

The mechanics of shuffle products and their siblings

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ABSTRACT

Nous poursuivons ici le travail commenc e dans (Enjalbert and Hoang Ngoc Minh, 2012) en d ecrivant des produits de m elanges d'alg ebres de plus "grandes" de fonctions sp eciales (issues d' equations diff erentielles  a p oles simples). Les  tudier nous conduit   d efinir une classe de produits de m elange, que nous nommons φ -shuffles. Nous  tudions cette classe d'un point de vue combinatoire, en commen ant par  tendre (sous conditions) le th eor eme de Radford   celle-ci, puis en construisant (toujours sous conditions) sa big ebre. Nous analysons les conditions des r esultats pr ecit es pour les simplifier en les rendant visible d es la d efinition du produit de m elange. Nous testons enfin ces conditions sur les produits introduits en d ebut d'article.

We carry on the investigation initiated in Enjalbert and Hoang Ngoc Minh (2012): we describe new shuffle products coming from some special functions and group them, along with other products encountered in the literature, in larger and larger classes of products, which we name φ -shuffle products. Our paper is dedicated to a study of the latter class, from a combinatorial standpoint. We consider first how to extend Radford's theorem to the products in that class, then how to construct their bi-algebras. As some conditions are necessary to carry that out, we study them closely and simplify them so that they can be seen directly from the definition of the product. We eventually test these conditions on the products mentioned above.

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1. Introduction

As a matter of fact, mathematics (in particular number theory), physics and other sciences provide, for their theories, algebras of special functions indexed by parameters,¹ with a product, defined at first as a function $X^* \times X^* \rightarrow A(X)$ and satisfying a simple recurrence of the type

$$\forall(a, b) \in X^2, \forall(u, v) \in (X^*)^2, \quad au \sqcup_{\varphi} vb = a(u \sqcup_{\varphi} bv) + b(au \sqcup_{\varphi} v) + \varphi(a, b)(u \sqcup_{\varphi} v), \quad (1)$$

the initialization being provided by the fact that 1_{X^*} should be a unit. Of course, we will address the question of the existence of such a product, and will extend it by linearity to $A(X)$.

However, recall that these special functions are indexed by parameters but, unfortunately, sometimes do not exist for some of their values: the prototype of this case is the Riemann zeta function $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ for $s = 1$. Nevertheless, if these

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¹ The combinatorial supports of these parameters will finally resolve themselves into words.

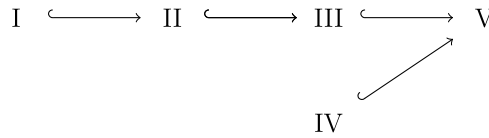


Fig. 1. Hasse diagram of the inclusions between classes.

Table 1

Name	Formula (recursion)	φ	Type
Shuffle [21]	$au \sqcup bv = a(u \sqcup bv) + b(au \sqcup v)$	$\varphi \equiv 0$	I
Stuffle [18]	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v)$ $+ x_{i+j}(u \sqcup v)$	$\varphi(x_i, x_j) = x_{i+j}$	I
Min-stuffle [5]	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v)$ $- x_{i+j}(u \sqcup v)$	$\varphi(x_i, x_j) = -x_{i+j}$	III
Muffle [12]	$x_i u \sqcup x_j v = x_i(u \sqcup x_j v) + x_j(x_i u \sqcup v)$ $+ x_{i \times j}(u \sqcup v)$	$\varphi(x_i, x_j) = x_{i \times j}$	I
q -shuffle [2]	$x_i u \sqcup_q x_j v = x_i(u \sqcup_q x_j v) + x_j(x_i u \sqcup_q v)$ $+ q x_{i+j}(u \sqcup_q v)$	$\varphi(x_i, x_j) = q x_{i+j}$	III
q -shuffle ₂	$x_i u \sqcup_q x_j v = x_i(u \sqcup_q x_j v) + x_j(x_i u \sqcup_q v)$ $+ q^{ij} x_{i+j}(u \sqcup_q v)$	$\varphi(x_i, x_j) = q^{ij} x_{i+j}$	II
LDIAG(1, q_s)			
[9] (non-crossed, non-shifted)	$au \sqcup bv = a(u \sqcup bv) + b(au \sqcup v)$ $+ q_s^{a b} a.b(u \sqcup v)$	$\varphi(a, b) = q_s^{a b}(a.b)$	II
q -Infiltration [8]	$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v)$ $+ q \delta_{a,b} a(u \uparrow v)$	$\varphi(a, b) = q \delta_{a,b} a$	III
AC-stuffle	$au \sqcup_\varphi bv = a(u \sqcup_\varphi bv) + b(au \sqcup_\varphi v)$ $+ \varphi(a, b)(u \sqcup_\varphi v)$	$\varphi(a, b) = \varphi(b, a)$ $\varphi(\varphi(a, b), c) = \varphi(a, \varphi(b, c))$	IV
Semigroup-stuffle	$x_r u \sqcup_\perp x_s v = x_r(u \sqcup_\perp x_s v) + x_s(x_r u \sqcup_\perp v)$ $+ x_{r \perp s}(u \sqcup_\perp v)$	$\varphi(x_r, x_s) = x_{r \perp s}$	I
φ -shuffle	$au \sqcup_\varphi bv = a(u \sqcup_\varphi bv) + b(au \sqcup_\varphi v)$ $+ \varphi(a, b)(u \sqcup_\varphi v)$	$\varphi(a, b)$ law of AAU	V

“functions” are seen formally, one can in many cases,² define a product on the indices which governs the effective product on the functions.³

Once the formal identity is obtained, there are many ways to write the divergent quantities as limits of terms which fulfil the same identities (truncated or power series).⁴ [19].

Returning to this family of products, we will use a typology based on examples frequently encountered in the literature as well as new ones that we supply in Section 2.

1. Type I: factor φ comes from a product (possibly with zero) between letters (i.e. $X \cup \{0\}$ is a semigroup).
2. Type II: factor φ comes from the deformation of a semigroup product by a bicharacter.
3. Type III: factor φ comes from the deformation of a semigroup product by a colour factor.
4. Type IV: factor φ is the commutative law of an associative algebra (CAA) on $A.X$
5. Type V: factor φ is the law of an associative algebra (AA) on $A.X$

These classes are ordered by the following (strict) inclusion diagram: See Fig. 1

We have collected examples from the literature, with the corresponding formulas, in Table 1.

Of course, the q -shuffle is the (classical) shuffle [3,21] when $q = 0$. As for the q -infiltration, when $q = 1$, one recovers the infiltration product defined in [4].

Many shuffle products arise in number theory when one studies polylogarithms, harmonic sums and polyzêtas: it was in order to study all these products that two of us introduced Type IV (see above) [10].

On the other hand, in combinatorial physics, one has coproducts with bi-multiplicative (and noncommutative) perturbation factors (see [7]).

The structure of the paper is the following: in part 2, we complete the first products of [10] with the description of products which come from Hurwitz polyzêta functions (the product given in [15] was not valid in all cases) and from generalized Polylerch functions. We are able to give the complete recursive relation which allows to define all kinds of products; we verify that it implies the existence and uniqueness of this product, which can be extended to $A(X)$. We examine the “known” and the “new” products in order to determine their classes. In part 3, we consider how to extend Radford’s theorem, which

² That includes in particular all the cases under consideration in our paper

³ That is the domain of symbolic computation in the vein of Euler and Arbogast [13,19].

⁴ That is the domain of renormalization and asymptotic analysis initiated by Du Bois-Reymond and Hardy [6,14].

we can prove in all cases (product \sqcup_φ commutative or not commutative, only associativity is required); moreover, we show that, if the φ -shuffle product is commutative, the Lyndon words constitute a pure transcendence basis of the corresponding commutative algebra (see Corollary 1). In all cases this algebra, endowed with deconcatenation coproduct, admits a Hopf algebra structure. The basis of Lyndon words is the key to effective computations on the algebra of special functions ruled by such products⁵ In part 4, we determine the necessary and sufficient conditions on φ so that \sqcup_φ belong to the class of AC-products; we give also necessary and sufficient conditions for such a product to be dualizable (i.e. to be the adjoint of a comultiplication).

Preliminary remark. It is worth emphasizing at the outset that, although some of the objects/results under review in the present paper have already been defined/proved elsewhere, we include them in our study to lay out as complete a picture as possible and to exemplify the rather ‘pedestrian’ approach we have adopted. In particular, we have refrained throughout the paper from using more sophisticated algebraic techniques.

Notation. In the sequel, X will denote an alphabet, and, except in Section 3, $\mathbb{Q} \subset K \subset A$ is a chain of commutative and associative algebras with unit (\mathbb{Q} -CAAU).

2. Hurwitz Polyzêtas and generalized polylerch functions

We remind the reader of some special functions introduced in [12] and complete their study: we prove that they follow a product law which we describe.

2.1. Some special functions and their products

The Riemann Polyzêta is the function which maps every composition $\mathbf{s} = (s_1, \dots, s_r) \in (\mathbb{N}_{\geq 1})^r$, to⁶

$$\zeta(\mathbf{s}) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}. \tag{2}$$

We now make an observation which, however simple, will appear in different disguises as a building block of many a construction of the paper: There is a (linear) bijection between the module freely generated by (all) compositions and $A\langle Y \rangle$ (where $Y = \{y_k\}_{k \geq 1}$) defined by

$$\beta_s : (s_1, \dots, s_r) \mapsto y_{s_1} \dots y_{s_r}. \tag{3}$$

So, if $\mathbf{s} = (s_1, \dots, s_r) \in (\mathbb{N}_{\geq 1})^r$, $s_1 > 1$ and $\mathbf{s}' = (s'_1, \dots, s'_r)$, $s'_1 > 1$ are compositions, one knows [12] that⁷

$$\zeta(\mathbf{s} \sqcup \mathbf{s}') = \zeta(\mathbf{s})\zeta(\mathbf{s}'). \tag{4}$$

That function ζ is well-known and is a special case of the following special functions.

2.1.1. Coloured Polyzêtas

The coloured polyzêta is the function which, to a composition $\mathbf{s} = (s_1, \dots, s_r)$ and a tuple of complex numbers of the same length $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$, associates

$$\zeta(\mathbf{s}, \boldsymbol{\xi}) = \sum_{n_1 > \dots > n_r > 0} \frac{\xi_1^{n_1} \dots \xi_r^{n_r}}{n_1^{s_1} \dots n_r^{s_r}}. \tag{5}$$

It should be noted that $\zeta(\mathbf{s}, \boldsymbol{\xi})$ appears –with the notation $\text{Li}_s(\boldsymbol{\xi})$ – in particle physics [23].

To describe the product here, we will use two alphabets $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$ and M be the (free) submonoid generated by $Y \times X$. One easily checks that⁸

$$M = \{(u, v) \in Y^* \times X^* \mid |u| = |v|\}.$$

As above, to make things rigorous (but slightly more difficult to read), one considers the (linear) bijection defined, on M , by

$$\beta_c : ((s_1, \dots, s_r), (\xi_1, \dots, \xi_r)) \mapsto (y_{s_1} \dots y_{s_r}, x_{\xi_1} \dots x_{\xi_r}).$$

The duffle product is defined as follows.

⁵ The decomposition algorithm (which we shall not describe in detail) is based on formula (36) of lemma 4.

⁶ The following series converges for $s_1 > 1$. Under that condition, the definition can be extended by linearity to the module generated by the set of so-called *admissible* compositions.

⁷ With a slight abuse of language. Strictly speaking, Eq. (4) actually reads

$$\zeta(\beta_s^{-1}(\beta_s(\mathbf{s}) \sqcup \beta_s(\mathbf{s}'))) = \zeta(\mathbf{s})\zeta(\mathbf{s}').$$

⁸ Throughout the paper $|w|$ stands for the length of the word w .

Definition 1 ([10]). (Product of coloured polyzêtas) Let $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$ and M be as above. The duffle is defined as a bilinear product over $A[M] = A\langle Y \times X \rangle$ such that

$$\begin{aligned} \forall w \in M^*, \quad w \sqcup 1_{M^*} = 1_{M^*} \sqcup w = w, \\ \forall y_i, y_j \in Y^2, \forall x_k, x_l \in X^2, \forall u, v \in M^{*2}, \quad (y_i, x_k).u \sqcup (y_j, x_l).v = (y_i, x_k)(u \sqcup (y_j, x_l)v) \\ + (y_j, x_l)((y_i, x_k)u \sqcup v) + (y_{i+j}, x_{k \times l})(u \sqcup v). \end{aligned}$$

Again, we will show that, under suitable conditions⁹

$$\zeta((\mathbf{s}, \boldsymbol{\xi}) \sqcup (\mathbf{s}', \boldsymbol{\xi}')) = \zeta(\mathbf{s}, \boldsymbol{\xi}) \zeta(\mathbf{s}', \boldsymbol{\xi}'). \tag{6}$$

2.1.2. Hurwitz Polyzetâs

The Hurwitz polyzêta is the function which, to a composition $\mathbf{s} = (s_1, \dots, s_r)$ and a tuple of parameters¹⁰ of the same length $\mathbf{t} = (t_1, \dots, t_r)$, associates

$$\zeta(\mathbf{s}, \mathbf{t}) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}}. \tag{7}$$

This series converges if and only if $s_1 > 1$ (for a “global” way to expand (7) as a meromorphic function of $\mathbf{s} \in \mathbb{C}^r$, see [11]). To be able to cope with the case $s_1 = 1$, we have to use the truncated Hurwitz polyzêtas function given by:

$$\forall N \in \mathbb{N}_{>0}, \quad \zeta_N(\mathbf{s}, \mathbf{t}) = \sum_{N \geq n_1 > \dots > n_N > 0} \frac{1}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}}. \tag{8}$$

In order to obtain the product law, we will use here two alphabets $Y = \{y_i\}_{i \in \mathbb{N}_{>0}}$, $Z = \{z_t\}_{t \in K}$, the (free) submonoid N generated by $Y \times Z$ and, as usual, the bijection

$$\beta_h : ((s_1, \dots, s_r), (t_1, \dots, t_r)) \mapsto (y_{s_1} \dots y_{s_r}, z_{t_1} \dots z_{t_r}) \tag{9}$$

suitably extended by linearity. We have now the following product

Definition 2. (Product of formal Hurwitz polyzêtas) Let $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $Z = \{z_t\}_{t \in K}$ and N be as above. The huffle is defined as a bilinear product over $A[N] = A\langle Y \times Z \rangle$ such that

$$\begin{aligned} \forall w \in N^*, \quad w \sqcup 1_{N^*} = 1_{N^*} \sqcup w = w, \\ \forall y_i, y_j \in Y^2, \forall z_t, z_{t'} \in Z^2, \forall u, v \in N^{*2}, \\ t = t' \Rightarrow (y_i, z_t)u \sqcup (y_j, z_t)v \\ = (y_i, z_t)(u \sqcup (y_j, z_t)v) + (y_j, z_t)((y_i, z_t)u \sqcup v) \\ + (y_{i+j}, z_t)(u \sqcup v) \\ t \neq t' \Rightarrow (y_i, z_t).u \sqcup (y_j, z_{t'}).v \\ = (y_i, z_t).(u \sqcup (y_j, z_{t'}).v) + (y_j, z_{t'}).((y_i, z_t).u \sqcup v) \\ + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t).(u \sqcup v) \\ + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'}).(u \sqcup v). \end{aligned}$$

We also will show that¹¹ for all integer N

$$\zeta_N((\mathbf{s}, \mathbf{t}) \sqcup (\mathbf{s}', \mathbf{t}')) = \zeta_N(\mathbf{s}, \mathbf{t}) \zeta_N(\mathbf{s}', \mathbf{t}'). \tag{10}$$

⁹ Again, rigorously speaking, the left-hand side of the following equation should read

$$\zeta(\beta_c^{-1}(\beta_c(\mathbf{s}, \boldsymbol{\xi}) \sqcup \beta_c(\mathbf{s}', \boldsymbol{\xi}'))).$$

¹⁰ All parameters in the tuple are taken in some subring K of \mathbb{C} and none of them is a strictly positive integer.

¹¹ Again, rigorously speaking, the left-hand side of the following equation should read

$$\zeta_N(\beta_h^{-1}(\beta_h(\mathbf{s}, \mathbf{t}) \sqcup \beta_h(\mathbf{s}', \mathbf{t}'))).$$

Remark 1. The functions we call ‘Hurwitz polyzêtas’, a term coined in the last century (see for example [15]), must not be confused with the monocenter polyzêtas, defined only for a composition \mathbf{s} and a parameter t by

$$\zeta(\mathbf{s}, t) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{(n_1 - t)^{s_1} \dots (n_r - t)^{s_r}}, \tag{11}$$

which follow a much simpler rule, namely the stuffle product on the compositions.

2.1.3. Generalized Polylerch functions

The generalized Polylerch function is the function which maps a composition $\mathbf{s} = (s_1, \dots, s_r)$, a tuple $\xi = (\xi_1, \dots, \xi_r)$ of complex numbers, and a tuple $\mathbf{t} = (t_1, \dots, t_r)$ of parameters (see footnote 10), all three of the same length, to

$$\zeta(\mathbf{s}, \mathbf{t}, \xi) = \sum_{n_1 > \dots > n_r > 0} \frac{\xi_1^{n_1} \dots \xi_r^{n_r}}{(n_1 - t_1)^{s_1} \dots (n_r - t_r)^{s_r}}. \tag{12}$$

Here, we will need three alphabets $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$, $Z = \{z_t\}_{t \in K}$ and the (free) submonoid T generated by $Y \times Z \times X$. The bijection

$$\beta_l : ((s_1, \dots, s_r), (t_1, \dots, t_r), (\xi_1, \dots, \xi_r)) \mapsto (y_{s_1} \dots y_{s_r}, z_{t_1} \dots z_{t_r}, x_{\xi_1} \dots x_{\xi_r}) \tag{13}$$

still extended by linearity. The product \boxplus is given by the following definition:

Definition 3. (Product of Generalized Polylerch functions)

Let $Y = \{y_i\}_{i \in \mathbb{N}^*}$, $X = \{x_i\}_{i \in \mathbb{C}^*}$, $Z = \{z_t\}_{t \in K}$ and T be the (free) submonoid generated by $Y \times Z \times X$.

The luffle is defined as the bilinear product over $A[T] = A(Y \times Z \times X)$ satisfying the following recursive relation:

$$\begin{aligned} \forall w \in A^*, \quad & w \boxplus 1_{A^*} = 1_{A^*} \boxplus w = w, \\ \forall (y_i, y_j) \in Y^2, \forall (z_t, z_{t'}) \in Z^2, \forall (x_k, x_l) \in X^2, \forall (u, v) \in A^{*2}, \\ t = t' \Rightarrow & (y_i, z_t, x_k).u \boxplus (y_j, z_t, x_l).v \\ & = (y_i, z_t, x_k). (u \boxplus (y_j, z_t).v) + (y_j, z_t, x_l). ((y_i, z_t).u \boxplus v) \\ & \quad + (y_{i+j}, z_t, x_{k \times l}). (u \boxplus v) \\ t \neq t' \Rightarrow & (y_i, z_t, x_k).u \boxplus (y_j, z_{t'}, x_l).v \\ & = (y_i, z_t, x_k). (u \boxplus (y_j, z_{t'}).v) + (y_j, z_{t'}, x_l). ((y_i, z_t).u \boxplus v) \\ & \quad + \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t, x_{k \times l}). (u \boxplus v) \\ & \quad + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'}, x_{k \times l}). (u \boxplus v). \end{aligned}$$

We also show¹²

$$\zeta((\mathbf{s}, \mathbf{t}, \xi) \boxplus (\mathbf{s}', \mathbf{t}', \xi')) = \zeta(\mathbf{s}, \mathbf{t}, \xi) \zeta(\mathbf{s}', \mathbf{t}', \xi'). \tag{15}$$

2.2. General framework of study

Other products from Table 1 belong to the same family as the examples examined so far, and so pertain to the same kind of approach. As we aim to offer as comprehensive a framework as possible, we now concentrate on the most general class of φ -products, i.e. class V, which emerges from definition 4. We will use a unitary ring as the ground set of scalars (and not a field as it would be expected in combinatorics) because some applications require to work with rings of (analytic or arithmetic) functions.

Proposition 1. Let A be a unitary commutative ring, X be an alphabet and $\varphi : X \times X \rightarrow A(X)$ is an arbitrary mapping. Then there exists a unique mapping $\star : X^* \times X^* \rightarrow A(X)$ satisfying the conditions:

$$(R) \begin{cases} \text{for any } w \in X^*, 1_{X^*} \star w = w \star 1_{X^*} = w, \\ \text{for any } a, b \in X \text{ and } u, v \in X^*, \\ \quad au \star bv = a(u \star bv) + b(au \star v) + \varphi(a, b)(u \star v). \end{cases} \tag{16}$$

¹² Again, rigorously speaking, the left-hand side of Eq. (15) should read

$$\zeta(\beta_l^{-1}(\beta_l(\mathbf{s}, \mathbf{t}, \xi) \boxplus \beta_l(\mathbf{s}', \mathbf{t}', \xi'))). \tag{14}$$

Proof. By recurrence over $n = |u| + |v|$. \square

Definition 4. With the notations of Proposition 1, the unique mapping from $X \times X$ to $A(X)$ satisfying conditions (R) will be denoted \sqcup_φ and will be called φ -shuffle product.

From now on, we suppose that φ takes its values in AX , the space of homogeneous polynomials of degree 1. We still denote by φ its linear extension to $AX \otimes AX$ given by

$$\varphi(P, Q) = \sum_{x,y \in X} \langle P|x \rangle \langle Q|y \rangle \varphi(x, y) \tag{17}$$

and \sqcup_φ the extension of the mapping of Definition 4 by linearity [1]¹³ to $A(X) \otimes A(X)$. Then \sqcup_φ becomes a law of algebra (with 1_{X^*} as unit) on $A(X)$.

2.3. Extending quasi-stuffle relations

The following elementary result can be found in any complex analysis textbook. It is freely used throughout this section.

Lemma 1. For any integers $s, r \geq 1$, for any complex numbers $a, b \neq a$:

$$\forall x \in \mathbb{C} \setminus \{a, b\}, \frac{1}{(x-a)^s(x-b)^r} = \sum_{k=1}^s \frac{a_k}{(x-a)^k} + \sum_{k=1}^r \frac{b_k}{(x-b)^k} \tag{18}$$

where, for all $k \in \{1, \dots, s\}$, $a_k = \binom{s+r-k-1}{r-1} \frac{(-1)^{s-k}}{(a-b)^{s+r-k}}$ and, for all $k \in \{1, \dots, r\}$, $b_k = \binom{s+r-k-1}{s-1} \frac{(-1)^{r-k}}{(b-a)^{s+r-k}}$.

Let $\mathbf{t} = (t_1, \dots, t_r)$ be a set of parameters (see footnote 10), $\mathbf{s} = (s_1, \dots, s_r)$ a composition, and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in \mathbb{C}^r$. We define, for $N \in \mathbb{N}_{>0}$,

$$M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N = \sum_{N \geq n_1 > \dots > n_r > 0} \prod_{i=1}^r \frac{\xi_i^{n_i}}{(n_i - t_i)^{s_i}} \tag{19}$$

and $M_{(0, \dots, 0)}^N = 1$.

Of course, it is a truncated series of $\zeta(\mathbf{s}; \mathbf{t}; \boldsymbol{\xi})$.

Proposition 2. For every composition \mathbf{s} , tuple $\boldsymbol{\xi}$ of complex numbers, tuple \mathbf{t} of parameters all of the same length $l \in \mathbb{N}$, and for every composition \mathbf{r} , tuple $\boldsymbol{\rho}$ of complex numbers, tuple \mathbf{t}' of parameters also of the same length $k \in \mathbb{N}$, one has

$$\forall N \in \mathbb{N}, M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \sqcup (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N \tag{20}$$

Proof. If $l = 0$ or $k = 0$, that is immediate.

Let $l \in \mathbb{N}^*, k \in \mathbb{N}^*$ and $\mathbf{s} = (s_1, \dots, s_l)$ and $\mathbf{r} = (r_1, \dots, r_k)$ two compositions, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l) \in \mathbb{C}^l$, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_k) \in \mathbb{C}^k$, and $\mathbf{t} = (t_1, \dots, t_l)$, $\mathbf{t}' = (t'_1, \dots, t'_k)$ two sets of parameters and put $\mathbf{s}_2 = (s_2, \dots, s_l)$, $\mathbf{r}_2 = (r_2, \dots, r_k)$, $\boldsymbol{\xi}_2 = (\xi_2, \dots, \xi_l)$, $\boldsymbol{\rho}_2 = (\rho_2, \dots, \rho_k)$, $\mathbf{t}_2 = (t_2, \dots, t_l)$ and $\mathbf{t}'_2 = (t'_2, \dots, t'_k)$,

- If $t'_1 = t_1$,

$$M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N = \sum_{N \geq n_1, N \geq n'_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}', \boldsymbol{\xi}', \mathbf{t}_2}^{n_1} \frac{\rho_1^{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{\mathbf{r}', \boldsymbol{\rho}', \mathbf{t}'_2}^{n'_1} \tag{21}$$

Classically, we decompose the sum $\sum_{N \geq n_1, N \geq n'_1}$ into three sums corresponding to the simplices $n_1 > n'_1$; $n'_1 > n_1$ and $n_1 = n'_1$ and get

$$\begin{aligned} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N &= \sum_{N \geq n_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^{n_1} M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^{n_1} \\ &+ \sum_{N \geq n'_1} \frac{\rho_1^{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^{n'_1} M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}'_2}^{n'_1} \\ &+ \sum_{N \geq m} \frac{(\xi_1 \rho_1)^m}{(m - t_1)^{s_1}} \frac{1}{(m - t'_1)^{r_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^m M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}'_2}^m \end{aligned} \tag{22}$$

¹³ We recall that, all tensor products being supposed over A , AX (resp. $A(X)$) admits X (resp. X^*) as linear basis, therefore $AX \otimes AX$ (resp. $A(X) \otimes A(X)$) is free with basis $X \times X$ (resp. $X^* \times X^*$) or more precisely, the image family $(x \otimes y)_{x,y \in X}$ (resp. $(u \otimes v)_{u,v \in X^*}$).

so that,

$$\forall N \in \mathbb{N}, \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \sqcup (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N. \tag{23}$$

- In the same way, when $t_1 \neq t'_1$

$$\begin{aligned} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N &= \sum_{N \geq n_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^{n_1} M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^{n_1} \\ &+ \sum_{N \geq n'_1} \frac{\rho_1^{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^{n'_1} M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}'_2}^{n'_1} \\ &+ \sum_{N \geq m} (\xi_1 \rho_1)^m \frac{1}{(m - t_1)^{s_1}} \frac{1}{(m - t'_1)^{r_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^m M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}'_2}^m \\ &= \sum_{N \geq n_1} \frac{\xi_1^{n_1}}{(n_1 - t_1)^{s_1}} M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^{n_1} M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^{n_1} \\ &+ \sum_{N \geq n'_1} \frac{\rho_1^{n'_1}}{(n'_1 - t'_1)^{r_1}} M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^{n'_1} M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}'_2}^{n'_1} \\ &+ \sum_{N \geq m} \left[\sum_{k=1}^{s_1} \binom{s_1 + r_1 - k - 1}{r_1 - 1} \frac{(-1)^{s_1 - k}}{(t_1 - t'_1)^{s_1 + r_1 - k}} \frac{(\xi_1 \rho_1)^m}{(m - t'_1)^k} \right. \\ &\left. + \sum_{k=1}^{r_1} \binom{s_1 + r_1 - k - 1}{s_1 - 1} \frac{(-1)^{r_1 - k}}{(t_1 - t'_1)^{s_1 + r_1 - k}} \right] M_{\mathbf{s}_2, \boldsymbol{\xi}_2, \mathbf{t}_2}^m M_{\mathbf{r}_2, \boldsymbol{\rho}_2, \mathbf{t}'_2}^m \end{aligned} \tag{24}$$

so

$$\forall N \in \mathbb{N}, \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \sqcup (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N. \quad \square \tag{25}$$

Remark 2. For an integer r , an r -tuple $\chi = (\chi_1, \dots, \chi_r)$ of multiplicative characters¹⁴ and $(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t})$ as above) let us define

$$M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N(\chi) = \sum_{N \geq n_1 > \dots > n_r > 0} \prod_{i=1}^r \frac{\chi_i^{n_i}(\xi_i)}{(n_i - t_i)^{s_i}}. \tag{26}$$

The same proof shows that, for any $(\mathbf{s}, \boldsymbol{\xi}) \in \mathbb{Z}_{>0}^l \times \mathbb{C}^l$ and $(\mathbf{r}, \boldsymbol{\rho}) \in \mathbb{Z}_{>0}^k \times \mathbb{C}^k$, for any l -tuple \mathbf{t} and k -tuple \mathbf{t}' of parameters (see footnote 10),

$$\forall N \in \mathbb{N}, \quad M_{\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}}^N(\chi) M_{\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}'}^N(\chi) = M_{(\mathbf{s}, \boldsymbol{\xi}, \mathbf{t}) \sqcup (\mathbf{r}, \boldsymbol{\rho}, \mathbf{t}')}^N(\chi). \tag{27}$$

This result allows us to deduce some product relations on the different multi-zêta functions.

Theorem 2.1. Let $\mathbf{s} = (s_1, \dots, s_l)$ and $\mathbf{r} = (r_1, \dots, r_k)$ be two compositions, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_l)$ and $\boldsymbol{\rho} = (\rho_1, \dots, \rho_k)$ be respectively a l -tuple and a k -tuple of complex numbers of which the first component has a modulus strictly less than 1, $\mathbf{t} = (t_1, \dots, t_s)$ and $\mathbf{t}' = (t'_1, \dots, t'_k)$ be two tuples of parameters not in $\mathbb{N}_{>0}$, and $N \in \mathbb{N}$

- (i) For the coloured polyzêta function:

$$\zeta(\mathbf{s}, \boldsymbol{\xi}) \zeta(\mathbf{r}, \boldsymbol{\rho}) = \zeta((\mathbf{s}, \boldsymbol{\xi}) \sqcup (\mathbf{r}, \boldsymbol{\rho})) \tag{28}$$

- (ii) For the truncated Hurwitz polyzêta function:

$$\zeta_N(\mathbf{s}, \mathbf{t}) \zeta_N(\mathbf{r}, \mathbf{t}') = \zeta_N((\mathbf{s}, \mathbf{t}) \sqcup (\mathbf{r}, \mathbf{t}')) \tag{29}$$

- (iii) In particular, for the monocentered polyzêta function:

$$\zeta(\mathbf{s}, (t, \dots, t)) \zeta(\mathbf{r}, (t, \dots, t)) = \zeta((\mathbf{s}, (t, \dots, t)) \sqcup (\mathbf{r}, (t, \dots, t))) \tag{30}$$

where t is a parameter s.t. $t \notin \mathbb{N}_{>0}$.

- (iv) For the Polylerch generalized function:

$$\zeta(\mathbf{s}, \mathbf{t}, \boldsymbol{\xi}) \zeta(\mathbf{r}, \mathbf{t}', \boldsymbol{\rho}) = \zeta((\mathbf{s}, \mathbf{t}, \boldsymbol{\xi}) \sqcup (\mathbf{r}, \mathbf{t}', \boldsymbol{\rho})) \tag{31}$$

¹⁴ Endomorphisms of the semigroup (\mathbb{C}, \times) .

Proof. (ii) comes directly from Proposition 2 because $\zeta_N(\mathbf{s}, \mathbf{t}) = M_{\mathbf{s},(1,\dots,1),\mathbf{t}}^N$; for (i), (iii) and (iv), apply Proposition 2 with, respectively, the functions

$$M_{\mathbf{s},\xi,(0,\dots,0)}^N, \quad M_{\mathbf{s},(1,\dots,1),(t,\dots,t)}^N \quad \text{and} \quad M_{\mathbf{s},\xi,\mathbf{t}}^N$$

and take both sides of the equality to the limit as N grows to infinity. \square

Remark 3. We cannot use this method for the Hurwitz polyzêtas because in the decomposition, some divergent terms (which have $s_1 = 1$!) appear: for example, for $t \neq t'$,

$$\begin{aligned} (y_2, z_t) \sqcup (y_3, z_{t'}) &= (y_2 y_3, z_t z_{t'}) + (y_3 y_2, z_{t'} z_t) + \sum_{n=0}^1 \binom{2+n}{2} \frac{(-1)^n}{(t-t')^{3+n}} (y_{2-n}, z_t) \\ &\quad + \sum_{n=0}^2 \binom{1+n}{1} \frac{(-1)^n}{(t-t')^{2+n}} (y_{3-n}, z_{t'}) \\ &= (y_2 y_3, z_t z_{t'}) + (y_3 y_2, z_{t'} z_t) + \frac{1}{(t-t')^3} (y_2, z_t) - \frac{3}{(t-t')^4} (y_1, z_t) \\ &\quad + \frac{1}{(t-t')^2} (y_3, z_{t'}) - \frac{2}{(t-t')^3} (y_1, z_{t'}) + \frac{3}{(t-t')^4} (y_1, z_{t'}). \end{aligned} \tag{32}$$

Separately, the terms $-\frac{3}{(t-t')^4} (y_1, z_t)$ and $\frac{3}{(t-t')^4} (y_1, z_{t'})$, corresponding respectively to $\frac{-3}{(t-t')^4} \frac{1}{n-t}$ and $\frac{3}{(t-t')^4} \frac{1}{n-t'}$ give a divergent series although all other terms correspond to convergent series. Of course, the sum of the two

$$\frac{3}{(t-t')^4} \left(-\frac{1}{n-t} + \frac{1}{n-t'} \right) = \frac{3}{(t-t')^4} \left(\frac{t'-t}{(n-t)(n-t')} \right) \tag{33}$$

is a term of a convergent series, but the series is not a Hurwitz Polyzêta.

3. Radford’s theorem for the AC-stuffle.

In this subsection, A is only supposed to be a ring (with unit); when we need it to be commutative or to contain the set of rational numbers, we will state it explicitly.

Let $<$ be a total ordering on the alphabet X , and $\mathcal{Lyn}(X)$ denote the family of Lyndon words [22] constructed from X^* w.r.t. this ordering. We will prove that the largest framework in which Radford’s theorem holds true [20] is when φ is commutative (and associative).

3.1. Computing φ -shuffle expressions using shuffles

In this subsection A is a ring with unit and $\varphi : AX \otimes AX \rightarrow AX$ an associative law. We can express the result of the φ -shuffle product thanks to the shuffle product (and some terms of lower degree). First, we observe what happens with the product of two words:

Lemma 2. For $u, v \in X^*$, there exists $(C_{u,v}^w)_{|w| < |u|+|v|} \in A^{(\mathbb{N})}$ such that:

$$u \sqcup_{\varphi} v = u \sqcup v + \sum_{|w| < |u|+|v|} C_{u,v}^w w.$$

Proof. Omitted. \square

Now, because the Lyndon words are candidates to be a transcendence basis [16], we see what happens when they are φ -shuffled.

Definition 5. Let $\star : A(X) \times A(X) \mapsto A(X)$ be an associative law with unit. For any $\alpha \in \mathbb{N}^{(\mathcal{Lyn}(X))}$ and $\{l_1, \dots, l_r\} \supset \text{supp}(\alpha)$ in strict decreasing order (i.e. $l_1 > \dots > l_r$), we set

$$\mathbb{X}^{\star\alpha} = l_1^{\star\alpha_1} \star \dots \star l_r^{\star\alpha_r}, \tag{34}$$

where $\alpha_i = \alpha(l_i)$ for all i and, for short, $\mathbb{X} = \mathcal{Lyn}(X)$.

One easily checks that the product (34) does not depend on the choice of the set $\{l_1, \dots, l_r\} \supset \text{supp}(\alpha)$. We will also need the following parameter (which will turn out to be the length of the dominant terms in the product)

$$\|\alpha\| = \sum_{l \in \mathcal{Lyn}(X)} \alpha(l) |l|. \tag{35}$$

Lemma 3. If \sqcup_φ is associative,

$$\forall \alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}, \exists (C_\beta^\alpha)_\beta \in A^{(\mathbb{N}^{(\mathcal{L}yn(X))})} / \mathbb{X}^{\sqcup_\varphi \alpha} = \mathbb{X}^{\sqcup \alpha} + \sum_{\substack{\beta \in \mathbb{N}^{(\mathcal{L}yn(X))} \\ \|\beta\| < \|\alpha\|}} C_\beta^\alpha \mathbb{X}^{\sqcup \beta}.$$

Proof. Omitted \square

3.2. Radford's theorem in φ -shuffle algebras

Lemma 4. If \sqcup_φ is associative,

$$\forall p \in \mathbb{N}^*, \text{span} \left((\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right) = \text{span} \left((\mathbb{X}^{\sqcup \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right). \tag{36}$$

Proof. Lemma 3 give

$$\forall p \in \mathbb{N}^*, \text{span} \left((\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right) \subset \text{span} \left((\mathbb{X}^{\sqcup \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right).$$

We just have to prove, for any $p \in \mathbb{N}^*$, the property $\mathcal{P}(p)$:

$$\text{span} \left((\mathbb{X}^{\sqcup \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right) \subset \text{span} \left((\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right) \tag{37}$$

- It is true for $p = 1$.
- Assume $\mathcal{P}(p)$ true for an integer p .
 Let $\alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}$ such that $\|\alpha\| < p + 1$.
 We can find $(C_\beta^\alpha)_\beta \in A^{(\mathbb{N}^{(\mathcal{L}yn(X))})}$ such that $\mathbb{X}^{\sqcup_\varphi \alpha} = \mathbb{X}^{\sqcup \alpha} + \sum_{\substack{\beta \in \mathbb{N}^{(\mathcal{L}yn(X))} \\ \|\beta\| < \|\alpha\|}} C_\beta^\alpha \mathbb{X}^{\sqcup \beta}$, so $\mathbb{X}^{\sqcup \alpha} = \mathbb{X}^{\sqcup_\varphi \alpha} - \sum_{\substack{\beta \in \mathbb{N}^{(\mathcal{L}yn(X))} \\ \|\beta\| < \|\alpha\|}} C_\beta^\alpha \mathbb{X}^{\sqcup \beta}$.
 But every term of the sum is of the form $C_\beta^\alpha \mathbb{X}^{\sqcup \beta}$ with $\beta \in \mathbb{N}^{(\mathcal{L}yn(X))}$ and $\|\beta\| < \|\alpha\| < p + 1$ so $\|\beta\| < p$.
 Consequently, they are in $\text{span} \left((\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right)$, and so is the sum. By the induction hypothesis, the sum is in $\text{span} \left((\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p} \right)$, therefore $\mathbb{X}^{\sqcup \alpha} \in \text{span} \left((\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X)), \|\alpha\| < p+1} \right)$. \square

Theorem 3.1. Let A be a commutative ring (with unit) such that ¹⁵ $\mathbb{Q} \subset A$ and $\sqcup_\varphi : A(X) \otimes A(X) \rightarrow A(X)$ is associative. If X is totally ordered by $<$, then $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}}$ is a linear basis of $A(X)$.

Proof. Since this family is a generating family by Lemma 4, only freeness remains to be proven. Let $\sum_{\alpha \in J} \beta_\alpha \mathbb{X}^{\sqcup_\varphi \alpha} = 0$ be a null linear combination of $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in \mathbb{N}^{(\mathcal{L}yn(X))}}$, with J a nonempty finite subset of $\mathbb{N}^{(\mathcal{L}yn(X))}$. Thanks to Lemma 3, for any $\alpha \in J$, we can find a finite family $B_\alpha \subset \mathbb{N}^{(\mathcal{L}yn(X))}$ and $(C_\beta^\alpha)_{\beta \in B_\alpha} \in A^{B_\alpha}$ such that

$$\mathbb{X}^{\sqcup_\varphi \alpha} = \mathbb{X}^{\sqcup \alpha} + \sum_{\substack{\beta \in B_\alpha \\ \|\beta\| < \|\alpha\|}} C_\beta^\alpha \mathbb{X}^{\sqcup \beta}.$$

Set $B = J \cup \left(\bigcup_{\alpha \in J} B_\alpha \right)$; B is a finite set. Then $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in J}$ is a triangular family for $|\cdot|$ with respect to the family $\mathcal{F} = (\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in B}$ in the vector space $\text{span}(\mathcal{F})$, which is of finite dimension. But \mathcal{F} is a basis, so $(\mathbb{X}^{\sqcup_\varphi \alpha})_{\alpha \in J}$ is free and $\forall \alpha \in J, \beta_\alpha = 0$. \square

Corollary 1. Under the same hypotheses, if in addition \sqcup_φ is commutative in A then

- (i) The algebra $\mathcal{A} = (A(X), \sqcup_\varphi, 1_{X^*})$ is a polynomial algebra.
- (ii) $\text{Lyn}(X)$ is a transcendence basis of \mathcal{A} .

Remark 4. It is necessary to suppose $\mathbb{Q} \subset A$ as, in case $\varphi \equiv 0$, one has

$$\forall n \in \mathbb{N}_{>0}, a^n = \frac{1}{n!} (a^{\sqcup n}) \tag{38}$$

Proof.

- (i) Immediate result.
- (ii) Comes from Theorems 3.1 and 4.1, which proves in an elementary (so independent) way that the commutativity of φ is equivalent to the commutativity of \sqcup_φ . \square

¹⁵ This condition amounts to ask that $\mathbb{N}^+. 1_A \subset A^\times$

3.3. Bialgebra structure

From now on, all tensor products are supposed over A (unless specifically mentioned, see the proof of prop. [Proposition 4](#)).

Definition 6. A law \star defined over $A\langle X \rangle$ is a dual law (or dualizable) if there exists a linear mapping $\Delta_\star : A\langle X \rangle \rightarrow A\langle X \rangle \otimes A\langle X \rangle$ such

$$\forall (u, v, w) \in X^* \times X^* \times X^*, \quad \langle u \star v | w \rangle = \langle u \otimes v | \Delta_\star(w) \rangle^{\otimes 2}. \tag{39}$$

In this case, Δ_\star will be called the comultiplication dual to \star .

Theorem 3.2. Let A be a commutative ring (with unit). We suppose that the product $\sqcup_\varphi : A\langle X \rangle \otimes A\langle X \rangle \rightarrow A\langle X \rangle$ is an associative and commutative law on $A\langle X \rangle$, then the algebra $(A\langle X \rangle, \sqcup_\varphi, 1_{X^*})$ can be endowed with the comultiplication Δ_{conc} dual to the concatenation

$$\Delta_{\text{conc}}(w) = \sum_{uv=w} u \otimes v \tag{40}$$

and the “constant term” character $\epsilon(P) = \langle P | 1_{X^*} \rangle$.

(i) With this setting

$$\mathcal{B}_\varphi = (A\langle X \rangle, \sqcup_\varphi, 1_{X^*}, \Delta_{\text{conc}}, \epsilon) \tag{41}$$

is a bialgebra¹⁶.

(ii) The bialgebra (41) is, in fact, a Hopf Algebra.

Proof.

(i) It is a classical combinatorial verification, done in [10]. The following identity remains to be proven:

$$\forall (w_1, w_2) \in X^*, \quad \Delta_{\text{conc}}(w_1 \sqcup_\varphi w_2) = \Delta_{\text{conc}}(w_1) \Delta_{\text{conc}}(w_2) \tag{42}$$

which can be done by a (lengthy) induction or by duality.

(ii) Let, classically, Δ_{conc}^+ be defined by

$$\forall w \in X^*, \quad \Delta_{\text{conc}}^+(w) = \sum_{\substack{uv=w \\ u, v \neq 1}} u \otimes v.$$

We remark that Δ_{conc}^+ is coassociative and locally nilpotent, i.e.

$$(\forall w \in X^*)(\exists n \in \mathbb{N}^*)(\Delta_{\text{conc}}^+)^{(n)}(w) = 0$$

which gives result. \square

4. Conditions for AC-shuffle and dualizability

4.1. Commutative and associative conditions

We have obtained an extended version of Radford’s theorem and other properties with conditions stated w.r.t. \sqcup_φ , we will see in this subsection that these conditions can be set uniquely in terms of properties of φ itself.

Definition 7. For $P \in A\langle X \rangle$, we denote $\text{supp}(P)$ the support of P and

$$\text{deg}(P) = \max\{|l|, l \in \text{supp}(P)\}$$

(with this convention $\text{deg}(0) = -\infty$).

Lemma 5. Let A be a commutative ring, X be an alphabet and $\varphi : X \times X \rightarrow A\langle X \rangle$ is an arbitrary mapping. Then

$$\forall (u, v) \in (X^*)^2, \quad \text{deg}(u \sqcup_\varphi v) \leq |u| + |v|. \tag{43}$$

¹⁶ Commutative and, when $|X| \geq 2$, noncocommutative.

Proof. If $|u| = 0$ or $|v| = 0$, then $u \sqcup_{\varphi} v$ is one of $\{u, v\}$ so its length is $|u| + |v|$.

Let X^+ be the set of nonempty words. We prove $\forall (u, v) \in (X^+)^2, \deg(u \sqcup_{\varphi} v) = |u| + |v|$ by induction on $|u| + |v|$.

For any letters a and $b, a \sqcup_{\varphi} b = ab + ba + \varphi(a, b)1_{A^*}$ so $\deg(a \sqcup_{\varphi} b) = 2 = |a| + |b|$.

One assumes the property true for all words $u, v \in X^+$ such that $|u| + |v| \leq n$, where n is an integer. Let u and v be now two words of X^+ such that $|u| + |v| = n + 1$.

There exist x, y in X, u', v' in X^* such that $u = xu', v = yv'$ (because $(u, v) \in (X^+)^2$). Then $|u| + |v'| = |u| + |v| - 1 \leq n$, so $\deg(y(u \sqcup_{\varphi} v')) \leq n + 1$. Also $|u'| + |v| = |u| - 1 + |v| \leq n$ so $\deg(x(u' \sqcup_{\varphi} v)) \leq n + 1$, and $|u'| + |v'| = |u| - 1 + |v| - 1 \leq n$ so $\deg(\varphi(x, y)u' \sqcup_{\varphi} v') \leq n + 1$. Hence, $\deg(u \sqcup_{\varphi} v) = n + 1$: the induction is proved. \square

Theorem 4.1. In the context of Definition 4,

- (i) The law \sqcup_{φ} is commutative if and only if the extension $\varphi : AX \otimes AX \rightarrow AX$ is commutative.
- (ii) The law \sqcup_{φ} is associative if and only if the extension $\varphi : AX \otimes AX \rightarrow AX$ is associative.

Proof. We give an elementary proof.

- (i) $[\sqcup_{\varphi} \text{ commutative} \implies \varphi \text{ commutative}]$

Let us suppose $\forall (u, v) \in (X^*)^2, u \sqcup_{\varphi} v = v \sqcup_{\varphi} u$.

In particular, $\forall (x, y) \in (X^*)^2, x \sqcup_{\varphi} y = x \sqcup_{\varphi} y$. But, for any $(x, y) \in X^2$,

$$x \sqcup_{\varphi} y = xy + yx + \varphi(x, y) \text{ and } y \sqcup_{\varphi} x = yx + xy + \varphi(y, x). \tag{44}$$

and so $(\forall x, y \in X)(\varphi(x, y) = \varphi(y, x))$.

$[\varphi \text{ commutative} \implies \sqcup_{\varphi} \text{ commutative}]$

Now let us suppose φ is commutative then let us prove by recurrence on $|uv|$ that \sqcup_{φ} is commutative:

- The previous computation proves that the recurrence holds for $|u| = |v| = 0$.
- Suppose the recurrence holds for any $u_1, v_1 \in X^*$ such that $|u_1 v_1| \leq n$ and let $u, v \in X^*$ such that $|uv| \leq n + 1$.
If $u = 1_{X^*}$ or $v = 1_{X^*}$ it is true, else let $u = xu'$ and $v = yv'$ with $x, y \in X$ and $u', v' \in X^*$. Then,

$$\begin{aligned} u \sqcup_{\varphi} v &= x(u' \sqcup_{\varphi} yv) + y(xu' \sqcup_{\varphi} v) + \varphi(x, y)(u' \sqcup_{\varphi} v') \\ &= x(yv \sqcup_{\varphi} u) + y(v' \sqcup_{\varphi} xu') + \varphi(y, x)(v' \sqcup_{\varphi} u') \\ &\quad \text{(by the induction hypothesis)} \\ &= v \sqcup_{\varphi} u. \end{aligned} \tag{45}$$

- (ii) $[\sqcup_{\varphi} \text{ associative} \implies \varphi \text{ associative}]$ Let us suppose

$$\forall u, v, w \in X^*, (u \sqcup_{\varphi} v) \sqcup_{\varphi} w = u \sqcup_{\varphi} (v \sqcup_{\varphi} w). \tag{46}$$

Then, for any $x, y, z \in X$, one has

$$(x \sqcup_{\varphi} y) \sqcup_{\varphi} z = x \sqcup_{\varphi} (y \sqcup_{\varphi} z). \tag{47}$$

But

$$\begin{aligned} (x \sqcup_{\varphi} y) \sqcup_{\varphi} z &= (xy + yx + \varphi(x, y)) \sqcup_{\varphi} z \\ &= xy \sqcup_{\varphi} z + yx \sqcup_{\varphi} z + \varphi(x, y) \sqcup_{\varphi} z \\ &= x(y \sqcup_{\varphi} z) + z(xy \sqcup_{\varphi} 1) + \varphi(x, z)y \\ &\quad + y(x \sqcup_{\varphi} z) + z(yx \sqcup_{\varphi} 1) + \varphi(y, z)x \\ &\quad + \varphi(x, y)z + z\varphi(x, y) + \varphi(\varphi(x, y), z) \\ &= x(yz + zy + \varphi(y, z)) + zxy + \varphi(x, z)y \\ &\quad + y(xz + zx + \varphi(x, z)) + zyx \\ &\quad + \varphi(y, z)x + \varphi(x, y)z + z\varphi(x, y) + \varphi(\varphi(x, y), z) \end{aligned} \tag{48}$$

$$\begin{aligned} x \sqcup_{\varphi} (y \sqcup_{\varphi} z) &= x \sqcup_{\varphi} (yz + zy + \varphi(y, z)) \\ &= x \sqcup_{\varphi} yz + x \sqcup_{\varphi} zy + x \sqcup_{\varphi} \varphi(y, z) \\ &= x(1 \sqcup_{\varphi} yz) + y(x \sqcup_{\varphi} z) + \varphi(x, y)z \\ &\quad + x(1 \sqcup_{\varphi} zy) + z(x \sqcup_{\varphi} y) + \varphi(x, z)y \\ &\quad + x\varphi(y, z) + \varphi(y, z)x + \varphi(x, \varphi(y, z)) \\ &= xyz + y(xz + zx + \varphi(x, z)) + \varphi(x, y)z \end{aligned} \tag{49}$$

$$\begin{aligned}
 &+ xzy + z(xy + yx + \varphi(x, y)) + \varphi(x, z)y \\
 &+ x\varphi(y, z) + \varphi(y, z)x + \varphi(x, \varphi(y, z)).
 \end{aligned} \tag{50}$$

One can then deduce that

$$(\forall x, y, z \in X)(x \sqcup_{\varphi} (y \sqcup_{\varphi} z) = (x \sqcup_{\varphi} y) \sqcup_{\varphi} z) \tag{51}$$

$$\iff (\forall x, y, z \in X)(\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)). \tag{52}$$

[φ associative $\implies \sqcup_{\varphi}$ associative] Now if φ is associative then let us prove by induction on $|u| + |v| + |w|$ that \sqcup_{φ} is associative:

- The previous equivalence proves that the induction holds for $|u| = |v| = |w| = 1$.
- Suppose the recurrence holds for any $u, v \in X^*$ such that $3 \leq |u| + |v| + |w| \leq n$ and $|u|, |v|, |w| \neq 1$.
- Let $u = xu', v = yv'$ and $w = zw'$ with $x, y, z \in X$ and $u', v', w' \in X^*$. Then,

$$\begin{aligned}
 &u \sqcup_{\varphi} (v \sqcup_{\varphi} w) \\
 &= u \sqcup_{\varphi} (y(v' \sqcup_{\varphi} w') + z(v \sqcup_{\varphi} w') + \varphi(y, z)(v' \sqcup_{\varphi} w')) \\
 &= x(u' \sqcup_{\varphi} y(v' \sqcup_{\varphi} w')) + y(u \sqcup_{\varphi} (v' \sqcup_{\varphi} w')) + \varphi(x, y)(u' \sqcup_{\varphi} (v' \sqcup_{\varphi} w')) \\
 &\quad + x(u' \sqcup_{\varphi} z(v \sqcup_{\varphi} w')) + z(u \sqcup_{\varphi} (v \sqcup_{\varphi} w')) + \varphi(x, z)(u' \sqcup_{\varphi} (v \sqcup_{\varphi} w')) \\
 &\quad + x(u' \sqcup_{\varphi} \varphi(y, z)(v' \sqcup_{\varphi} w')) + \varphi(y, z)(u \sqcup_{\varphi} (v' \sqcup_{\varphi} w')) \\
 &\quad + \varphi(x, \varphi(y, z))u' \sqcup_{\varphi} (v' \sqcup_{\varphi} w') \\
 &= x(u' \sqcup_{\varphi} (v \sqcup_{\varphi} w)) \\
 &\quad + y(u \sqcup_{\varphi} (v' \sqcup_{\varphi} w)) + \varphi(x, y)(u' \sqcup_{\varphi} (v' \sqcup_{\varphi} w)) \\
 &\quad + z(u \sqcup_{\varphi} (v \sqcup_{\varphi} w')) + \varphi(x, z)(u' \sqcup_{\varphi} (v \sqcup_{\varphi} w')) \\
 &\quad + \varphi(y, z)(u \sqcup_{\varphi} (v' \sqcup_{\varphi} w')) + \varphi(x, \varphi(y, z))u' \sqcup_{\varphi} (v' \sqcup_{\varphi} w')
 \end{aligned} \tag{53}$$

and

$$\begin{aligned}
 &(u \sqcup_{\varphi} v) \sqcup_{\varphi} w \\
 &= (x(u' \sqcup_{\varphi} v) + y(u \sqcup_{\varphi} v') + \varphi(x, y)(u' \sqcup_{\varphi} v')) \sqcup_{\varphi} w \\
 &= x((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w) + z(x(u' \sqcup_{\varphi} v) \sqcup_{\varphi} w') + \varphi(x, z)((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w') \\
 &\quad + y((u \sqcup_{\varphi} v') \sqcup_{\varphi} w) + z(y(u \sqcup_{\varphi} v') \sqcup_{\varphi} w') + \varphi(y, z)((u \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
 &\quad + \varphi(x, y)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w) + z(\varphi(x, y)(u' \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
 &\quad + \varphi(\varphi(x, y), z)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
 &= x((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w) + \varphi(x, z)((u' \sqcup_{\varphi} v) \sqcup_{\varphi} w') \\
 &\quad + y((u \sqcup_{\varphi} v') \sqcup_{\varphi} w) + \varphi(y, z)((u \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
 &\quad + \varphi(x, y)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w) + \varphi(\varphi(x, y), z)((u' \sqcup_{\varphi} v') \sqcup_{\varphi} w') \\
 &\quad + z(u \sqcup_{\varphi} v) \sqcup_{\varphi} w'.
 \end{aligned} \tag{54}$$

Indeed, thanks to the induction hypothesis and the commutativity of φ , the terms $u \sqcup_{\varphi} (v \sqcup_{\varphi} w)$ and $(u \sqcup_{\varphi} v) \sqcup_{\varphi} w$ are equal. \square

4.2. Dualizability conditions

Proposition 3. We call $\gamma_{x,y}^z := \langle \varphi(x, y)|z \rangle$ the structure constants of φ (w.r.t. the basis X).

The product \sqcup_{φ} is a dual law if and only if $(\gamma_{x,y}^z)_{x,y,z \in X}$ is dualizable in the following sense

$$(\forall z \in X)(\#\{(x, y) \in X^2 | \gamma_{x,y}^z \neq 0\} < +\infty). \tag{55}$$

Proof. (\sqcup_{φ} dual law $\implies \gamma_{x,y}^z$ dualizable). Let Δ be the dual of \sqcup_{φ} , that is, for all $u, v, w \in X^*$

$$\langle u \sqcup_{\varphi} v | w \rangle = \langle u \otimes v | \Delta(w) \rangle^{\otimes 2}. \tag{56}$$

For all $z \in X$, one must have $\Delta(z) = \sum_{i=1}^n \alpha_i u_i \otimes v_i$. On the other hand, for all $x, y \in X$, one has $(x \sqcup_{\varphi} y) - (xy + yx) = \varphi(x, y)$. Hence

$$\begin{aligned}
 \gamma_{x,y}^z &= \langle \varphi(x, y) | z \rangle = \langle (x \sqcup_{\varphi} y) - (xy + yx) | z \rangle = \langle (x \sqcup_{\varphi} y) | z \rangle - \langle (xy + yx) | z \rangle \\
 &= \langle (x \otimes y) | \Delta(z) \rangle = \langle (x \otimes y) | \sum_{i=1}^n \alpha_i u_i \otimes v_i \rangle.
 \end{aligned} \tag{57}$$

We can deduce from the preceding argument that

$$\gamma_{x,y}^z \neq 0 \implies (x \in \cup_{i=1}^n Au_i \text{ and } y \in \cup_{i=1}^n Av_i)$$

which proves the point.

[$\gamma_{x,y}^z$ dualizable $\implies \sqcup_\varphi$ dual law]). This is, combinatorially speaking, the most interesting point. We first define a comultiplication Δ on $A\langle X \rangle$ by transposing the structure constants of \sqcup_φ by

$$\Delta(z) := z \otimes 1 + 1 \otimes z + \sum_{x,y \in X} \gamma_{x,y}^z x \otimes y \tag{58}$$

and, as the sum is finite (see however the comment after this theorem), this quantity belongs to $A\langle X \rangle \otimes A\langle X \rangle$. One then has a linear mapping $\Delta : AX \rightarrow A\langle X \rangle \otimes A\langle X \rangle$ which is extended, by universal property, into a morphism of algebras $\Delta : A\langle X \rangle \rightarrow A\langle X \rangle \otimes A\langle X \rangle$. Explicitly, for all $w = z_1 z_2 \cdots z_n$, one has

$$\Delta(z_1 z_2 \cdots z_n) = \Delta(z_1) \Delta(z_2) \cdots \Delta(z_n). \tag{59}$$

Now, we prove that the dual law of the latter coproduct is exactly \sqcup_φ .

First remark: by (58) and (59), one has

$$\Delta(w) = w \otimes 1 + 1 \otimes w + \sum_{u,v \in X^+} \beta(u,v) u \otimes v \tag{60}$$

the last sum being finitely supported. This shows by duality that

$$u \sqcup_\Delta 1 = 1 \sqcup_\Delta u = u \tag{61}$$

(here, \sqcup_Δ stands for the dual law of Δ). Moreover

$$\begin{aligned} au \sqcup_\Delta bv &= \sum_{w \in X^*} \langle au \sqcup_\Delta bv | w \rangle w \\ &= \sum_{w \in X^*} \langle au \otimes bv | \Delta(w) \rangle w \\ &= \langle au \otimes bv | 1 \otimes 1 \rangle 1 + \sum_{w \in X^+} \langle au \otimes bv | \Delta(w) \rangle w \\ &= \sum_{x \in X; m \in X^*} \langle au \otimes bv | \Delta(xm) \rangle xm \\ &= \sum_{x \in X; m \in X^*} \langle au \otimes bv | \Delta(x) \Delta(m) \rangle xm \\ &= \sum_{x \in X; m \in X^*} \left\langle au \otimes bv \left| \left(x \otimes 1 + 1 \otimes x + \sum_{y,z \in X} \langle \Delta(x) | y \otimes z \rangle y \otimes z \right) \Delta(m) \right. \right\rangle xm \\ &= \sum_{x \in X; m \in X^*} \langle au \otimes bv | (x \otimes 1) \Delta(m) \rangle xm \\ &\quad + \sum_{x \in X; m \in X^*} \langle au \otimes bv | (1 \otimes x) \Delta(m) \rangle xm \\ &\quad + \sum_{x \in X; m \in X^*} \left\langle au \otimes bv \left| \sum_{y,z \in X} \langle \Delta(x) | y \otimes z \rangle y \otimes z \Delta(m) \right. \right\rangle xm \\ &= \sum_{m \in X^*} \langle au \otimes bv | (a \otimes 1) \Delta(m) \rangle am + \sum_{m \in X^*} \langle au \otimes bv | (1 \otimes b) \Delta(m) \rangle bm \\ &\quad + \sum_{x \in X; m \in X^*} \langle au \otimes bv | \langle \Delta(x) | a \otimes b \rangle a \otimes b \Delta(m) \rangle xm \\ &= \sum_{m \in X^*} \langle u \otimes bv | \Delta(m) \rangle am + \sum_{m \in X^*} \langle au \otimes v | \Delta(m) \rangle bm \\ &\quad + \sum_{x \in X; m \in X^*} \langle \Delta(x) | a \otimes b \rangle \langle u \otimes v | \Delta(m) \rangle xm \\ &= a \sum_{m \in X^*} \langle u \otimes bv | \Delta(m) \rangle m + b \sum_{m \in X^*} \langle au \otimes v | \Delta(m) \rangle m \end{aligned}$$

$$\begin{aligned}
 & + \sum_{m \in X^*} \left(\sum_{x \in X} \langle \Delta(x) | a \otimes b \rangle x \right) \langle u \otimes v | \Delta(m) \rangle m \\
 & = a(u \sqcup_{\Delta} bv) + b(au \sqcup_{\Delta} v) + \varphi(a, b)(u \sqcup_{\Delta} v).
 \end{aligned} \tag{62}$$

This proves that the dual law \sqcup_{Δ} equals \sqcup_{φ} and we are done. \square

4.3. The Hopf–Hurwitz algebra

In Section 2.1.2, we provided the law on indices followed by the product of Formal Hurwitz polyzêtas, we now prove that the law φ associated with it is associative. The “centres” will be taken from a subfield K of \mathbb{C} and the set of coefficients A is a K -CAAU.

Proposition 4.

(i) The law $\varphi : AN \otimes AN \rightarrow AN$ associated to \circ is defined, on the basis N , by the multiplication table $\mathcal{T}_{\text{Formal Hurwitz}}$

$$\begin{aligned}
 & \text{if } t = t'; \quad \varphi((y_i, z_t), (y_j, z_{t'})) = (y_{i+j}, z_t) \\
 & \text{if } t \neq t'; \quad \varphi((y_i, z_t), (y_j, z_{t'})) = \sum_{n=0}^{i-1} \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{i-n}, z_t) \\
 & \quad + \sum_{n=0}^{j-1} \binom{i-1+n}{i-1} \frac{(-1)^n}{(t'-t)^{i+n}} (y_{j-n}, z_{t'}).
 \end{aligned} \tag{63}$$

(ii) The product \sqcup is associative, commutative and unital, making $(A\langle N \rangle, \circ, 1_N)$ into a A -CAAU.

Proof. (i) Let first $j : KN \rightarrow K\langle X \rangle$ be the linear mapping defined by $j((y_i, z_t)) = \frac{1}{(X-t)^i}$. In fact, as the $\{\frac{1}{(X-t)^i}\}$ are linearly independent, j is injective. On the other hand, j is a morphism of K -AAU due to the fact that the multiplication table is identical. Hence φ is a law of K -CAAU on $A \otimes_K KN$. (ii) Is a consequence of the general theorems. \square

Now, we have the following bialgebra

$$\mathcal{H}_{\text{Formal Hurwitz}} = (A\langle N \rangle, \sqcup_{\varphi}, 1_{N^*}, \Delta_{\text{conc}}, \epsilon) \tag{64}$$

which is a Hopf algebra. Note that \sqcup_{φ} is not dualisable which means that the adjoint

$$\Delta_{\sqcup_{\varphi}} : N^* \rightarrow A\langle\langle N^* \otimes N^* \rangle\rangle \tag{65}$$

does not have its image in $A\langle N \rangle \otimes A\langle N \rangle$. See next paragraph for tools and proofs.

Corollary 2. The product \sqcup is associative, commutative and unital, making $(A\langle N \rangle, \sqcup, 1_N)$ into a A -CAAU.

Proof. It comes that the product \sqcup is a direct product of the products \circ and \sqcup . \square

5. Conclusion

We have been able to give a useful extended version of Radford’s theorem. Let us observe that:

- For the shuffle product, $\varphi_{\sqcup} \equiv 0$, so the shuffle \sqcup is associative, commutative and dualizable.
- The stuffle product over an alphabet indexed by \mathbb{N} is associative and commutative because $\varphi_{\sqcup}(x_i, x_j) = x_{i+j}$ is so; moreover it is dualizable.
- The muffle product over an alphabet indexed by \mathbb{C} is associative and commutative because $\varphi_{\sqcup}(x_i, x_j) = x_{i \times j}$ is so; it is not dualizable because for all $n \in \mathbb{N}_{>0}$, $x_1 = \varphi_{\sqcup}(x_{1/n}, x_n)$.

However, there are multiplicative subsemigroups S of \mathbb{C} such that φ restricted to the alphabet $(x_i)_{i \in S}$ is dualizable. Such an example is given where S is the set of r th root of unity, the case is described in [17] example 2.

- The duffle product over an alphabet indexed by $\mathbb{N}^* \times \mathbb{C}^*$ is associative and commutative because $\varphi_{\sqcup}((y_i, x_k), (y_j, x_l)) = (y_{i+j}, x_{k \times l})$ is associative and commutative; it is not dualizable either (for the same reason).

But we can make the same remark as the muffle about the possibility to restrict the alphabet so that φ becomes dualizable.

- The Formal Polyzêta product is associative, commutative but, for all $j \in \mathbb{N}_{>0}$, with $t \neq t'$,

$$\begin{aligned} \varphi \circ ((y_2, z_t), (y_j, z_{t'})) &= \sum_{n=0}^1 \binom{j-1+n}{j-1} \frac{(-1)^n}{(t-t')^{j+n}} (y_{2-n}, z_t) \\ &\quad + \sum_{n=0}^{j-1} \binom{1+n}{1} \frac{(-1)^n}{(t'-t)^{2+n}} (y_{j-n}, z_{t'}) \end{aligned} \quad (66)$$

and then $\forall j \in \mathbb{N}_{>0}$, $(\varphi \circ ((y_2, z_t), (y_j, z_{t'})) | (y_1, z_t)) = -\binom{j}{j-1} \frac{1}{(t-t')^{j+1}} \neq 0$ which implies that \circ is not dualizable.

- The Lerch product is associative, commutative and not dualizable (for the same reason as \circ).

So, if we work in the Riemann polyzêta algebra, in the coloured polyzêta algebra, or in the Generalized Lerch polyzêta algebra, we can use a representation with the Lyndon set as a transcendence basis. Moreover, the Riemann polyzêta algebra and the truncated Hurwitz polyzêta algebra can both be completed into Hopf algebras.

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References

- [1] N. Bourbaki, *Algebra I Chapters 1-3*, Springer, Berlin, 1989.
- [2] C. Bui, *Hopf Algebras of Shuffles and Quasi-Shuffles. Constructions of Dual Bases* (Master dissertation), Villetaneuse, 2012.
- [3] K.T. Chen, Iterated path integrals, *Bull. Amer. Math. Soc.* 83 (1977) 831–879.
- [4] K.T. Chen, R.H. Fox, R.C. Lyndon, Free differential calculus, IV. The quotient groups of the lower central series, *Ann. of Math.* 68 (1) (1958) 81–95.
- [5] C. Costermans, Calcul symbolique non commutatif: analyse des constantes d'arbres de fouille, in: Thèse de doctorat, Université Lille I, 2008.
- [6] P. Du Bois-Reymond, Théorie générale des fonctions, in: Traduit de L'Allemand par G. Milhaud et a. Giro, Imprimerie niçoise, Nice, 1887.
- [7] G.H.E. Duchamp, P. Blasiak, A. Horzela, K.A. Penson, A.I. Solomon, A three-parameter Hopf deformation of the algebra of Feynman-like diagrams, *J. Russ. Laser Res.* 31 (1) (2010) 162–181.
- [8] G. Duchamp, M. Flouret, É. Laugerotte, J.-G. Luque, Direct and dual laws for automata with multiplicities, *Theoret. Comput. Sci.* 267 (1–2) (2001) 105–120.
- [9] G.H.E. Duchamp, C. Tollu, K.A. Penson, G.A. Koshevoy, Deformations of Algebras: Twisting and Perturbations, *Sém. Lothar. Combin.* B62e (2010) 14.
- [10] J.-Y. Enjalbert, V. Hoang Ngoc Minh, Combinatorial study of colored Hurwitz polyzêtas, *Discrete Math.* 312 (24) (2012) December.
- [11] J.-Y. Enjalbert, V. Hoang Ngoc Minh, Propriétés combinatoires et prolongement analytique effectif de polyzêtas de hurwitz et de leurs homologues, *J. Théor. Nombres Bordeaux* 19 (3) (2007).
- [12] J.-Y. Enjalbert, V. Hoang Ngoc Minh, Analytic and combinatoric aspects of hurwitz polyzêtas, *J. Théor. Nombres Bordeaux* 23 (2) (2011).
- [13] M. Fréchet, *Biographie du mathématicien alsacien arbogast, Thalès 4* (1940).
- [14] G.H. Hardy, Orders of Infinity: "The Infinitärcalcul" of Paul Du Bois-Reymond, in: *Cambridge Tracts in Mathematics and Mathematical Physics*, vol. 12, Cambridge University Press, 1910, p. viii.
- [15] V. Hoang Ngoc Minh, G. Jacob, M. Petitot, N.E. Oussous, De l'algèbre des ζ de Riemann multivariées à l'algèbre des ζ de Hurwitz multivariées, *Sém. Lothar. Combin.* B44i (2001).
- [16] M.E. Hoffman, The algebra of multiple harmonic series, *J. Algebra* 194 (1997).
- [17] M.E. Hoffman, *J. Algebraic Combin.* 11 (2000).
- [18] M.E. Hoffman, Quasi-symmetric functions, multiple zeta values, and rooted trees, *Oberwolfach Reports* 3 (2006).
- [19] L.A. Lusternik, S.S. Petrova, Les premières étapes du calcul symbolique, *Rev. Hist. Sci.* 25 (3) (1972).
- [20] D.E. Radford, A natural ring basis for shuffle algebra and an application to group schemes, *J. Algebra* 58 (2) (1979) June.
- [21] R. Ree, Lie elements and an algebra associated with shuffles, *Ann. of Math.* 68 (2) (1958) September.
- [22] C. Reutenauer, *Free Lie Algebras*, in: *London Mathematical Society Monographs*, vol. 7, Clarendon Press, Oxford, 1993, p. 286 pages.
- [23] S. Weinzierl, Hopf algebras and Dyson–Schwinger equations. lectures given at the workshop "Dyson–Schwinger Equations in Modern Physics and Mathematics, September 2014, 30 pages, <http://arxiv.org/abs/1506.09119>.