About Some Drinfel'd Associators

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Abstract. We study, by means of a fragment of theory about noncommutative differential equations, existence and unicity of Drinfel'd solutions $G_i, i = 0, 1$ (with asymptotic conditions). From there, we give examples of Drinfel'd series with rational coefficients.

Keywords: Drinfel'd series · Harmonic sums
Knizhnik-Zamolodchikov equations polylogarithms · Polyzetas
Regularization · Renormalization
Noncommutative differential equations
Noncommutative generating series

1 Knizhnik-Zamolodchikov Differential Equations and Coefficients of Drinfel'd Associators

In 1986 [11], in order to study the linear representations of the braid group B_n coming from the monodromy of the Knizhnik-Zamolodchikov differential equations, Drinfel'd introduced a class of formal power series Φ on noncommutative variables over the finite alphabet $X = \{x_0, x_1\}$. Such power series Φ are called *Drinfel'd series* (or *associators*). For $n = 3$, it leads to the following fuchsian differential equation with three regular singularities in $\{0, 1, +\infty\}$:

$$(DE)\ \ dG(z) = \left(x_0 \frac{dz}{z} + x_1 \frac{dz}{1-z}\right) G(z).$$

This is connected to the fact that the pure braid group on three strands P_3 is the semi-direct product of the pure braid group on two strands (a copy of $\mathbb{Z}$) with a copy of the free group on two generators. Although this interpretation of (DE) does not play an explicit role below, it can be kept in mind with a view towards applications.

Solutions of (DE) are power series, with coefficients which are mono-valued functions on the simply connected domain $\Omega := \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$ and can be seen as multi-valued over[1] $\mathbb{C} \setminus \{0, 1\}$ on noncommutative variables on X.

[1] In fact, we have mappings from the universal covering $\widetilde{\mathbb{C} \setminus \{0, 1\}}$.

V. P. Gerdt et al. (Eds.): CASC 2018, LNCS 11077, pp. 146–163, 2018.
https://doi.org/10.1007/978-3-319-99639-4_10

Drinfel'd proved that (DE) admits two particular mono-valued solutions on Ω [12,13]

$$G_0(z) \underset{z\rightsquigarrow 0}{\widetilde{}} \exp[x_0 \log(z)] \text{ and } G_1(z) \underset{z\rightsquigarrow 1}{\widetilde{}} \exp[-x_1 \log(1-z)]. \tag{1}$$

and the existence of an associator $\Phi_{KZ} \in \mathbb{R}\langle\langle X\rangle\rangle$ such that $G_0 = G_1\Phi_{KZ}$ [12,13] but he did not make explicit neither G_0 and G_1 nor Φ_{KZ}. After that, via representations of the chord diagram algebras, Lê and Murakami [26] expressed the coefficients of Φ_{KZ} as *linear* combinations of special values of several complex variables *zeta* functions, $\{\zeta_r\}_{r\in\mathbb{N}_+}$,

$$\zeta_r : \mathcal{H}_r \longrightarrow \mathbb{R},\ (s_1,\ldots,s_r) \longmapsto \sum_{n_1>\ldots>n_k>0} \frac{1}{n_1^{s_1}\ldots n_k^{s_r}}, \tag{2}$$

where $\mathcal{H}_r := \{(s_1,\ldots,s_r) \in \mathbb{C}^r \,|\, \forall m = 1,..,r, \sum_{i=1}^m \Re(s_i) > m\}$.

For $(s_1,\ldots,s_r) \in \mathcal{H}_r$, one has two ways of thinking $\zeta_r(s_1,\ldots,s_r)$ as limits, fulfilling identities [1,20,21]. Firstly, they are limits of *polylogarithms* and secondly, as truncated sums, they are limits of *harmonic sums*:

$$\mathrm{Li}_{s_1,\ldots,s_k}(z) = \sum_{n_1>\ldots>n_k>0} \frac{z^{n_1}}{n_1^{s_1}\ldots n_k^{s_k}}, \text{ for } z \in \mathbb{C}, |z| < 1, \tag{3}$$

$$\mathrm{H}_{s_1,\ldots,s_k}(N) = \sum_{n_1>\ldots>n_k>0}^{N} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}}, \text{ for } N \in \mathbb{N}_+. \tag{4}$$

More precisely, if $(s_1,\ldots,s_r) \in \mathcal{H}_r$ then, after a theorem by Abel, one has

$$\lim_{z\to 1} \mathrm{Li}_{s_1,\ldots,s_k}(z) = \lim_{n\to\infty} \mathrm{H}_{s_1,\ldots,s_k}(n) =: \zeta_r(s_1,\ldots,s_k) \tag{5}$$

else it does not hold, for $(s_1,\ldots,s_r) \notin \mathcal{H}_r$, while $\mathrm{Li}_{s_1,\ldots,s_k}$ is well defined over $\{z \in \mathbb{C}, |z| < 1\}$ and so is $\mathrm{H}_{s_1,\ldots,s_k}$, as Taylor coefficients of the following function

$$\frac{\mathrm{Li}_{s_1,\ldots,s_k}(z)}{1-z} = \sum_{n\geq 1} \mathrm{H}_{s_1,\ldots,s_k}(n) z^n, \text{ for } z \in \mathbb{C}, |z| < 1. \tag{6}$$

For $r = 1$, ζ_1 is nothing else but the famous Riemann zeta function and, for $r = 0$, it is convenient to set ζ_0 to the constant function $s \mapsto 1_{\mathbb{R}}$. In all the sequel, for simplification, we will adopt the notation ζ for $\zeta_r, r \in \mathbb{N}$.

In this work, we will describe the regularized solutions of (DE). Remark also that replacing letters $\{x_i\}_{i=0,1}$ by constant matrices $\{M_i\}_{i=0,1}$ (resp. analytical vector fields $\{A_i\}_{i=0,1}$), one deals with linear (resp. nonlinear) differential equations [3,5,19,25] (resp. [6,9,22]). Hence, (DE) can also be considered as the universal linear and nonlinear differential equation with three singularities. Therefore these computations can undergo an automatic treatment (see, for instance [16] and the subsequent sessions).

For that, we are considering the alphabets $X := \{x_0, x_1\}$ and $Y_0 := \{y_s\}_{s\geq 0}$ equipped with the total ordering $x_0 < x_1$ and $y_0 > y_1 > y_2 > \ldots$, respectively. Let us also consider $Y := Y_0 \setminus \{y_0\}$.

The free monoid generated by X (resp. Y, Y_0) is denoted by X^* (resp. Y^*, Y_0^*) and admits 1_{X^*} (resp. $1_{Y^*}, 1_{Y_0^*}$) as unit. The sets of polynomials and formal power series, with coefficients in a commutative $\mathbb{Q}$-algebra A, over X^* (resp. Y^*, Y_0^*) are denoted by $A\langle X\rangle$ (resp. $A\langle Y\rangle, A\langle Y_0\rangle$) and $A\langle\langle X\rangle\rangle$ (resp. $A\langle\langle Y\rangle\rangle, A\langle\langle Y_0\rangle\rangle$), respectively.

The sets of polynomials are A-modules and endowed with the associative concatenation, the associative commutative shuffle (resp. quasi-shuffle) product, over $A\langle X\rangle$ (resp. $A\langle Y\rangle, A\langle Y_0\rangle$). Their associated coproducts are denoted, respectively, $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\uplus}$.

The shuffle algebra $(A\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})$ and quasi-shuffle algebra $(A\langle Y\rangle, \uplus, 1_{Y^*})$ admit the sets of Lyndon words denoted, respectively, by $\mathcal{L}ynX$ and $\mathcal{L}ynY$, as transcendence bases [27] (resp. [22,23]).

Now, for $Z = X$ or Y, denoting $\mathrm{Lie}_A\langle Z\rangle$ and $\mathrm{Lie}_A\langle\langle Z\rangle\rangle$ the sets of, respectively, Lie polynomials and Lie series, the enveloping algebra $\mathcal{U}(\mathrm{Lie}_A\langle Z\rangle)$ is isomorphic to the (Hopf) bialgebra

$$\mathcal{H}_{\sqcup\!\sqcup}(Z) := (A\langle Z\rangle, ., 1_{Z^*}, \Delta_{\sqcup\!\sqcup}, \mathsf{e}). \tag{7}$$

We get also

$$\mathcal{H}_{\uplus}(Y) := (A\langle Y\rangle, ., 1_{Y^*}, \Delta_{\uplus}, \mathsf{e}) \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\uplus}(Y))), \tag{8}$$

where $\mathrm{Prim}(\mathcal{H}_{\uplus}(Y)) = \mathrm{span}_A\{\pi_1(w) | w \in Y^*\}$ and, for any $w \in Y^*$ [2,22,23],

$$\pi_1(w) = \sum_{k=1}^{(w)} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k \in Y^+} \langle w | u_1 \uplus \ldots \uplus u_k\rangle u_1 \ldots u_k. \tag{9}$$

The paper is organised as follows: Sect. 1 is devoted to setting the combinatorial framework of noncommutative differential Knizhnik-Zamolodchikov equations and Drinfel'd associators. Afterwards, in Sect. 2, we recall some algebraic structures about polylogarithms and harmonic sums, through their indexing by words. In Sect. 3, we will study, by means of a fragment of theory about noncommutative differential equations[2], existence and unicity of Drinfel'd solutions (1). Finally, in Sect. 4, we will renormalize solutions of (DE) and will regularize them at singularites. Also some examples of Drinfel'd series with rational coefficients are provided. Some results in this paper have been presented in [10], as preprint, but never published before (see also [24]).

2 Indexing Polylogarithms and Harmonic Sums by Words and Their Generating Series

For any $r \in \mathbb{N}$, any combinatorial composition $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ can be associated with words

$$x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \in X^*x_1 \text{ and } y_{s_1} \ldots y_{s_r} \in Y^*. \tag{10}$$

[2] The main theorem, although not very difficult once the correct setting has been implemented, is very powerful and new here in its two-sided version (see Subsect. 3.1).

Similarly, any multi-index[3] $(s_1,\ldots,s_r) \in \mathbb{N}^r$ can be associated with words $y_{s_1}\ldots y_{s_r} \in Y_0^*$. Then let us index polylogarithms and harmonic sums by words [2,21]:

$$\mathrm{Li}_{x_0^r}(z) := \frac{(\log(z))^r}{r!}, \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1} := \mathrm{Li}_{s_1,\ldots,s_r}, \mathrm{H}_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{s_1,\ldots,s_r}. \quad (11)$$

Similarly, let $\mathrm{Li}_{-s_1,\ldots,-s_k}$ and $\mathrm{H}_{-s_1,\ldots,-s_k}$ be indexed by words[4] as follows [7,8]:

$$\mathrm{Li}^-_{y_0^r}(z) := \left(\frac{z}{1-z}\right)^r, \mathrm{Li}^-_{y_{s_1}\ldots y_{s_r}} := \mathrm{Li}_{-s_1,\ldots,-s_r} \text{ and} \quad (12)$$

$$\mathrm{H}^-_{y_0^r}(n) := \binom{n}{r} = \frac{(n)_r}{r!}, \mathrm{H}^-_{y_{s_1}\ldots y_{s_r}} := \mathrm{H}_{-s_1,\ldots,-s_r}.$$

There exists a law of algebra, denoted by $\top$, in $\mathbb{Q}\langle\langle Y_0\rangle\rangle$, such that the morphism (14) of algebras is <u>*surjective*</u>. With this, we get the following [7]

$$\mathrm{H}^-_\bullet : (\mathbb{Q}\langle Y_0\rangle, ⧢ , 1_{Y_0^*}) \longrightarrow (\mathbb{Q}\{\mathrm{H}^-_w\}_{w\in Y_0^*}, \times, 1),\ w \longmapsto \mathrm{H}^-_w, \quad (13)$$

$$\mathrm{Li}^-_\bullet : (\mathbb{Q}\langle Y_0\rangle, \top, 1_{Y_0^*}) \longrightarrow (\mathbb{Q}\{\mathrm{Li}^-_w\}_{w\in Y_0^*}, \times, 1),\ w \longmapsto \mathrm{Li}^-_w, \quad (14)$$

such that [7]

$$\ker \mathrm{H}^-_\bullet = \ker \mathrm{Li}^-_\bullet = \mathbb{Q}\langle\{w - w\top 1_{Y_0^*} | w \in Y_0^*\}\rangle. \quad (15)$$

Moreover, the families $\{\mathrm{H}^-_{y_k}\}_{k\ge 0}$ and $\{\mathrm{Li}^-_{y_k}\}_{k\ge 0}$ are $\mathbb{Q}$-linearly independent.

On the other hand, the following morphisms of algebras are <u>*injective*</u>

$$\mathrm{H}_\bullet : (\mathbb{Q}\langle Y\rangle, ⧢ , 1_{Y^*}) \longrightarrow (\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, \times, 1),\ w \longmapsto \mathrm{H}_w, \quad (16)$$

$$\mathrm{Li}_\bullet : (\mathbb{Q}\langle X\rangle, ⧢, 1_{X^*}) \longrightarrow (\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, \times, 1),\ w \longmapsto \mathrm{Li}_w. \quad (17)$$

Moreover, the families $\{\mathrm{H}_w\}_{w\in Y^*}$ and $\{\mathrm{Li}_w\}_{w\in X^*}$ are $\mathbb{Q}$-linearly independent and the families $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ and $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ are $\mathbb{Q}$-algebraically independent. But at singularities of $\{\mathrm{Li}_w\}_{w\in X^*}, \{\mathrm{H}_w\}_{w\in Y^*}$, the following convergent values

$$\forall u \in Y^* - y_1Y^*, \zeta(u) := \mathrm{H}_u(+\infty) \text{ and } \forall v \in x_0X^*x_1, \zeta(v) := \mathrm{Li}_v(1) \quad (18)$$

are no longer linearly independent and the values $\{\mathrm{H}_l(+\infty)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$ (resp. $\{\mathrm{Li}_l(1)\}_{l\in\mathcal{L}ynX\setminus X}$) are no longer algebraically independent [21,28].

The graphs of the isomorphisms of algebras, $\mathrm{Li}_\bullet$ and $\mathrm{H}_\bullet$, as generating series, read then [3,4,21]

$$\mathrm{L} := \sum_{w\in X^*} \mathrm{Li}_w w = \prod^{\searrow}_{l\in\mathcal{L}ynX} e^{\mathrm{Li}_{S_l} P_l} \text{ and } \mathrm{H} := \sum_{w\in Y^*} \mathrm{H}_w w = \prod^{\searrow}_{l\in\mathcal{L}ynY} e^{\mathrm{H}_{\Sigma_l} \Pi_l}, \quad (19)$$

[3] The weight of $(s_1,\ldots,s_r) \in \mathbb{N}_+^r$ (resp. $\mathbb{N}^r$) is defined as the integer $s_1+\ldots+s_r$ which corresponds to the weight, denoted (w), of its associated word $w \in Y^*$ (resp. Y_0^*) and also (in the case of Y) to the length, denoted by $|u|$, of its associated word $u \in X^*$.

[4] Note that, all these $\{\mathrm{Li}^-_w\}_{w\in Y_0^*}$ and $\{\mathrm{H}^-_w\}_{w\in Y_0^*}$ diverge at their singularities.

where the PBW basis $\{P_w\}_{w\in X^*}$ (resp. $\{\Pi_w\}_{w\in Y^*}$) is expanded over the basis of $\mathcal{U}(\mathrm{Lie}_A\langle X\rangle)$ (resp. $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\uplus}(Y)))$, $\{P_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\Pi_l\}_{l\in\mathcal{L}ynY}$), and $\{S_w\}_{w\in X^*}$ (resp. $\{\Sigma_w\}_{w\in Y^*}$) is the basis of the shuffle $(\mathbb{Q}\langle Y\rangle,\sqcup\!\sqcup,1_{X^*})$ (resp. the quasi-shuffle $(\mathbb{Q}\langle Y\rangle,\uplus,1_{Y^*})$) containing the transcendence basis $\{S_l\}_{l\in\mathcal{L}ynX}$ (resp. $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$).

By termwise differentiation, L satisfies the noncommutative differential equation (DE) with the boundary condition $\mathrm{L}(z)\widetilde{_{z\to0^+}}e^{x_0\log(z)}$. It is immediate that the power series H and L are group-like, for $\Delta_{\uplus}$ and $\Delta_{\sqcup\!\sqcup}$, respectively. Hence, the following noncommutative generating series are well defined and are group-like, for $\Delta_{\uplus}$ and $\Delta_{\sqcup\!\sqcup}$, respectively [21–23]:

$$Z_{\uplus}:=\prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow}e^{\mathrm{H}_{\Sigma_l}(+\infty)\Pi_l}\text{ and }Z_{\sqcup\!\sqcup}:=\prod_{l\in\mathcal{L}ynX\setminus X}^{\searrow}e^{\mathrm{Li}_{S_l}(1)P_l}. \tag{20}$$

Definitions (5) and (18) lead then to the following <u>*surjective*</u> poly-morphism

$$\zeta:\begin{matrix}(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1,\sqcup\!\sqcup,1_{X^*})\\(\mathbb{Q}1_{Y^*}\oplus(Y-\{y_1\})\mathbb{Q}\langle Y\rangle,\uplus,1_{Y^*})\end{matrix}\twoheadrightarrow(\mathcal{Z},\times,1), \tag{21}$$

$$\begin{matrix}x_0x_1^{r_1-1}\ldots x_0x_1^{r_k-1}\\y_{s_1}\ldots y_{s_k}\end{matrix}\longmapsto\sum_{n_1>\ldots>n_k>0}n_1^{-s_1}\ldots n_k^{-s_k}, \tag{22}$$

where $\mathcal{Z}$ is the $\mathbb{Q}$-algebra generated by $\{\zeta(l)\}_{l\in\mathcal{L}ynX\setminus X}$ (resp. $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX\setminus X}$), or equivalently, generated by $\{\zeta(l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$ (resp. $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY\setminus\{y_1\}}$).

Now, let $t_i\in\mathbb{C},|t_i|<1,i\in\mathbb{N}$. For $z\in\mathbb{C},|z|<1$, we have [18]

$$\sum_{n\ge0}\mathrm{Li}_{x_0^n}(z)\,t_0^n=z^{t_0}\text{ and }\sum_{n\ge0}\mathrm{Li}_{x_1^n}(z)\,t_1^n=\frac{1}{(1-z)^{t_1}}. \tag{23}$$

These suggest to extend the morphism $\mathrm{Li}_\bullet$ over $(\mathrm{Dom}(\mathrm{Li}_\bullet),\sqcup\!\sqcup,1_{X^*})$, via *Lazard's elimination*, as follows (subjected to be convergent):

$$\mathrm{Li}_S(z)=\sum_{n\ge0}\langle S|x_0^n\rangle\frac{\log^n(z)}{n!}+\sum_{k\ge1}\sum_{w\in(x_0^*x_1)^kx_0^*}\langle S|w\rangle\mathrm{Li}_w(z), \tag{24}$$

with $\mathbb{C}\langle X\rangle\sqcup\!\sqcup\mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle\sqcup\!\sqcup\mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle\subset\mathrm{Dom}(\mathrm{Li}_\bullet)\subset\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ denotes the closure, of $\mathbb{C}\langle X\rangle$ in $\mathbb{C}\langle X\rangle X$, by $\{+,.,^*\}$. For example [18,19],

1. For any $x,y\in X$ and for any $i,j\in\mathbb{N}_+,u,v\in\mathbb{C}$ such that $|u|<1$ and $|v|<1$, since

$$(ux+vy)^*=(xx)^*\sqcup\!\sqcup(vy)^*\text{ and }(x^*)^{\sqcup\!\sqcup i}=(ix)^* \tag{25}$$

then

$$\mathrm{Li}_{(x_0^*)^{\sqcup\!\sqcup i}\sqcup\!\sqcup(x_1^*)^{\sqcup\!\sqcup j}}(z)=\frac{z^i}{(1-z)^j}. \tag{26}$$

2. For $a \in \mathbb{C}, x \in X, i \in \mathbb{N}_+$, since

$$(ax)^{*i} = (ax)^* ⧢ (1+ax)^{i-1} \tag{27}$$

then

$$\begin{aligned}\mathrm{Li}_{(ax_0)^{*i}}(z) &= z^a \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(a\log(z))^k}{k!},\\ \mathrm{Li}_{(ax_1)^{*i}}(z) &= \frac{1}{(1-z)^a} \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{(a\log((1-z)^{-1})^k}{k!}.\end{aligned} \tag{28}$$

3. Let $V = (t_1x_0)^{*s_1} x_0^{s_1-1} x_1 \ldots (t_rx_0)^{*s_r} x_0^{s_r-1} x_1$, for $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$. Then

$$\mathrm{Li}_V(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{(n_1-t_1)^{s_1} \ldots (n_r-t_r)^{s_r}}. \tag{29}$$

In particular, for $s_1 = \ldots = s_r = 1$, one has

$$\begin{aligned}\mathrm{Li}_V(z) &= \sum_{n_1, \ldots, n_r > 0} \mathrm{Li}_{x_0^{n_1-1} x_1 \ldots x_0^{n_r-1} x_1}(z)\, t_0^{n_1-1} \ldots t_r^{n_r-1}\\ &= \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{(n_1-t_1) \ldots (n_r-t_r)}.\end{aligned} \tag{30}$$

4. From the previous points, one gets

$$\begin{aligned}\{\mathrm{Li}_S\}_{S \in \mathbb{C}\langle X\rangle ⧢ \mathbb{C}[x_0^*] ⧢ \mathbb{C}[(-x_0^*)] ⧢ \mathbb{C}[x_1^*]} &= \mathrm{span}_{\mathbb{C}} \left\{ \frac{z^a}{(1-z)^b} \mathrm{Li}_w(z) \right\}_{w \in X^*}^{a \in \mathbb{Z}, b \in \mathbb{N}}\\ &\subset \mathrm{span}_{\mathbb{C}} \{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r \in \mathbb{Z}^r} \oplus \mathrm{span}_{\mathbb{C}} \{z^a | a \in \mathbb{Z}\},\end{aligned} \tag{31}$$

$$\begin{aligned}\{\mathrm{Li}_S\}_{S \in \mathbb{C}\langle X\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle ⧢ \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle} &= \mathrm{span}_{\mathbb{C}} \left\{ \frac{z^a}{(1-z)^b} \mathrm{Li}_w(z) \right\}_{w \in X^*}^{a,b \in \mathbb{C}}\\ &\subset \mathrm{span}_{\mathbb{C}} \{\mathrm{Li}_{s_1,\ldots,s_r}\}_{s_1,\ldots,s_r \in \mathbb{C}^r} \oplus \mathrm{span}_{\mathbb{C}} \{z^a | a \in \mathbb{C}\}.\end{aligned} \tag{32}$$

3 Noncommutative Evolution Equations

As was previously said, Drinfel'd proved that (DE) admits two particular solutions on Ω. These new tools and results can be considered as pertaining to the domain of *noncommutative evolution equations*. We will, here, only mention what is relevant for our needs.

Even for one sided [5] differential equations, in order to cope with limit initial conditions (see applications below), one needs the two sided version.

[5] As the left (DE) for instance (see [6]).

Let then $\Omega \subset \mathbb{C}$ be open simply connected and $\mathcal{H}(\Omega)$ denotes the algebra of holomorphic functions on Ω. We suppose we are given two series (called *multipliers*) without constant term $M_1, M_2 \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$ (X is an alphabet and the subscript indicates that the series have no constant term). Let then

$$(DE_2)\ \ \mathbf{d}S = M_1 S + S M_2.$$

be our two sided differential equation. A solution of it is a series $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$ such that (DE_2) is satisfied.

In the sequel, we will use of the following lemma.

Lemma 1. *Let $\mathcal{B}$ be a filter basis on Ω and S a solution of (DE_2) such that $\lim_{\mathcal{B}}\langle S(z)|w\rangle = 0$, for all $w \in X^*$, then $S \equiv 0$.*

Proof. Let us suppose $S \not\equiv 0$ and w be a word of minimal length of $\mathrm{supp}(S)$. Then for this word, one has

$$\frac{d}{dz}\langle S|w\rangle = \langle M_1 S + S M_2|w\rangle = 0,$$

due to the fact that M_i have no constant term. Then, for this word, $z \mapsto \langle S(z)|w\rangle$ is constant on Ω. But, due to the fact that $\lim_{\mathcal{B}}\langle S|w\rangle = 0$, one must have this constant to be zero in contradiction with the reasoning on the support.

3.1 The Main Theorem

The following theorem, although not very difficult to establish once the correct setting has been implemented, is very powerful and new here in its two-sided version.[6]

Theorem 1. *(i) Solutions of (DE_2) form a $\mathbb{C}$-vector space.*

(ii) Solutions of (DE_2) have their constant term (as coefficient of 1_{X^*}) which are constant functions (on Ω); there exist solutions with constant coefficient 1_Ω (hence invertible).

(iii) If two solutions coincide at one point $z_0 \in \overline{\Omega}$, they coincide everywhere.

(iv) Let be the following one-sided equations

$$(DE^{(1)})\quad \mathbf{d}S = M_1 S \ \ \textit{and}\ \ (DE^{(2)})\quad \mathbf{d}S = S M_2,$$

and let $S_i, i = 1, 2$ be a solution of $(DE^{(i)})$. Then $S_1 S_2$ is a solution of (DE_2). Conversely, every solution of (DE_2) can be constructed so.

(v) If $M_i, i = 1, 2$ are primitive and if S, a solution of (DE_2), is group-like at one point, (or, even at one limit point) it is globally group-like.

Proof. (i) Straightforward.

[6] It implies the previous (one-sided) version [6] which was aimed at the linear independence of coordinate functions.

(ii) One can use Lemma 1 or directly remark that the map $S \mapsto \langle S|1_{X^*}\rangle = \epsilon(S)$ is a character (of $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$) which commutes with the derivations, i.e.,

$$\epsilon(\mathbf{d}S) = \frac{d}{dz}\epsilon(S).$$

Hence, as $\epsilon(M_i) = 0$, for every solution of (DE_2), one has $\frac{d}{dz}(\epsilon(S)) = 0$ whence the claim, as Ω is connected.
Now, for each $z_0 \in \Omega$, one can construct the unique solution of (DE_2) such that $S(z_0) = 1_{X^*}$ by the following process (Picard's process)

$$S_0 = 1_{X^*},\ \ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M_1(s)S_n(s) + S_n(s)M_2(s)ds$$

(term by term integration). Due to the fact that $M_i(s), i = 1, 2$ has no constant term, its limit $S_{Pic}^{z_0} := \lim_{n\to\infty} S_n$ exists and is such that $S_{Pic}^{z_0}(z_0) = 1_{X^*}$. Then its constant term is everywhere $1_{\mathbb{C}}$ (i.e. $\langle S_{Pic}^{z_0}|1_{X^*}\rangle = 1_{\Omega}$) and therefore $S_{Pic}^{z_0}$ is invertible in $\mathcal{H}(\Omega)\langle\langle X\rangle\rangle$.

(iii) In fact, the previous reasoning can be carried over for any length (in point "ii" it was for length 0). The claim is an easy consequence of Lemma 1.

(vi) The fact that the product S_1S_2 (for $S_i, i = 1, 2$ solutions of $(DE^{(i)})$) is a solution of (DE_2) is straightforward. Let us now suppose S to be a solution of (DE_2) and set, here for short, $S_2 := S_{Pic}^{z_0}$, the corresponding Picard solution of $(DE^{(2)})$ (notation as above). We now compute with $T := S(S_2)^{-1}$

$$\begin{aligned}\mathbf{d}T = \mathbf{d}(S(S_2)^{-1}) &= \mathbf{d}S(S_2)^{-1} + S\mathbf{d}(S_2)^{-1}\\ &= (M_1S + SM_2) + S(-S_2)^{-1}\mathbf{d}S_2(S_2)^{-1} = M_1T,\end{aligned}$$

which proves the claim (as $S = TS_2$).

(v) One first remarks that the two preceding points hold if (DE_2) is stated for series over any locally finite monoid [15]. Such a monoid M has the property (and in fact is defined by it) that every element $x \in M$ has a finite number of factorizations

$$x = x_1 \ldots x_n \text{ with } x_i \in M \setminus \{1_M\},$$

and the length above is replaced by $l(x) := \sup(n)$ for all factorisations as above[7]. Series over M are just functions $S \in R^M$ (the ring R here is $R = \mathcal{H}(\Omega)$ and $\langle S|m\rangle$ is another notation for the image of m by S), polynomials are finitely supported series $S \in R^{(M)}$ and the canonical pairing *series-polynomials*, $\langle S|P\rangle$ reads

$$\langle S|P\rangle := \sum_{m\in M} \langle S|m\rangle\langle m|P\rangle.$$

[7] For example $l(1_M) = 0$ and $l(x) = 1$ for $x \in M_+ \setminus (M_+)^2$ (with $M_+ = M \setminus \{1_M\}$) the minimal set of generators of M [15]).

Now, we return to the monoid X^* but we will reason on $M = X^* \otimes X^* \simeq X^* \times X^*$ (direct product, thus also locally finite). Let S be a solution of (DE_2) with $M_i, i = 1, 2$ primitive (hence without constant term). One has

$$\begin{aligned}
\mathbf{d}(S \otimes S) &= \mathbf{d}S \otimes S + S \otimes \mathbf{d}S \\
&= (M_1 S + S M_2) \otimes S + S \otimes (M_1 S + S M_2) \\
&= (M_1 \otimes 1 + 1 \otimes M_1)(S \otimes S) + (S \otimes S)(M_2 \otimes 1 + 1 \otimes M_2), \\
\mathbf{d}(\Delta_{\sqcup\!\sqcup}(S)) &= (\Delta_{\sqcup\!\sqcup}(\mathbf{d}S)) = \Delta_{\sqcup\!\sqcup}(M_1 S + S M_2) = \Delta_{\sqcup\!\sqcup}(M_1)\Delta_{\sqcup\!\sqcup}(S) + \Delta_{\sqcup\!\sqcup}(S)\Delta_{\sqcup\!\sqcup}(M_2)
\end{aligned}$$

(again $M = X^* \otimes X^* \subset \mathcal{H}(\Omega)\langle X \rangle \otimes \mathcal{H}(\Omega)\langle X \rangle$, all tensor products are over $\mathcal{H}(\Omega)$). Hence, we see that $S \otimes S$ and $\Delta_{\sqcup\!\sqcup}(S)$ (double series, i.e., series over $X^* \otimes X^*$) satisfy two-sided differential equations with the same multipliers (left$= M_1 \otimes 1 + 1 \otimes M_1 = \Delta_{\sqcup\!\sqcup}(M_1)$ and right$= M_2 \otimes 1 + 1 \otimes M_2 = \Delta_{\sqcup\!\sqcup}(M_2)$), then it suffices that they coincide at one point of $\bar{\Omega}$ in order that $\Delta_{\sqcup\!\sqcup}(S) = S \otimes S$ (the property $\langle S | 1_{X^*} \rangle = 1$ is granted from the fact that S is group-like at one point of $\bar{\Omega}$).

Remark 1. – Every holomorphic series $S(z) \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ which is group-like ($\Delta(S) = S \otimes S$ and $\langle S | 1_{X^*} \rangle$) is a solution of a one-sided dynamics with primitive multiplier (take $M_1 = (\mathbf{d}S)S^{-1}$ and $M_2 = 0$, or $M_2 = S^{-1}(\mathbf{d}S)$ and $M_1 = 0$).
– Invertible solutions of an equation of type $S' = M_1 S$ are on the same orbit by multiplication on the right by *invertible constant series*, i.e., let $S_i, i = 1, 2$ be invertible solutions of $(DE^{(1)})$, then there exists an unique invertible $T \in \mathbb{C}\langle\langle X \rangle\rangle$ such that $S_2 = S_1 T$. From this and point (iv) of the theorem, one can parametrize the set of invertible solutions of (DE_2).

3.2 Application : Unicity of Solutions with Asymptotic Conditions

In a previous work [6], we proved that asymptotic group-likeness, for a series, implies[8] that the series in question is group-like everywhere. The process above (Theorem 1, Picard's process) can still be performed, under certain conditions with improper integrals. We then construct the series L recursively as

$$\langle \mathrm{L}(z) | w \rangle = \begin{cases} \dfrac{\log^n(z)}{n!} & \text{if } w = x_0^n \\ \displaystyle\int_0^z \frac{ds}{1-z} \langle \mathrm{L}(z) | u \rangle & \text{if } w = x_1 u \\ \displaystyle\int_0^z \frac{ds}{z} \langle \mathrm{L}(z) | u x_1 x_0^n \rangle & \text{if } w = x_0 u x_1 x_0^n. \end{cases} \tag{33}$$

[8] Under the condition that the multiplier be primitive, result extended as point (v) of the theorem above.

One can show that (see [6] for details):

- this process is well defined at each step and computes the series L as below;
- L is solution of (DE), is exactly G_0 and is group-like.

We here only prove that G_0 is unique using the theorem above. Consider the series

$$T(z) = \mathrm{L}(z)e^{-x_0 \log(z)}. \tag{34}$$

Then T is solution of an equation of the type (DE_2)

$$T'(z) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)T(z) + T(z)\frac{-x_0}{z}, \tag{35}$$

but

$$\lim_{z \to z_0} G_0(z)e^{-x_0 \log(z)} = 1, \tag{36}$$

so, by Theorem 1, one has

$$G_0(z)e^{-x_0 \log(z)} = \mathrm{L}(z)e^{-x_0 \log(z)} \tag{37}$$

and then[9]

$$G_0 = \mathrm{L}. \tag{38}$$

A similar (and symmetric) argument can be performed for G_1 and then, in this interpretation and context, Φ_{KZ} is unique.

4 Double Global Regularization of Associators

4.1 Global Renormalization by Noncommutative Generating Series

Global singularities analysis leads to to the following global renormalization [3,4]:

$$\begin{aligned} &\lim_{z \to 1} \exp\left(-y_1 \log \frac{1}{1-z}\right)\pi_Y(\mathrm{L}(z)) \\ &\qquad = \lim_{n \to \infty} \exp\left(\sum_{k \geq 1} \mathrm{H}_{y_k}(n)\frac{(-y_1)^k}{k}\right)\mathrm{H}(n) = \pi_Y(Z_{\sqcup\!\sqcup}). \end{aligned} \tag{39}$$

Thus, the coefficients $\{\langle Z_{\sqcup\!\sqcup}|u\rangle\}_{u\in X^*}$ (i.e. $\{\zeta_{\sqcup\!\sqcup}(u)\}_{u\in X^*}$) and $\{\langle Z_{\uplus}|v\rangle\}_{v\in Y^*}$ (i.e. $\{\zeta_{\uplus}(v)\}_{v\in Y^*}$) represent the finite part of the asymptotic expansions, in $\{(1-z)^{-a}\log^b(1-z)\}_{a,b\in\mathbb{N}}$ (resp. $\{n^{-a}\mathrm{H}_1^b(n)\}_{a,b\in\mathbb{N}}$) of $\{\mathrm{Li}_w\}_{u\in X^*}$ (resp. $\{\mathrm{H}_w\}_{v\in Y^*}$). On the other way, by a transfer theorem [17], let $\{\gamma_w\}_{v\in Y^*}$ be the

[9] See also [24].

finite parts of $\{\mathrm{H}_w\}_{v\in Y^*}$, in $\{n^{-a}\log^b(n)\}_{a,b\in\mathbb{N}}$, and let Z_γ be their noncommutative generating series. Hence,

$$\gamma_\bullet : (\mathbb{Q}\langle Y\rangle, \,\uplus\,, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1),\ w \longmapsto \gamma_w, \tag{40}$$

is a character and Z_γ is group-like, for $\Delta_{\uplus}$. Moreover [22,23],

$$Z_\gamma = \exp(\gamma y_1) \prod_{l\in\mathcal{L}ynY\setminus\{y_1\}}^{\searrow} \exp(\zeta(\Sigma_l)\Pi_l) = \exp(\gamma y_1) Z_{\uplus}. \tag{41}$$

The asymptotic behavior leads to the bridge[10] equation [3,4,22,23]

$$Z_\gamma = B(y_1)\pi_Y(Z_{\sqcup\!\sqcup}), \text{ or equivalently } Z_{\sqcup\!\sqcup} = B'(y_1)\pi_Y(Z_{\sqcup\!\sqcup}), \tag{42}$$

where (see [3,4] and [22,23])

$$B(y_1)=\exp\left(\gamma y_1 - \sum_{k\geq 2}(-y_1)^k\frac{\zeta(k)}{k}\right) \text{ and } B'(y_1)=\exp\left(-\sum_{k\geq 2}(-y_1)^k\frac{\zeta(k)}{k}\right). \tag{43}$$

Similarly, there is $C_w^- \in \mathbb{Q}$ and $B_w^- \in \mathbb{N}$, such that [7]

$$\mathrm{H}_w^-(N) \underset{N\to+\infty}{\widetilde{\quad}} N^{(w)+|w|}C_w^- \text{ and } \mathrm{Li}_w^-(z) \underset{z\to 1}{\widetilde{\quad}} (1-z)^{-(w)-|w|}B_w^-. \tag{44}$$

Moreover,

$$C_w^- = \prod_{w=uv, v\neq 1_{Y_0^*}} ((v)+|v|)^{-1} \text{ and } B_w^- = ((w)+|w|)!C_w^-. \tag{45}$$

Now, one can then consider the following noncommutative generating series:

$$\mathrm{L}^- := \sum_{w\in Y_0^*} \mathrm{Li}_w^- w,\ \mathrm{H}^- := \sum_{w\in Y_0^*} \mathrm{H}_w^- w,\ C^- := \sum_{w\in Y_0^*} C_w^- w. \tag{46}$$

Then H^- and C^- are group-like for, respectively, $\Delta_{\uplus}$ and $\Delta_{\sqcup\!\sqcup}$ and [7]

$$\lim_{z\to 1} h^{\odot -1}((1-z)^{-1}) \odot \mathrm{L}^-(z) = \lim_{N\to+\infty} g^{\odot -1}(N) \odot \mathrm{H}^-(N) = C^-, \tag{47}$$

$$h(t) = \sum_{w\in Y_0^*} ((w)+|w|)!t^{(w)+|w|}w \quad \text{and} \quad g(t) = \left(\sum_{y\in Y_0} t^{(y)+1}y\right)^*. \tag{48}$$

4.2 Global Regularization by Noncommutative Generating Series

Next, for any $w \in Y_0^*$, there exists a unique polynomial $p \in (\mathbb{Z}[t], \times, 1)$ of degree $(w)+|w|$ such that [7]

$$\mathrm{Li}_w^-(z) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{(1-z)^k} = \sum_{k=0}^{(w)+|w|} p_k e^{-k\log(1-z)} \in (\mathbb{Z}[(1-z)^{-1}], \times, 1), \tag{49}$$

$$\mathrm{H}_w^-(n) = \sum_{k=0}^{(w)+|w|} p_k \binom{n+k-1}{k-1} = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!}(n)_k \in (\mathbb{Q}[(n)_\bullet], \times, 1), \tag{50}$$

[10] This equation is different from Jean Écalle's one [14].

where[11]

$$(n)_\bullet : \mathbb{N} \longrightarrow \mathbb{Q},\ \ i \longmapsto (n)_i = n(n-1)\ldots(n-i+1). \tag{51}$$

In other terms, for any $w \in Y_0^*, k \in \mathbb{N}, 0 \le k \le (w) + |w|$, one has

$$\langle \mathrm{Li}_w^- | (1-z)^{-k} \rangle = k! \langle \mathrm{H}_w^- | (n)_k \rangle. \tag{52}$$

Hence, denoting $\tilde{p}$ the exponential transform of the polynomial p, one has

$$\mathrm{Li}_w^-(z) = p((1-z)^{-1}) \text{ and } \mathrm{H}_w^-(n) = \tilde{p}((n)_\bullet) \tag{53}$$

with

$$p(t) = \sum_{k=0}^{(w)+|w|} p_k t^k \in (\mathbb{Z}[t], \times, 1) \text{ and } \tilde{p}(t) = \sum_{k=0}^{(w)+|w|} \frac{p_k}{k!} t^k \in (\mathbb{Q}[t], \times, 1). \tag{54}$$

Let us then associate p and $\tilde{p}$ with the polynomial $\check{p}$ obtained as follows:

$$\check{p}(t) = \sum_{k=0}^{(w)+|w|} k! p_k t^k = \sum_{k=0}^{(w)+|w|} p_k t^{\sqcup\!\sqcup k} \in (\mathbb{Z}[t], \sqcup\!\sqcup, 1). \tag{55}$$

Let us recall also that, for any $c \in \mathbb{C}$, one has

$$(n)_c \underset{n \to +\infty}{\widetilde{\ \ \ }} n^c = e^{c \log(n)}$$

and, with the respective scales of comparison, one has the following finite parts

$$\mathrm{f.p.}_{z \to 1} c \log(1-z) = 0,\ \ \{(1-z)^a \log^b((1-z)^{-1})\}_{a \in \mathbb{Z}, b \in \mathbb{N}}, \tag{56}$$

$$\mathrm{f.p.}_{n \to +\infty} c \log n = 0,\ \ \{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}. \tag{57}$$

Hence, using the notations given in (49) and (50), one can see, from (56) and (57), that the values $p(1)$ and $\tilde{p}(1)$ obtained in (54) represent the following finite parts:

$$\mathrm{f.p.}_{z \to 1} \mathrm{Li}_w^-(z) = \mathrm{f.p.}_{z \to 1} \mathrm{Li}_{R_w}(z) = p(1) \in \mathbb{Z}, \tag{58}$$

$$\mathrm{f.p.}_{n \to +\infty} \mathrm{H}_w^-(n) = \mathrm{f.p.}_{n \to +\infty} \mathrm{H}_{\pi_Y(R_w)}(n) = \tilde{p}(1) \in \mathbb{Q}. \tag{59}$$

One can use then these values $p(1)$ and $\tilde{p}(1)$, instead of the values B_w^- and C_w^-, to regularize, respectively, $\zeta_{\sqcup\!\sqcup}(R_w)$ and $\zeta_\gamma(\pi_Y(R_w))$ as showed Theorem 2 below because, essentially, $B_\bullet^-$ and $C_\bullet^-$ do not realize characters for, respectively, $(\mathbb{Q}\langle X \rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \mathsf{e})$ and $(\mathbb{Q}\langle Y \rangle, \uplus, 1_{Y^*}, \Delta_{\uplus}, \mathsf{e})$ [7].

Now, in virtue of the extension of $\mathrm{Li}_\bullet$, defined as in (23) and (24), and of the Taylor coefficients, the previous polynomials $p, \tilde{p}$ and $\check{p}$ given in (54)–(55) can be determined explicitly thanks to

[11] Here, it is also convenient to denote $\mathbb{Q}[(n)_\bullet]$ the set of "polynomials" expanded as follows

$$\forall p \in, \mathbb{Q}[(n)_\bullet],\ p = \sum_{k=0}^{d} p_k (n)_k,\ \deg(p) = d.$$

Proposition 1 **([24]).**

1. *The following morphisms of algebras are* *bijective:*

$$\lambda : (\mathbb{Z}[x_1^*], ш, 1_{X^*}) \longrightarrow (\mathbb{Z}[(1-z)^{-1}], \times, 1),\ R \longmapsto \mathrm{Li}_R,$$
$$\eta : (\mathbb{Q}[y_1^*], ⧢, 1_{Y^*}) \longrightarrow (\mathbb{Q}[(n)_\bullet], \times, 1),\quad S \longmapsto \mathrm{H}_S.$$

2. *For any $w = y_{s_1}, \ldots y_{s_r} \in Y_0^*$, there exists a unique polynomial R_w belonging to $(\mathbb{Z}[x_1^*], ш, 1_{X^*})$ of degree $(w) + |w|$, such that*

$$\mathrm{Li}_{R_w}(z) = \mathrm{Li}_w^-(z) = p((1-z)^{-1}) \in (\mathbb{Z}[(1-z)^{-1}], \times, 1),$$
$$\mathrm{H}_{\pi_Y(R_w)}(n) = \mathrm{H}_w^-(n) = \quad \tilde{p}((n)_\bullet) \quad \in (\mathbb{Q}[(n)_\bullet], \times, 1).$$

In particular, via the extension, by linearity, of $R_\bullet$ over $\mathbb{Q}\langle Y_0\rangle$ and via the linear independent family $\{\mathrm{Li}_{y_k}^-\}_{k\ge0}$ in $\mathbb{Q}\{\mathrm{Li}_w^-\}_{w\in Y_0^*}$, one has

$$\forall k, l \in \mathbb{N},\ \mathrm{Li}_{R_{y_k} ш R_{y_l}} = \mathrm{Li}_{R_{y_k}} \mathrm{Li}_{R_{y_l}} = \mathrm{Li}_{y_k}^- \mathrm{Li}_{y_l}^- = \mathrm{Li}_{y_k \top y_l}^- = \mathrm{Li}_{R_{y_k \top y_l}}.$$

3. *For any w, one has $\check{p}(x_1^*) = R_w$.*
4. *More explicitly, for any $w = y_{s_1}, \ldots y_{s_r} \in Y_0^*$, there exists a unique polynomial R_w belonging to $(\mathbb{Z}[x_1^*], ш, 1_{X^*})$ of degree $(w) + |w|$, given by*

$$R_{y_{s_1}\ldots y_{s_r}} = \sum_{k_1=0}^{s_1} \sum_{k_2=0}^{s_1+s_2-k_1} \cdots \sum_{k_r=0}^{\substack{(s_1+\ldots+s_r)-\\(k_1+\ldots+k_{r-1})}} \binom{s_1}{k_1} \cdots \binom{s_1+\ldots+s_r-k_1-\ldots-k_{r-1}}{k_r} \rho_{k_1} ш \ldots ш \rho_{k_r},$$

where, for any $i = 1, \ldots, r$, if $k_i = 0$ then $\rho_{k_i} = x_1^* - 1_{X^*}$ else, for $k_i > 0$, denoting the Stirling numbers of second kind by $S_2(k, j)$'s, one has

$$\rho_{k_i} = \sum_{j=1}^{k_i} S_2(k_i, j)(j!)^2 \sum_{l=0}^{j} \frac{(-1)^l}{l!} \frac{(x_1^*)^{ш(j-l+1)}}{(j-l)!}.$$

Proposition 2 **([3,4,22,23]).** *With notations of (21), similar to the character $\gamma_\bullet$, the poly-morphism ζ can be extended as follows*

$$\zeta_ш : (\mathbb{Q}\langle X\rangle, ш, 1_{X^*}) \longrightarrow (\mathcal{Z}, \times, 1) \text{ and } \zeta_{⧢} : (\mathbb{Q}\langle Y\rangle, ⧢, 1_{Y^*}) \longrightarrow (\mathcal{Z}, \times, 1),$$

satisfying, for any $\in \mathcal{L}ynY \setminus \{y_1\}$,

$$\zeta_ш(\pi_X(l)) = \zeta_{⧢}(l) = \gamma_l = \zeta(l)$$

and, for the generators of length (resp. weight) one, for X^* (resp. Y^*),

$$\zeta_ш(x_0) = \zeta_ш(x_1) = \zeta_{⧢}(y_1) = 0.$$

Now, to regularize $\{\zeta(s_1, \ldots, s_r)\}_{(s_1,\ldots,s_r)\in\mathbb{C}^r}$, we use

Lemma 2 ([7]).

1. *The family* $\{x_0^*, x_1^*\}$ *is algebraically independent over* $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})$ *within* $(\mathbb{C}\langle\langle X\rangle\rangle, ⧢, 1_{X^*})$.
   In particular, the power series x_0^* *and* x_1^* *are transcendent over* $\mathbb{C}\langle X\rangle$.
2. *The module* $(\mathbb{C}\langle X\rangle, ⧢, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ *is* $\mathbb{C}\langle X\rangle$*-free and a* $\mathbb{C}\langle X\rangle$*-basis of it is given by the family* $\{(x_0^*)^{⧢k} ⧢ (x_1^*)^{⧢l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}$.
   Hence, $\{w ⧢ (x_0^*)^{⧢k} ⧢ (x_1^*)^{⧢l}\}^{(k,l)\in\mathbb{Z}\times\mathbb{N}}_{w\in X^*}$ *is a* $\mathbb{C}$*-basis of it.*
3. *One has, for any* $x_i \in X$, $\mathbb{C}^{\mathrm{rat}}\langle\langle x_i\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(tx_i)^* ⧢ \mathbb{C}\langle x_i\rangle | t \in \mathbb{C}\}$.

Since, for any $t \in \mathbb{C}, |t| < 1$, one has $\mathrm{Li}_{(tx_1)^*}(z) = (1-z)^{-t}$ and [3,4]

$$\mathrm{H}_{\pi_Y(tx_1)^*} = \sum_{k\geq 0} \mathrm{H}_{y_1^k} t^k = \exp\left(-\sum_{k\geq 1} \mathrm{H}_{y_k} \frac{(-t)^k}{k}\right) \tag{60}$$

then, in virtue of Proposition 1, we obtain successively

Proposition 3 ([7]). *The characters* $\zeta_{⧢}$ *and* $\gamma_\bullet$ *can be extended as follows:*

$$\zeta_{⧢} : (\mathbb{C}\langle X\rangle ⧢ \mathbb{C}[x_1^*], ⧢, 1_{X^*}) \longrightarrow (\mathbb{C}, \times, 1_{\mathbb{C}}) \text{ and}$$
$$\gamma_\bullet : (\mathbb{C}\langle Y\rangle \,\text{⧢}\!\!\!\!-\, \mathbb{C}[y_1^*], \,\text{⧢}\!\!\!\!-\,, 1_{Y^*}) \longrightarrow (\mathbb{C}, \times, 1_{\mathbb{C}}),$$

such that, for any $t \in \mathbb{C}$ *such that* $|t| < 1$*, one has*

$$\zeta_{⧢}((tx_1)^*) = 1_{\mathbb{C}} \text{ and } \gamma_{(ty_1)^*} = \exp\left(\gamma t - \sum_{n\geq 2} \zeta(n) \frac{(-t)^n}{n}\right) = \frac{1}{\Gamma(1+t)}.$$

Theorem 2 ([24]).

1. *For any* $(s_1, \ldots, s_r) \in \mathbb{N}_+^r$ *associated with* $w \in Y^*$*, there exists a unique polynomial* $p \in \mathbb{Z}[t]$ *of valuation* 1 *and of degree* $(w) + |w|$ *such that*

$$\begin{aligned}
\check{p}(x_1^*) &= R_w && \in (\mathbb{Z}[x_1^*], ⧢, 1_{X^*}),\\
p((1-z)^{-1}) &= \mathrm{Li}_{R_w}(z) && \in (\mathbb{Z}[(1-z)^{-1}], \times, 1),\\
\tilde{p}((n)_\bullet) &= \mathrm{H}_{\pi_Y(R_w)}(n) && \in (\mathbb{Q}[(n)_\bullet], \times, 1),\\
\zeta_{⧢}(-s_1, \ldots, -s_r) = p(1) &= \zeta_{⧢}(R_w) && \in (\mathbb{Z}, \times, 1),\\
\gamma_{-s_1,\ldots,-s_r} = \tilde{p}(1) &= \gamma_{\pi_Y(R_w)} && \in (\mathbb{Q}, \times, 1).
\end{aligned}$$

2. *Let* $\Upsilon(n) \in \mathbb{Q}[(n)_\bullet]\langle\langle Y\rangle\rangle$ *and* $\Lambda(z) \in \mathbb{Q}[(1-z)^{-1}][\log(z)]\langle\langle X\rangle\rangle$ *be the non-commutative generating series of* $\{\mathrm{H}_{\pi_Y(R_w)}\}_{w\in Y^*}$ *and* $\{\mathrm{Li}_{R_{\pi_Y(w)}}\}_{w\in X^*}$*:*

$$\Upsilon := \sum_{w\in Y^*} \mathrm{H}_{\pi_Y(R_w)} w \text{ and } \Lambda := \sum_{w\in X^*} \mathrm{Li}_{R_{\pi_Y(w)}} w, \text{ with } \langle \Lambda(z) | x_0\rangle = \log(z).$$

Then Υ *and* Λ *are group-like, for respectively* $\Delta_{\text{⧢}\!\!\!\!-}$ *and* $\Delta_{⧢}$*, and:*

$$\Upsilon = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\mathrm{H}_{\pi_Y(R_{\Sigma_l})}\Pi_l} \text{ and } \Lambda = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{R_{\pi_Y(S_l)}}P_l}.$$

3. Let $Z_\gamma^- \in \mathbb{Q}\langle\langle Y\rangle\rangle$ and $Z_ш^- \in \mathbb{Z}\langle\langle X\rangle\rangle$ be the noncommutative generating series of $\{\gamma_{\pi_Y(R_w)}\}_{w\in Y^*}$ and[12] $\{\zeta_ш(R_{\pi_Y(w)})\}_{w\in X^*}$, respectively:

$$Z_\gamma^- := \sum_{w\in Y^*} \gamma_{\pi_Y(R_w)} w \text{ and } Z_ш^- := \sum_{w\in X^*} \zeta_ш(R_{\pi_Y(w)}) w.$$

Then Z_γ^- and $Z_ш^-$ are group-like, for respectively $\Delta_{\uplus}$ and $\Delta_ш$, and:

$$Z_\gamma^- = \prod_{l\in\mathcal{L}ynY}^{\searrow} e^{\gamma_{\pi_Y(R_{\Sigma_l})}\Pi_l} \text{ and } Z_ш^- = \prod_{l\in\mathcal{L}ynX}^{\searrow} e^{\zeta_ш(\pi_Y(S_l))P_l}.$$

Moreover,

$$\mathrm{F.P.}_{n\to+\infty}\Upsilon(n) = Z_\gamma^- \text{ and } \mathrm{F.P.}_{z\to1}\Lambda(z) = Z_ш^-, \tag{61}$$

meaning that, for any $v \in Y^*$ and $u \in X^*$, one has

$$\mathrm{f.p.}_{n\to+\infty}\langle\Upsilon(n)|v\rangle = \langle Z_\gamma^-|v\rangle \text{ and } \mathrm{f.p.}_{z\to1}\langle\Lambda(z)|u\rangle = \langle Z_ш^-|u\rangle. \tag{62}$$

To end this section, let us recall that the function Γ is meromorphic, admits no zeroes and simple poles in $-\mathbb{N}$. Hence, Γ^{-1} is entire and admits simple zeros in $-\mathbb{N}$.

Moreover, using the incomplete beta function, i.e., for $z, a, b \in \mathbb{C}$ such that $|z| < 1, \Re a > 0, \Re b > 0$,

$$\begin{aligned}\mathrm{B}(z; a, b) &:= \int_0^z dt\, t^{a-1}(1-t)^{b-1}\\ &= \mathrm{Li}_{x_0[(ax_0)^*ш((1-b)x_1)^*]}(z)\\ &= \mathrm{Li}_{x_1[((a-1)x_0)^*ш(-bx_1)^*]}(z),\end{aligned} \tag{63}$$

and setting

$$\begin{aligned}\mathrm{B}(a, b) &:= \mathrm{B}(1; a, b)\\ &= \zeta_ш(x_0[(ax_0)^*ш((1-b)x_1)^*])\\ &= \zeta_ш(x_1[((a-1)x_0)^*ш(-bx_1)^*]).\end{aligned} \tag{64}$$

we have, on the one hand, the following Euler's formula

$$\mathrm{B}(a, b)\Gamma(a + b) = \Gamma(a)\Gamma(b) \tag{65}$$

[12] On the one hand, by Proposition 2, one has $\langle Z_ш^-|x_0\rangle = \zeta_ш(x_0) = 0$.
On the other hand, since $R_{y_1} = (2x_1)^* - x_1^*$ then $\mathrm{Li}_{R_{y_1}}(z) = (1-z)^{-2} - (1-z)^{-1}$ and $\mathrm{H}_{\pi_Y(R_{y_1})}(n) = \binom{n}{2} - \binom{n}{1}$. Hence, one also has $\langle Z_ш^-|x_1\rangle = \zeta_ш(R_{\pi_Y(y_1)}) = 0$ and $\langle Z_\gamma^-|x_1\rangle = \gamma_{\pi_Y(R_{y_1})} = -1/2$.

and, on the other hand[13], in virtue of Proposition 3,

$$\exp\Bigl(\sum_{n\geq 2}\zeta(n)\frac{(u+v)^n-(u^n+v^n)}{n}\Bigr)=\frac{\Gamma(1-u)\Gamma(1-v)}{\Gamma(1-u-v)} \tag{66}$$
$$=\frac{\gamma_{(-(u+v)y_1)^*}}{\gamma_{(-uy_1)^*}\gamma_{(-vy_1)^*}}=\frac{\gamma_{(-(u+v)y_1)^*}}{\gamma_{(-uy_1)^*\,\uplus\,(-vy_1)^*}}.$$

Hence, it follows that

Corollary 1 ([24]). *For any $u,v\in\mathbb{C}$ such that $|u|<1,|v|<1$ and $|u+v|<1$, one has*

$$\begin{aligned}\gamma_{(-(u+v)y_1)^*}&=\gamma_{(-uy_1)^*\,\uplus\,(-vy_1)^*}\zeta_{\sqcup\!\sqcup}(x_0[(-ux_0)^*\sqcup\!\sqcup(-(1+v)x_1)^*])\\&=\gamma_{(-uy_1)^*\,\uplus\,(-vy_1)^*}\zeta_{\sqcup\!\sqcup}(x_1[(-(1+u)x_0)^*\sqcup\!\sqcup(-vx_1)^*]).\end{aligned}$$

Remark 2 By (25), for any $u,v\in\mathbb{C}$ such that $|u|<1,|v|<1$ and $|u+v|<1$, one also has

$$\zeta_{\sqcup\!\sqcup}((-(u+v)x_1)^*)=\zeta_{\sqcup\!\sqcup}((-ux_1)^*\sqcup\!\sqcup(-vx_1)^*)=\zeta_{\sqcup\!\sqcup}((-ux_1)^*)\zeta_{\sqcup\!\sqcup}((-vx_1)^*)=1.$$

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[13] The first equality of (66) is already presented in [13].
Since $(-uy_1)^*\uplus(uy_1)^*=(-u^2y_2)^*$ then, letting $v=-u$ in (66), we have

$$\exp\Bigl(-\sum_{n\geq 1}\zeta(2n)\frac{u^{2n}}{n}\Bigr)=\Gamma(1-u)\Gamma(1+u)=\frac{1}{\gamma_{(-uy_1)^*\,\uplus\,(uy_1)^*}}=\frac{1}{\gamma_{(-u^2y_2)^*}}.$$

It is also a consequence obtained by expanding identities like (60) [3,4]

$$\forall y_r\in Y,\ \ y_r^k=\frac{(-1)^k}{k!}\sum_{\substack{s_1,\ldots,s_k>0\\ s_1+\ldots+ks_k=k}}\frac{(-y_r)^{\uplus s_1}}{1^{s_1}}\uplus\ldots\uplus\frac{(-y_{kr})^{\uplus s_k}}{k^{s_k}}.$$

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