu(w_1)\gamma \quad \cdots \quad \mu(w_m)\gamma)).$$

These matrices have polynomial entries in the data. In view of what precedes, minimality is equivalent to the nonnullity of some $n \times n$-minor of L and of M.

Then, for every choice $I \subset [1, m]$ with $|I| = n$, let Δ_I^L (resp. Δ_I^M) be the corresponding $n \times n$ minor. The mapping $P : K^{|A| \times n^2 + 2n} \to K^s$ with $s = \binom{m}{n}^2$ given by the formula

(family of products)

$$P(\lambda,\mu,\gamma) = (\Delta_I^L \,.\, \Delta_J^M)_{\substack{I,J\subset[1,m]\\ |I|=|J|=n}}$$

is polynomial and such that

$$P(\lambda,\mu,\gamma) = 0 \Leftrightarrow \mathscr{A} = (\lambda,\mu,\gamma) \text{ not minimal.} \qquad \square$$

End of the proof of Theorem 6. (1) For the two first operations, let $P_{\boxplus} = P(\mathscr{A}_1 \boxplus \mathscr{A}_2)$, $P_{\boxdot} = P(\mathscr{A}_1 \boxdot \mathscr{A}_2)$, and prove that $P_{\boxplus}$ (resp. $P_{\boxdot}$) is not trivial using $\tau(S_{\alpha,n}) = \mathscr{A}_1$ and $\tau(S_{\beta,n}) = \mathscr{A}_2$, $\alpha \neq \beta$ (resp. $\tau(S_{\alpha,n}) = \mathscr{A}_1$ and $\tau(S_{\alpha,m}) = \mathscr{A}_2$) extended to the alphabet A in view of Remark 4. For the star operation, prove that $P_{\boxed{*}} = P(\mathscr{A}_1^{\boxed{*}})$ is not trivial using $\tau(T_n) = \mathscr{A}_1$.

(2) Now, if $\phi : K^r \to K^s$ is polynomial and not trivial, let v be the uniform probability measure on the product of disks, then the probability that $\phi(v) \neq 0$ is 1 since $\phi^{-1}\{0\}$ is closed with empty interior. $\square$

Corollary 8. *Let A be a finite alphabet and $\mathscr{A} = (\lambda,\mu,\gamma)$ be a K-automaton choosen at random within bounded nontrivial disks of K ($K = \mathbb{R}$ or $\mathbb{C}$). Then the probability that $\mathscr{A}$ be minimal is* 1.

Proof. Straightforward, as P is not trivial. $\square$

4. Dual laws

In this section, we prove that the laws of second kind which are (traditionally) known to preserve rationality (Hadamard, shuffle and infiltration product [22]) are in fact, up to a deformation, the only ones to be both subalphabet compatible and defined by duality (the general deformation will be denoted $\boxed{\uparrow}_q$). The last section is the beginning of a discussion over the compatibility between the geometry of K-automata and the dual laws.

4.1. Discussion

Let $a, b \in A$, $u, v \in A^*$, and $\odot_{\varepsilon,q}$ be the law defined recursively by

$$1 \odot_{\varepsilon,q} 1 = 1, \quad a \odot_{\varepsilon,q} 1 = 1 \odot_{\varepsilon,q} a = \varepsilon a,$$

$$au \odot_{\varepsilon,q} bv = \varepsilon(a(u \odot_{\varepsilon,q} bv) + b(au \odot_{\varepsilon,q} v)) + q\delta_{a,b} a(u \odot_{\varepsilon,q} v)$$

with $\delta_{a,b}$ the Kronecker delta.

One immediately checks that this law is associative if and only if $\varepsilon \in \{0,1\}$. We get, here, the well-known shuffle ($\sqcup\!\sqcup = \odot_{1,0}$), infiltration ($\uparrow = \odot_{1,1}$) and Hadamard ($\odot = \odot_{0,1}$) products ([6, 16]). Then, $\odot_{1,q}$ is a continuous deformation between shuffle and infiltration. These laws can be called "dual laws" as they proceed from the same template that we now describe. We use an implementable realisation of the lexicographically ordered tensor product. Let us recall that the tensor product of two spaces U and V with bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ is $U\otimes V$, with basis $(u_i\otimes v_j)_{(i,j)\in I\times J}$, and for the sake of computation, we impose that the set $I\times J$ be lexicographically ordered.

Let $K\langle A\rangle\otimes K\langle A\rangle$ be the "double" noncommutative polynomial algebra that is the set of finite sums $P=\sum_{u,v\in A^*}\langle P|u\otimes v\rangle u\otimes v$, the product being given by $(u_1\otimes v_1)(u_2\otimes v_2)=u_1u_2\otimes v_1v_2$.

The construction of dual laws is based on the following pattern:

Let $c:K\langle A\rangle\to K\langle A\rangle\otimes K\langle A\rangle$, if for all $w\in A^*$, the set $\{w\colon\ \langle u\otimes v|c(w)\rangle\neq 0\}$ is finite (in which case c itself will be called *locally finite*), then the sum

$$u\,\square_\alpha v=\sum_{w\in A^*}\langle u\otimes v|c_\alpha(w)\rangle w$$

exists and defines a (binary) law $\square_\alpha$ on $K\langle A\rangle$, dual to c_α. Then, this extends to series by

$$\langle R\,\square_\alpha S|w\rangle=\langle R\otimes S|c_\alpha(w)\rangle.$$

One can show easily that the three laws $\odot$, $\sqcup\!\sqcup$ and $\uparrow$ come from coproducts defined on the words by

(1) $c_\alpha(a_1a_2\cdots a_n)=c_\alpha(a_1)c_\alpha(a_2)\cdots c_\alpha(a_n)$,

(2) $c_\odot(a)=a\otimes a$, $c_{\sqcup\!\sqcup}(a)=a\otimes 1+1\otimes a$, $c_\uparrow(a)=a\otimes 1+1\otimes a+a\otimes a$

and, generally, $c_{\varepsilon,q}(a)=\varepsilon(a\otimes 1+1\otimes a)+qa\otimes a$.

The preceding computation scheme has an immediate consequence on the implementation of the laws.

Proposition 9. *Let* $R:(\lambda^r,\mu^r,\gamma^r)$ *and* $S:(\lambda^s,\mu^s,\gamma^s)$. *Then*

$$R\,\square_\alpha S\colon\ (\lambda^r\otimes\lambda^s,\mu^r\otimes\mu^s\circ c_\alpha,\gamma^r\otimes\gamma^s).$$

Proof. We verify it by duality. Indeed, for $w\in A^*$,

$$\begin{aligned}\langle R\otimes S|c_\alpha(w)\rangle&=\sum_{u,v\in A^*}\langle\lambda^r\otimes\lambda^s(\mu^r\otimes\mu^s(u\otimes v))\gamma^r\otimes\gamma^s\times u\otimes v|c_\alpha(w)\rangle\\&=\sum_{u,v\in A^*}\lambda^r\otimes\lambda^s(\mu^r\otimes\mu^s(u\otimes v))\gamma^r\otimes\gamma^s.\langle u\otimes v|c_\alpha(w)\rangle\\&=\lambda^r\otimes\lambda^s\left(\sum_{u,v\in A^*}\mu^r\otimes\mu^s\langle u\otimes,v|c_\alpha(w)\rangle(u\otimes v)\right)\gamma^r\otimes\gamma^s\\&=\lambda^r\otimes\lambda^s\left(\mu^r\otimes\mu^s\sum_{u,v\in A^*}\langle u\otimes v|c_\alpha(w)\rangle(u\otimes v)\right)\gamma^r\otimes\gamma^s\\&=\lambda^r\otimes\lambda^s(\mu^r\otimes\mu^s c_\alpha(w))\gamma^r\otimes\gamma^s.\qquad\square\end{aligned}$$

Associated with, let us study among laws which ones are associative.

Proposition 10. *Let K be a field, and $c_\alpha : K\langle A\rangle \to K\langle A\rangle \otimes K\langle A\rangle$ the alphabetic morphism defined on the letters of A by*

$$c_\alpha(a) = \sum_{p,q\geqslant 0} \alpha_{p,q} a^p \otimes a^q$$

with $c_\alpha(1) = 1\otimes 1$ ($\alpha_{p,q} = \alpha_{p,q}(a)$ may vary from one letter to one another).
(1) *The morphism c_α is locally finite if and only if $\alpha_{0,0} = 0$.*
(2) *Providing $\alpha_{0,0} = 0$, the following assertions are equivalent:*
   (a) *The law $\Box_\alpha$ defined by $\langle u \Box_\alpha v | w\rangle = \langle u\otimes v | c_\alpha(w)\rangle$ ($u, v, w \in A^*$) is associative.*
   (b) *The coefficients $\alpha_{p,q}$ satisfy the relations $\alpha_{p,q} = 0$ for p or $q \geqslant 2$, $\alpha_{0,1}, \alpha_{1,0} \in \{0,1\}$ and $\alpha_{0,1}\alpha_{1,1} = \alpha_{1,0}\alpha_{1,1}$.*
(3) *Providing* (2.2b), *the element 1_{A^*} is a unit for $\Box_\alpha$ if and only if $\alpha_{0,1} = \alpha_{1,0} = 1$.*

Proof. (1) We have $c_\alpha(a) = \alpha_{0,0} 1\otimes 1 + \sum_{p+q\geqslant 1} \alpha_{p,q} a^p \otimes a^q$, and then for all $n \geqslant 0$, $c_\alpha(a^n) = \alpha_{0,0}^n 1\otimes 1 + \sum_{p+q\geqslant 1} \beta_{p,q} a^p \otimes a^q$ for some $\beta_{p,q}$. If $\alpha_{0,0}$ were not zero, the term $1\otimes 1$ would appear in an infinity of words, and then c_α would not be locally finite.

Conversely, if $\alpha_{0,0}(a) = 0$ (for every letter), then $c_\alpha(a) = \sum_{p+q\geqslant 1} \alpha_{p,q} a^p \otimes a^q$ and for all word $w = a_1 \cdots a_n \in A^*$,

$$c_\alpha(w) = \sum_{\substack{p_i+q_i\geqslant 1\\ 1\leqslant i\leqslant n}} \left(\prod_{i=1}^{n} \alpha_{p_i,q_i}(a_i)\right) a_1^{p_1} \cdots a_n^{p_n} \otimes a_1^{q_1} \cdots a_n^{q_n}.$$

As $p_i + q_i \geqslant 1$, we have $\sum_{i=1}^{n} (p_i + q_i) \geqslant n$, that is to say $|w| \leqslant |u| + |v|$ and $Alph(w) = Alph(u) \cup Alph(v)$ with $u = a_1^{p_1} \cdots a_n^{p_n}$ and $v = a_1^{q_1} \cdots a_n^{q_n}$.

To summarize, the set

$$S = \{w / \langle u\otimes v | c_\alpha(w)\rangle \neq 0\}$$

has bounded lengths and its alphabet is finite, S is then finite.

(2) First, remark that (2.2a) is equivalent to the condition

$$(Id\otimes c_\alpha)\circ c_\alpha = (c_\alpha \otimes Id)\circ c_\alpha. \tag{4}$$

The law $\Box_\alpha$ is associative if and only if for all words $u_1, u_2, u_3 \in A^*$, we have

$$(u_1 \Box_\alpha u_2) \Box_\alpha u_3 = u_1 \Box_\alpha (u_2 \Box_\alpha u_3)$$

that is to say that, for all $w \in A^*$,

$$\langle (u_1 \Box_\alpha u_2) \Box_\alpha u_3 | w\rangle = \langle u_1 \Box_\alpha (u_2 \Box_\alpha u_3) | w\rangle.$$

But one has

$$\begin{aligned}\langle (u_1 \Box_\alpha u_2) \Box_\alpha u_3 | w\rangle &= \langle (u_1 \Box_\alpha u_2) \otimes u_3 | c_\alpha(w)\rangle \\ &= \langle u_1 \otimes u_2 \otimes u_3 | (c_\alpha \otimes Id)\circ c_\alpha(w)\rangle\end{aligned}$$

and

$$\begin{aligned}\langle u_1 \,\square_\alpha (u_2 \,\square_\alpha u_3)|w\rangle &= \langle u_1 \otimes (u_2 \,\square_\alpha u_3)|c_\alpha(w)\rangle \\ &= \langle u_1 \otimes u_2 \otimes u_3|(Id \otimes c_\alpha)\circ c_\alpha(w)\rangle.\end{aligned}$$

As u_1, u_2, u_3, w are arbitrary, we get $(c_\alpha \otimes Id)\circ c_\alpha = (Id \otimes c_\alpha)\circ c_\alpha$.

To show the equivalence between (2.2b) and (4), suppose first that (4) holds. We endow $\mathbb{N}^k$ with the lexicographic order (reading from left to right for instance) which is compatible with addition and will be denoted $\prec$ (here, $k=2,3$). Then, if it is not zero, $c_\alpha(a)$ can be written as

$$\alpha_{\bar{p},\bar{q}} a^{\bar{p}} \otimes a^{\bar{q}} + \sum_{(p,q)\prec(\bar{p},\bar{q})} \alpha_{p,q} a^p \otimes a^q,$$

$(\bar{p},\bar{q})$ being the highest couple of exponents in the support. Then,

$$\begin{aligned}(c_\alpha \otimes Id)\circ c_\alpha(a) &= \alpha_{\bar{p},\bar{q}} c_\alpha(a^{\bar{p}}) \otimes a^{\bar{q}} + \sum_{(p,q)\prec(\bar{p},\bar{q})} \alpha_{p,q} c_\alpha(a^p)\otimes a^q \\ &= \alpha_{\bar{p},\bar{q}}^{\bar{p}+1} a^{(\bar{p})^2} \otimes a^{\bar{p}\bar{q}} \otimes a^{\bar{q}} + \sum_{(p,q,r)\prec(\bar{p}^2,\bar{p}\bar{q},\bar{q})} \beta_{p,q,r} a^p \otimes a^q \otimes a^r,\end{aligned}$$

but

$$\begin{aligned}(Id \otimes c_\alpha)\circ c_\alpha(a) &= \alpha_{\bar{p},\bar{q}} a^{\bar{p}} \otimes c_\alpha(a^{\bar{q}}) + \sum_{(p,q)\prec(\bar{p},\bar{q})} \alpha_{p,q} a^p \otimes c_\alpha(a^q) \\ &= \alpha_{\bar{p},\bar{q}}^{\bar{q}+1} a^{\bar{p}} \otimes a^{\overline{pq}} \otimes a^{(\bar{q})^2} + \sum_{(p,q,r)\prec(\bar{p},\overline{pq},\bar{q}^2)} \beta_{p,q,r} a^p \otimes a^q \otimes a^r.\end{aligned}$$

Necessarily, $\bar{p}=\bar{p}^2$ and $\bar{q}=\bar{q}^2$, which is only possible when $\bar{p}\in\{0,1\}$ and $\bar{q}\in\{0,1\}$ and then $\alpha_{p,q}=0$ for p or $q \geqslant 2$. The equality now reads

$$\begin{aligned}&\alpha_{1,0} a\otimes 1\otimes 1 + \alpha_{0,1}^2 1\otimes 1\otimes a + \alpha_{0,1}\alpha_{1,1} a\otimes 1\otimes a \\ &\quad = \alpha_{1,0}^2 a\otimes 1\otimes 1 + \alpha_{0,1} 1\otimes 1\otimes a + \alpha_{1,0}\alpha_{1,1} a\otimes 1\otimes a,\end{aligned}$$

which implies (2.2b). The converse is a straightforward computation.

(3) The condition 1_{A^*} is a unit for $\square_\alpha$ implies that, for $a\in A$, we have

$$\begin{aligned}1\,\square_\alpha a = a\,\square_\alpha 1 = a &\Leftrightarrow \langle 1\,\square_\alpha a|a\rangle = \langle a\,\square_\alpha 1|a\rangle = 1 \\ &\Leftrightarrow \langle 1\otimes a|c_\alpha(a)\rangle = \langle a\otimes 1|c_\alpha(a)\rangle = 1 \\ &\Leftrightarrow \begin{cases} \left\langle 1\otimes a \middle| \sum_{p,q\geqslant 0} \alpha_{p,q} a^p \otimes a^q \right\rangle = 1 \\ \left\langle a\otimes 1 \middle| \sum_{p,q\geqslant 0} \alpha_{p,q} a^p \otimes a^q \right\rangle = 1 \end{cases} \\ &\Leftrightarrow \alpha_{0,1} = \alpha_{1,0} = 1.\end{aligned}$$

Conversely, the latter implies that, for each $w\in A^*$, $1\,\square_\alpha w = w\,\square_\alpha 1 = w$. $\square$

Remark 11. As for as commutative laws are concerned, the condition $\alpha_{p,q} = \alpha_{q,p}$ is sufficient. Moreover, the condition (2.2b) implies $\alpha_{0,1},\ \alpha_{1,0} \in \{0,1\}$.

In fact, the only dual laws which are associative ones are

$$c_{\varepsilon,q}(a) = \varepsilon(a \otimes 1 + 1 \otimes a) + qa \otimes a$$

with parameters ε and q such that $\varepsilon \in \{0,1\}$ and $q \in K$.

4.2. Usual dual laws

(a) Shuffle and infiltration product ($\varepsilon = 1,\ q \in \{0,1\}$)

Proposition 12. *Let R (resp. S) be a rational series and $\mathscr{A}_r = (\lambda_r, \mu_r, \gamma_r)$ (resp. $\mathscr{A}_s = (\lambda_s, \mu_s, \gamma_s)$) be a K-automaton which recognizes R (resp. S). Let n (resp. m) be the dimension of $\mathscr{A}_r$ (resp. $\mathscr{A}_s$).*
(1) *The automata corresponding to shuffle and infiltration products are, respectively:*

$$R \sqcup\!\sqcup S\colon\ (\lambda_r \otimes \lambda_s, (\mu_r(a) \otimes I_r + I_s \otimes \mu_s(a))_{a \in A}, \gamma_r \otimes \gamma_s), \tag{5}$$

$$R \uparrow S\colon\ (\lambda_r \otimes \lambda_s, (\mu_r(a) \otimes I_s + I_r \otimes \mu_s(a) + \mu_r(a) \otimes \mu_s(a))_{a \in A}, \gamma_r \otimes \gamma_s). \tag{6}$$

(2) *The bound nm is sharp in both the cases.*
(3) *The density result of Theorem 6 holds.*

Proof. Concerning point (2), an example reaching the bound for any rank is given by the families of series $S_n = a^{n-1}$ and $T_n = b^{n-1}$ of rank n. The shuffle product $S_n \sqcup\!\sqcup T_m = a^{n-1} \sqcup\!\sqcup b^{m-1}$ ($a \neq b \in A$) has a minimal linear representation of rank nm. The same example is valid for the infiltration product since, for $a \neq b$, $a^n \uparrow b^m = a^n \sqcup\!\sqcup b^m$.

To prove (3), it suffices to consider $P_{\boxed{\sqcup\!\sqcup}} = P(\mathscr{A}_1 \boxed{\sqcup\!\sqcup} \mathscr{A}_2)$ which is nontrivial using $\tau(S_n) = \mathscr{A}_1$ and $\tau(T_m) = \mathscr{A}_2$. $\square$

The proposition yields the following.

Definition 13. Let $\mathscr{A}_i = (\lambda_i, \rho_i, \gamma_i)$ with $i = 1, 2$ then we define $\mathscr{A}_1 \boxed{\sqcup\!\sqcup} \mathscr{A}_2$ and $\mathscr{A}_1 \boxed{\uparrow} \mathscr{A}_2$ by the formulas 5 and 6.

Remark 14. These laws are already associative at the level of K-automata.

(b) Hadamard product ($\varepsilon = 0,\ q = 1$)

We recall that the Hadamard product [8, 26] of two series is the pointwise product of the corresponding functions (on words). We can use the machinery above to describe a K-automaton for it.

Proposition 15. *Let R (resp. S) be a rational series and $\mathscr{A}_r = (\lambda^r, \mu^r, \gamma^r)$ ($\mathscr{A}_s = (\lambda^s, \mu^s, \gamma^s)$) be a K-automaton which recognizes R (resp. S). Let n (resp. m) be the dimension of $\mathscr{A}_r$ (resp. $\mathscr{A}_s$). A representation of the Hadamard product is*

$$R \odot S\colon\ (\lambda^r \otimes \lambda^s, (\mu^r(a) \otimes \mu^s(a))_{a\in A}, \gamma^r \otimes \gamma^s)$$

and the bound is asymptotically sharp.

Proof. Let $\beta(n,m) = \sup_{\substack{rank(R)=n \\ rank(S)=m}} rank(R \odot S)$. We claim that

$$\limsup_{n,m\to+\infty} \frac{\beta(n,m)}{nm} = 1,$$

(what we mean by "asymptotically sharp").

Indeed, supposing K is a field, let us consider the Hadamard product of two series of the family

$$S_n = \sum_{k\geqslant 0} a^{nk} = \frac{1}{(1-a^n)}.$$

The rank of S_n is n, and

$$\begin{aligned} S_n \odot S_m &= \sum_{k\geqslant 0} a^{nk} \odot \sum_{k'\geqslant 0} a^{mk'} = \sum_{p\geqslant 0} \langle S_n | a^p\rangle \langle S_m | a^p\rangle a^p \\ &= \sum_{k\geqslant 0} a^{lcm(n,m)k} = S_{lcm(n,m)}. \end{aligned}$$

Thus, for n and m coprime, the rank of the product is nm, which proves the claim. □

If $K = \mathbb{R}$ or $\mathbb{C}$, the theorem of density holds when n and m are coprime.

4.3. Compatibility between the dual laws and the geometry of K-automata

Here we just indicate the beginning of a discussion which will be published at length in a forthcoming paper. This discussion deals with compatibility between laws and the geometry of K-automata. Let A be an alphabet and θ be a dependence graph on it. [2]

The following fact is well known. If two automata $\mathscr{A}_1$ and $\mathscr{A}_2$ are compatible with the commutations of θ (that is, for every state q and $(a,b)\in\theta$, $q.ab = q.ba$), then so is their product shuffle $\mathscr{A}_1 \boxed{\sqcup\!\sqcup} \mathscr{A}_2$. We can address the problem in a more general way.

Definition 16. Let $R \subset A^* \times A^*$, we say that a (finite) K-automaton is R-compatible if and only if for every $q \in Q$, $(u,v) \in R$, $q.u = q.v$.

Now, due to the fact that for $K = \mathbb{B}$ or a field the minimal model is the K-automaton of quotients, we get the following result [17].

[2] This notion has been extensively used in the modellization of parallelism [19] and job shop [11].

Proposition 17. *Let K be a semiring and $R \subset A^* \times A^*$. Let us consider the two following assertions*:

(I) *The K-automaton $\mathscr{A} = (\lambda, \mu, \gamma)$ is R-compatible.*

(II) *The behaviour $S = \sum_{w \in A^*} \lambda\mu(w)\gamma w$ is R-saturated* (*i.e. for every $u, w \in A^*$, if $(v_1, v_2) \in R$, we have $\langle S|uv_1w\rangle = \langle S|uv_2w\rangle$*).

Then

(i) *Statement I implies II.*

(ii) *If $K = \mathbb{B}$ and $\mathscr{A}$ is the minimal DFA or if K is a field and $\mathscr{A}$ is minimal, then statements I and II are equivalent.*

Proof. (i) It is easily seen that the fact that $\mathscr{A}$ is R-compatible is equivalent to the fact that μ is R-compatible, that is to say $(v_1, v_2) \in R \Rightarrow \mu(v_1) = \mu(v_2)$. Then, assuming $(v_1, v_2) \in R$, we have

$$\langle S|uv_1w\rangle = \lambda\mu(u)\mu(v_1)\mu(w)\gamma = \lambda\mu(u)\mu(v_2)\mu(w)\gamma = \langle S|uv_2w\rangle.$$

(ii) If K is a field (resp. $K = \mathbb{B}$), the minimal K-automaton (resp. the minimal DFA) is, up to an isomorphism [3] which does not change the conclusion, the automaton of quotients with

- *States*: Left quotients $q_w = (w^{-1}S)_{w \in F}$, F being finite.
- *Transitions*: Given by $q_w.\, a = a^{-1}q_w$, which gives in turn $q_w.\, u = u^{-1}q_w$.

Thus, if $(u_1, u_2) \in R$, we have

$$\langle q_w.\, u_1|v\rangle = \langle q_w.\, u_1|v\rangle = \langle S|wu_1v\rangle = \langle S|wu_2v\rangle = \langle q_w.\, u_2|v\rangle$$

which proves the compatibility of $\mathscr{A}$. $\square$

Now, one can set up informally the compatibility problem in the following general frame.

Let $\diamondsuit$ be an n-ary law on $K^{rat}\langle\langle A\rangle\rangle$ and $\boxed{\diamond}$ be its – more nor less – natural lifting to the level of automata.

Problem. What are the relators $R \subset A^* \times A^*$ such that

$$\mathscr{A}_1, \ldots, \mathscr{A}_n \text{ are } R\text{-compatible } \textbf{implies } \diamond(\mathscr{A}_1, \ldots, \mathscr{A}_n) \text{ is } R\text{-compatible}$$

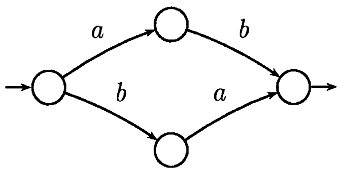
We begin to review the rational laws, focusing on partial commutations. Table 1 summarizes the first results [17].

[3] A set theoretical morphism in the boolean case and a vector space conjugacy in the case of a field. Then, in every case, K-linear.

Table 1

Kind	Rational laws	Compatibility with commutations	Compatibility with other congruences
First	$\times$	Yes	All
	$+$	Yes	All
	$.$	No	?
	$*$	No	?
Second	$\odot$	Yes	All
	$\sqcup\!\sqcup$	Yes	Depends on K
	$\uparrow$	Yes	as $\sqcup\!\sqcup$
	$\uparrow_q$	Yes	as $\sqcup\!\sqcup$

Remark 18. Let us consider the automaton $\mathscr{A}$ given by the following graph:



The automaton $\mathscr{A}$ is compatible with the commutation $ab \equiv ba$. This automaton is the representation of the series $S_1 = ab + ba$ in $K\langle\langle a,b\rangle\rangle$ and $S_2 = ab$ in $K[[a,b]]$.

More generally, if a K-automaton $\mathscr{A}$ is compatible with a congruence $\equiv$, then it is the linear representation of a series $S \in K[[A^*/_{\equiv}]]$ (function space of the quotient monoid). Of course, according to Proposition 17, the converse holds when K is a field (or a PID) and $\mathscr{A}$ is minimal or K is $\mathbb{B}$ and $\mathscr{A}$ the minimal DFA.

5. Conclusion

Many computations over rational series can be lifted to the level of K-automata and these classical constructions have been proved to be generically optimal. The implementation of classical dual laws (shuffle, Hadamard and infiltration products) has suggested to us that other laws (which also preserve rationality) could exist. In fact, we have proved that, under some natural hypothesis, there is no other choice than a deformation of the classical case.

The study of the shuffle product over K-automata raises the question of the compatibility with some geometric patterns of the K-automaton (this is well known for dependence relations). The answer to this problem is of course coefficient dependent and deserves a deeper study which will be the subject of a forthcoming paper.

Acknowledgements

We would like to thank J.M. Champarnaud for fruitful discussions and constructive remarks.

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