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INTERTWINING SPACES ASSOCIATED WITH q-ANALOGUES OF THE YOUNG SYMMETRIZERS IN THE HECKE ALGEBRA

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Abstract. Let $\mathcal{H}$ be the Hecke algebra of the symmetric group. With each subset $S \subset [1, n-1]$, we associate two idempotents $\square_S$ and ∇_S which are q-deformations of the symmetrizer and antisymmetrizer relative to the Young subgroup $\mathfrak{S}_{I_S}$ generated by the simple transpositions $\{(i, i+1)\}_{i \in S}$. We give here explicit bases for the intertwining space $\square_{S_1}\mathcal{H}\nabla_{S_2}$, indexed by the double classes $\mathfrak{S}_{I_{S_1}} \backslash \mathfrak{S}n/\mathfrak{S}_{I_{S_2}}$. We also compute bases and characters of the right ideals

$$\mathfrak{I}(I, J) = \square_{S_1}\mathcal{H}\nabla_{S_2}\mathcal{H}.$$

Résumé. Soit $\mathcal{H}$, l'algèbre de Hecke du groupe symétrique. À chaque sous ensemble $S \subset [1, n-1]$, on associe deux idempotents $\square_S$ et ∇_S qui sont les q-déformations des symétriseur et antisymétriseur du sous groupe de Young $\mathfrak{S}_{I_S}$ engendré par les transpositions simples $\{(i, i+1)\}_{i \in S}$. Nous donnons ici des bases explicites pour le sous espace d'entrelacement $\square_{S_1}\mathcal{H}\nabla_{S_2}$, indexées par les doubles classes $\mathfrak{S}_{I_{S_1}} \backslash \mathfrak{S}n/\mathfrak{S}_{I_{S_2}}$. Nous calculons également des bases et les caractères des idéaux

$$\mathfrak{I}(I, J) = \square_{S_1}\mathcal{H}\nabla_{S_2}\mathcal{H}.$$

Introduction. A question of importance, in representation theory of groups, is the construction of primitive idempotents in a group algebra. For the symmetric group, after the early construction by A. Young (see [17], [9] for details), several complete sets of primitive idempotents (i.e. generating irreducible ideals) have been proposed ([10], [14], [15] and [3], [6], [16] for the Hecke algebra).

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The original idea of Young (see e.g. [5], [9]) for computing a primitive idempotent in the algebra of the symmetric group consists in starting from a tableau without repetition

5	7	
2	4	
1	3	6

and deriving two idempotents, the row symmetrizer P (i.e. the sum of the permutations that preserve each row, in the picture $\mathfrak{S}_{\{1,3,6\}} \times \mathfrak{S}_{\{2,4\}} \times \mathfrak{S}_{\{5,7\}}$) and the column antisymmetrizer N (i.e. the alternating sum of the permutations that preserve each column, in the picture $\mathfrak{S}_{\{1,2,5\}} \times \mathfrak{S}_{\{3,4,7\}} \times \mathfrak{S}_{\{6\}}$). Now, PN is (up to a scalar) an idempotent and the associated representation is irreducible, of index equal to the shape of the tableau.

The idempotence of PN is due to two facts, the first being that $P.\mathbb{C}[\mathfrak{S}_n].N$ is one dimensional and the second being that there is a natural isomorphism

$$P.\mathbb{C}[\mathfrak{S}_n].N \cong \mathrm{Hom}_{\mathfrak{S}_n}(N.\mathfrak{S}_n, P.\mathfrak{S}_n)$$

Let $\mathcal{H}$ be the Hecke algebra of the symmetric group with basis $(T_w)_{w\in\mathfrak{S}_n}$. The preceding construction does not give in general idempotents in this algebra (even in the simplest case of a tableau of shape (1,2)). However, for a single row (or column), we get the following q-analogues of Young symmetrizers and antisymmetrizers

$$\square = \sum_{w\in\mathfrak{S}_n} T_w \quad \text{and} \quad \nabla = \sum_{w\in\mathfrak{S}_n} (-q)^{l_{\max}-l(w)} T_w$$

with $l_{\max} = \frac{n(n-1)}{2}$ and $l(w)$ being the length of the permutation w.

We can get also, with the same formulas, analogues of the Young row (resp. column) symmetrizers providing that the filling of the tableau is done with consecutive numbers in the rows (resp. columns). This explains why Young's construction cannot be adapted trivially. The elements so obtained, denoted by $\square_{\omega_I}$ and ∇_{ω_I}, are particular cases of the $\square_\mu$ and $\nabla_\mu,\ \mu \in \mathfrak{S}_n$ constructed elsewhere using solutions of the Yang-Baxter equation [4].

A complete set of primitive idempotents for the generic Hecke algebra have been defined (see [3], [4], [6]) with q-analogues of Young symmetrizers associated with partitions.

The idempotence of these elements is based (as for the symmetric group) on the combinatorial fact that, for I and J conjugate, the spaces $\square_{\omega_I}\mathcal{H}\nabla_{\omega_J}$ and $\nabla_{\omega_J}\mathcal{H}\square_{\omega_I}$ are both one-dimensional.

The preceding property still holds when I and J are no longer partitions but compositions associated with conjugate partitions. (This fact is characteristic free and independent of the choice of q.) Then, when the Hecke algebra is semi-simple, we can derive primitive idempotents of the form $\square_{\omega_I}h'\nabla_{\omega_J}h''$ (see Corollary 3 below).

Our main concern here is the combinatorial structure of the spaces $\square_{\omega_I}\mathcal{H}\nabla_{\omega_J}$ and of the right ideals

$$\mathfrak{I}(I,J) = \square_{w_I}\mathcal{H}\nabla_{w_J}\mathcal{H},$$

i.e. the sum of the ideals $\square_{w_I}h\nabla_{w_J}\mathcal{H}$ for $h \in \mathcal{H}$ ($I,\ J$ are arbitrary compositions). A motivation also to the study of these more general spaces is the exploration of alternative computations for elements of Kazdhan-Lusztig basis as it can be easily checked that the elements ∇_{ω_I} and $\square_{\omega_I}$ belong to invariant bases.

The paper is organized as follows. In the first section, we give the classical isomorphism

$$e.\mathcal{H}.f \cong \mathrm{Hom}_{\mathcal{H}}(f.\mathcal{H}, e.\mathcal{H}).$$

Dimensions and bases of the intertwining space $\Box_{\omega_I}.\mathcal{H}.\nabla_{\omega_J}$ are then given. These bases are indexed by well-known combinatorial objects, namely matrices with entries in $\{0,1\}$ and prescribed row (and column) sums.

The second part is devoted to the study of the structure of $\mathfrak{I}(I,J)$ (isotypic decomposition, bases and characters).

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I. Intertwining spaces.

I.1. *Intertwining two right direct summand ideals.* Let $\mathcal{H}$ be a k-algebra (k is a ring) and U,V two right $\mathcal{H}$-modules. In classical representation theory [2] $\mathrm{Hom}_{\mathcal{H}}(U,V)$ is called the *intertwining space* between U and V. In case U,V are principal (right) ideals of $\mathcal{H}$ generated by idempotents (i.e. factors of decompositions $\mathcal{H}=\mathcal{I}\oplus\mathcal{J}$ where $\mathcal{I}$ and $\mathcal{J}$ are ideals) the space $\mathrm{Hom}_{\mathcal{H}}(U,V)$ has a concrete realization within $\mathcal{H}$.

PROPOSITION 1. *Let e,f be two idempotents of $\mathcal{H}$. Then the mapping*

$$\psi : e\mathcal{H}f \longrightarrow \mathrm{Hom}_{\mathcal{H}}(f\mathcal{H}, e\mathcal{H})$$

defined by

$$\psi(ehf) : fx \longrightarrow ehfx$$

is an isomorphism.

Proof. It is obvious that $\psi(ehf)\in\mathrm{Hom}_{\mathcal{H}}(f\mathcal{H},e\mathcal{H})$. Now, for $\theta\in\mathrm{Hom}_{\mathcal{H}}(f\mathcal{H},e\mathcal{H})$ we have $\theta(f)\in e\mathcal{H}$, but $\theta(f)=\theta(f^2)=\theta(f)f\in e\mathcal{H}f$. The reader may then easily verify that

$$\phi : \mathrm{Hom}_{\mathcal{H}}(f\mathcal{H}, e\mathcal{H}) \longrightarrow e\mathcal{H}f$$

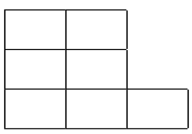
defined by $\phi(\theta)=\theta(f)$ is the inverse isomorphism. ∎

I.2. *Background for symmetrizers in the Hecke algebra.* Let us begin with some standard facts about combinatorics of partitions.

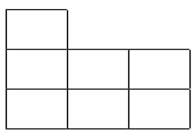
Let $I=(i_1,i_1,\cdots,i_k)$ be a vector with (strictly) positive integral entries. This object is called a *composition* of $|I|=\sum_{1\le r\le k} i_r$ whereas the entries (i_r) are called the *parts* of I. The set

$$CS(I)=\{i_1,i_1+i_2,\cdots,|I|\}=\Big\{\sum_{1\le r\le s} i_r\Big\}_{1\le s\le k}$$

is the set of *cumulated sums* of I. A decreasing composition J is called a *partition* of $|J|$ and we write $J\vdash|J|$. Partitions can be represented by plane diagrams of boxes, called *Ferrers' diagrams.* Such a diagram can be transposed (i.e. transformed by the orthogonal symmetry with axis $x=y$), the result being the diagram of the *conjugate partition* $I^{\sim}$. For example



Partition 322



331, conjugate of 322

Now, to each composition, there corresponds a unique partition I^0 obtained by reordering the parts and we will say that two compositions $I,\ J$ are *conjugate* iff $I^0 = J^{0\,\sim}$ (see [1]).

We will also need some well known results about the Hecke algebra of the symmetric group (which is a deformation of the group algebra).

Let k be a ring, $q \in k,\ n \in \mathbb{N}^*$. The Hecke algebra $\mathcal{H}_{n,q}^k$ is the associative k-algebra with unit presented with generators $(T_i)_{1\le i<n}$ and relations

$$T_iT_j = T_jT_i, \quad |i-j| \ge 2$$
$$(T_i - 1)(T_i + q) = 0, \quad 1 \le i < n$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad 1 \le i < n-1.$$

As a consequence of Murphy and Diper-James papers [15], [16], [3] it turns out that, if $q.[n]!_q$ is invertible in k ($[r]_q := 1 + q + \cdots q^{r-1}$ and $[n]!_q = [1]_q.[2]_q.\cdots[n]_q$), then $\mathcal{H}_{n,q}^k$ is a direct product of full matrix algebras.

$$\mathcal{H}_{n,q}^k \cong \prod_{\lambda\vdash n} \mathcal{M}(n_\lambda, k) \tag{I.1}$$

n_λ being the dimension of the irreducible representation associated with the partition λ of the symmetric group (the numbers n_λ can be computed directly by the "hook formula" [5]).

If k is a field, (I.1) implies that $\mathcal{H}_{n,q}^k$ is semi-simple and the decomposition (I.1) is unique. Each component $\mathcal{M}(n_\lambda, k)$ is called a *block* of $\mathcal{H}_{n,q}^k$, its neutral element is a minimal central idempotent that we will denote by e_λ and call the *central idempotent of shape* λ. This *idempotent* splits into indecomposables ones that are *primitive idempotents of shape* λ. We here briefly indicate how to construct an exhaustive (this means at least one for each shape) family of them.

Let $I = (i_1, i_2, \cdots i_k)$ be a composition of n. With I we associate a decomposition

$$[1, n] = I_1 \uplus I_2 \uplus \cdots \uplus I_k$$

where I_s is the interval $[1 + i_1 + \cdots + i_{s-1},\ i_1 + i_2 + \cdots i_s]$.

The subgroup

$$\mathfrak{S}_I = \{\sigma \in \mathfrak{S}_n \mid \forall s \le k\ \sigma(I_s) = I_s\}$$

is also the subgroup generated by the elementary transpositions $\sigma_i = (i, i+1)$ with $i \notin CS(I)$. This subgroup is called the *Young subgroup* associated with I. $\mathfrak{S}_I$ is naturally isomorphic (by restrictions) to

$$\mathfrak{S}_{i_1} \times \mathfrak{S}_{i_2} \cdots \times \mathfrak{S}_{i_k} \tag{I.2}$$

and, in particular, its cardinality is $I! := i_1!i_2!\cdots i_k!$. There is a unique longest element ω_I in $\mathfrak{S}_I$ which acts by reversion in every interval I_s. More precisely, with $0 \le h \le i_s - 1$

$$\omega_I(1 + i_1 + \cdots i_{s-1} + h) = i_1 + i_2 \cdots + i_s - h.$$

Define elements

$$\Box_I := \sum_{\mu\in\mathfrak{S}_I} T_\mu; \quad \nabla_I := \sum_{\mu\in\mathfrak{S}_I} (-q)^{l(\omega_I)-l(\mu)} T_\mu = \sum_{\mu\in\mathfrak{S}_I} (-q)^{l(\omega_I\mu)} T_\mu$$

In the spirit of [5], they will be called the *symmetrizer* (resp. *antisymmetrizer*) relative to $\mathfrak{S}_I$. These elements act as scalars on the elementary generators of $\mathfrak{S}_I$ and therefore define idempotents (see [4]).

PROPOSITION 2. *For $i \notin CS(I)$ we have*

i) $T_i\square_I = \square_I T_i = q\square_I$

ii) $T_i\nabla_I = \nabla_I T_i = -\nabla_I$

iii) *If $q, I! \in k^*$ then $\frac{1}{q.I!}\square_I$ and $\frac{1}{I!}\nabla_I$ are idempotents.*

iv) *If I, J are conjugate, the spaces $\square_I.\mathcal{H}.\nabla_J$ and $\nabla_J.\mathcal{H}.\square_I$ are one-dimensional.*

Proof. i), ii) For every $i \notin CS(I)$ let R_i (resp. D_i) be the set of $\mu \in \mathfrak{S}_I$ such that $l(\sigma_i\mu) = l(\mu) + 1$ (resp. $l(\sigma_i\mu) = l(\mu) - 1$). It is straightforward to see that $D_i = \sigma_i.R_i$. Then

$$\square_I = (\sum_{\mu\in R_i} T_\mu + \sum_{\mu\in D_i} T_\mu) = (1 + T_i)(\sum_{\mu\in R_i} T_\mu),$$

and the claim follows from the fact that $T_i(T_i + 1) = q(T_i + 1)$. The other identities of i) and ii) have a similar proof.

iii) is an immediate consequence of i) and ii).

iv) If I, J are conjugate, then it is easy to verify that (see also below Remark 6) there is only one $\nu \in \mathfrak{S}_n$ such that

$$i \notin CS(I),\ j \notin CS(J) \Longrightarrow l(\sigma_i\nu) = l(\nu) + 1,\ l(\nu\sigma_j) = l(\nu) + 1.$$

It follows from i) and ii) that $\square_I.T_\nu.\nabla_J$ is a generator of $\square_I.\mathcal{H}.\nabla_J$. Now, from the general theory of Coxeter groups, one has $l(\omega_I.\nu.\omega_J) = l(\omega_I) + l(\nu) + l(\omega_J) = m$ and then

$$\square_I.T_\nu.\nabla_J = T_{\omega_I\nu\omega_J} + \sum_{l(\mu)<m} R_\mu T_\mu.$$

This proves that $\square_I.T_\nu.\nabla_J$ is torsion free and then the claim follows. ∎

By virtue of the general theory of semi-simple algebras, Proposition 2.iv) above implies the corollary:

COROLLARY 3. *Suppose that k is a field and $\mathcal{H}^k_{n,q}$ is semi-simple. Let I, J be two conjugate compositions. Then there exist elements $\mu', \mu'' \in \mathfrak{S}_n$ such that $\square_I T_{\mu'}\nabla_J T_{\mu''}$ is a primitive idempotent of shape I.*

In the previous case, the elements $\square_I h'$ and $\nabla_J h''$ can be defined by means of a solution of the Yang-Baxter equation (see Remark 6.ii) below). Let us say now a word on this construction.

Recall that the action of $\mathfrak{S}_n$ from the right on the words of length n, is defined by

$$(a_1a_2\cdots a_n).\sigma := a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}.$$

Let $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_p}$ be a reduced decomposition of the permutation μ. The fact that this decomposition is reduced implies that the sequence $w_0, w_1, \dots$ defined inductively by:

$$\begin{cases} w_0 := 12\cdots n \\ w_k := w_{k-1}.\sigma_{i_k}, \end{cases}$$

is such that, at each step, $w_k = uijv$ and $w_{k+1} = ujiv$ with $i < j$. Then supposing that $[r]_q \in k^*$ for $r < n$ and setting $s_k := \frac{[j-i-1]_q}{[j-i]_q}$ we deduce that the products

$$(\nabla_{i_1} + s_1)(\nabla_{i_2} + s_2)\cdots(\nabla_{i_p} + s_p)$$
$$\text{and } (\square_{i_1} - s_1)(\square_{i_2} - s_2)\cdots(\square_{i_p} - s_p)$$

do not depend on the choice of the reduced decomposition of μ, but only of μ (see [4] for a complete proof). We will denote the first product by ∇_μ and the second one by $\square_\mu$.

The link with the symmetrizers become transparent with the following formulas [4]

$$\Box_{\omega_I} = \sum_{h\in\mathfrak{S}_I} T_h \qquad \nabla_{\omega_I} = \sum_{h\in\mathfrak{S}_I} (-q)^{l(\omega_I h)} T_h$$

R e m a r k 4. It is not hard to see that the leading term of $\Box_\mu$ (resp. ∇_μ) is T_μ. In other words, we have

$$\Box_\mu = T_\mu + \sum_{l(\nu)<l(\mu)} R_\nu(q) T_\nu \qquad \nabla_\mu = T_\mu + \sum_{l(\nu)<l(\mu)} S_\nu(q) T_\nu$$

with $R_\nu, S_\nu \in k(q)$.

From now on, we will write $\Box_{\omega_I}$ (resp. ∇_{ω_I}) instead of $\Box_I$ (resp. ∇_I).

I.3. *Bases for* $\Box_{\omega_I}.\mathcal{H}.\nabla_{\omega_J}$. In this section, we construct a basis of $\Box_{\omega_I}\mathcal{H}\nabla_{\omega_J}$ consisting of elements of the type $\Box_{\omega_I} T_\sigma \nabla_{\omega_J}$. This basis is closely related to the *minimal representatives* of the double classes $\mathfrak{S}_I\backslash\mathfrak{S}_n/\mathfrak{S}_J$ (see [11]). Let us state their properties.

PROPOSITION 5. *Let I, J be two compositions of n, and $W(I,J)$ be the set of $\sigma \in \mathfrak{S}_n$ such that*

$$l(\omega_I \sigma \omega_J) = l(\omega_I) + l(\sigma) + l(\omega_J). \tag{I.3}$$

Then

i) *$W(I,J)$ is the set of elements of minimal length in each double class $\mathfrak{S}_I.\mu.\mathfrak{S}_J$ and, hence, $W(I,J)$ is a set of representatives for $\mathfrak{S}_I\backslash\mathfrak{S}_n/\mathfrak{S}_J$.*

ii) *$(\Box_{\omega_I} T_\sigma \nabla_{\omega_J})_{\sigma\in W(I,J)}$ is a basis of the k-module $\Box_{\omega_I}\mathcal{H}\nabla_{\omega_J}$.*

P r o o f. i) is classical.

ii) The family $(\Box_{\omega_I} T_\mu \nabla_{\omega_J})_{\mu\in W(I,J)}$ is linearly independent over k as the dominant term of $\Box_{\omega_I} T_\mu \nabla_{\omega_J}$ is $T_{\omega_I \mu \omega_J}$. Now, let M be the linear span of the elements $(\Box_{\omega_I} T_\mu \nabla_{\omega_J})_{\mu\in W(I,J)}$. It is sufficient to prove that every $\Box_{\omega_I} T_\nu \nabla_{\omega_J}$ belongs to M for $\nu \in \mathfrak{S}_n$. Suppose that it is not the case, then there exists $\alpha \in \mathfrak{S}_n$ of minimal length with $\Box_{\omega_I} T_\alpha \nabla_{\omega_J} \notin M$. Now, $\alpha \notin W(I,J)$ and then $\alpha = \sigma_i \alpha_1$ for some $i \notin CS(I)$ or $\alpha = \alpha_2 \sigma_j$ for some $j \notin CS(J)$. Then, for example in the first case,

$$\Box_{\omega_I} T_{\sigma_i \alpha_1} \nabla_{\omega_J} = \Box_{\omega_I} T_{\sigma_i} T_{\alpha_1} \nabla_{\omega_J} = q \Box_{\omega_I} T_{\alpha_1} \nabla_{\omega_J}$$

— a contradiction. The other case is similar. ■

R e m a r k 6. i) Let $I = (i_1, i_2, \cdots i_r)$, $J = (j_1, j_2, \cdots j_s)$. Then the set $W(I,J)$ is in one-to-one correspondence with the set of $r \times s$-matrices with entries in $\{0,1\}$ and column (resp. row) sums given by I (resp. J). We first need a description of $\omega_I.W(I,J)$, the elements of which have a simple characterization in terms of subwords on consecutive letters and increasing conditions. Namely an element μ belongs to $\omega_I.W(I,J)$ iff:

(a) μ has the decreasing subwords of ω_I.

(b) μ is increasing on each interval defined by J.

We now describe the correspondence on an example (for practical reasons, as the parts involved here will be less than 10, we will write $I = 1311$ instead of $I = (1,3,1,1)$).

For example, with $I = 1311$, $J = 231$, the permutations μ are (written as words) $143562, 451362, 461352$. The matrices to be found have 3 rows and 4 columns. Index the columns by the subwords (condition (a) above), here $(1,\ 432,\ 5,\ 6)$ and the rows by the intervals of places where the permutation has to be increasing (condition (b) above), here $12, 345, 6$. For the first permutation we have

subwords	1	432	5	6	↓ *places*
	•	•	–	–	12
	–	•	•	•	345
	–	•	–	–	6

Permutation 143562

And, for the two remaining elements of $\omega_{1311}.W(1311, 231)$

1	432	5	6	
–	•	•	–	12
•	•	–	•	345
–	•	–	–	6

Permutation 451362

1	432	5	6	
–	•	–	•	12
•	•	•	–	345
–	•	–	–	6

Permutation 461352

ii) For $\nu \in W(I, J)$, the leading term of $\Box_{\omega_I} T_\nu \nabla_{\omega_J}$ is $T_{\omega_I \nu \omega_J}$, because the lengths add (see Proposition 5). So any family of the form

$$(\Box_{\omega_I} B_\nu \nabla_{\omega_J})_{\nu \in W(I,J)}$$

such that the leading term of B_ν is T_ν, is also a basis of $\Box_{\omega_I} \mathcal{H} \nabla_{\omega_J}$. Its transition matrix with respect to

$$(\Box_{\omega_I} T_\nu \nabla_{\omega_J})_{\nu \in W(I,J)}$$

is triangular. In particular, this is the case of $\Box_{\omega_I \nu} \nabla_{\omega_J}$ as one has the factorization $\Box_{\omega_I \nu} = \Box_{\omega_I} B_\nu$ for some B_ν with leading term T_ν (see [4] for details).

iii) In general, we deduce from Propositions 1 and 5 that

$$|W(I, J)| = \sum_{\lambda \vdash n} m'_\lambda . m''_\lambda$$

where m'_λ (resp. m''_λ) is the number of irreducible components of shape λ in $\Box_{\omega_I}.\mathcal{H}$ (resp. $\nabla_{\omega_J}\mathcal{H}$). In particular, if I, J are conjugate compositions then $|W(I, J)| = 1$ and $\Box_{\omega_I} \mathcal{H} \nabla_{\omega_J}$ is one-dimensional. In the semi-simple case, this allows to construct primitive idempotents for all possible shapes (see [4]).

iv) Notice that it can happen that $W(I, J) = \emptyset$. This is the case, for example, with $I = J = [n],\ n \geq 2$.

I.4. *Characters.* When k is of characteristic 0 and $q = 1$ one can compute the characteristics of $\Box_{\omega_I} \mathcal{H}$ and $\nabla_{\omega_J} \mathcal{H}$ considered as right $\mathfrak{S}_n$-modules. Here, we have

$$\mathrm{char}(\Box_{\omega_I} \mathcal{H}) = S^I := S_{i_1} S_{i_2} \cdots S_{i_k}, \quad \mathrm{char}(\nabla_{\omega_J} \mathcal{H}) = \Lambda^J := \Lambda_{i_1} \Lambda_{i_2} \cdots \Lambda_{i_k}.$$

If I, J are conjugate, these symmetric functions have only one common irreducible component S_λ, λ being the diagram with the row lengths given by I and the column lengths given by J (in fact, we could write $\lambda = I$) [12], [13]. We recover the fact that $\Box_{\omega_I} \mathcal{H} \nabla_{\omega_J}$ is one-dimensional in this case.

I.5. *Permutation of parts.* Suppose that the symmetric group $\mathfrak{S}_k$ acts on compositions of n into k parts by

$$I^\sigma = (i_1, i_2 \cdots i_k)^\sigma = (i_{\sigma(1)}, i_{\sigma(2)}, \cdots i_{\sigma(k)}).$$

Here we show that $\Box_{\omega_I}$ (resp. ∇_{ω_I}) and $\Box_{\omega_{I^\sigma}}$ (resp. $\nabla_{\omega_{I^\sigma}}$) are conjugate.

PROPOSITION 7. *Let I be a composition of n belonging to $\mathbb{N}^k$. For every $\sigma \in \mathfrak{S}_k$ there exists $\tau(\sigma) \in \mathfrak{S}_n$ such that*

$$\square_{\omega_{I^\sigma}} = T_{\tau(\sigma)} \square_{\omega_I} T_{\tau(\sigma)}^{-1} \qquad \nabla_{\omega_{I^\sigma}} = T_{\tau(\sigma)} \nabla_{\omega_I} T_{\tau(\sigma)}^{-1}.$$

Proof. This is a direct consequence of the fact that the Young subgroups $\mathfrak{S}_{\omega_{I^\sigma}}$ and $\mathfrak{S}_{\omega_I}$ are conjugate. ∎

COROLLARY 8. $\mathfrak{I}(I^{\sigma_1}, J^{\sigma_2}) = T_{\tau(\sigma_1)} \mathfrak{I}(I, J)$.

This explains how the results for general I, J can be computed from the case when I, J are partitions.

II. Structure of $\mathfrak{I}(I, J)$.

II.1. *Isotypic decomposition of* $\mathfrak{I}(I, J)$. From now on we suppose that k is a field and $q.[n!]_q \neq 0$ (this implies that $\mathcal{H}_{n,q}$ is semi-simple). The following result is classical and can be derived from the thesis of Hoefsmith [7] or the orthogonal family of idempotents of Jucys-Murphy [10], [16].

PROPOSITION 9. *There is a family of orthogonal central idempotents $(e_\lambda)_{\lambda \vdash n}$. Hence*

$$\mathcal{H}_{n,q} = \bigoplus_{\lambda \vdash n} e_\lambda . \mathcal{H}_{n,q},$$

each summand being a matrix algebra

$$e_\lambda . \mathcal{H}_{n,q} \cong \mathcal{M}(n_\lambda, k)$$

where n_λ is the dimension of the representation associated with the partition λ.

We derive the following decomposition of $\mathfrak{I}(I, J)$.

THEOREM 10. $\mathfrak{I}(I, J) = \bigoplus_{\substack{\lambda \vdash n \\ (S_\lambda, \Lambda^J) \neq 0}} \square_{\omega_I} \mathcal{H}.e'_\lambda.\mathcal{H}$ *where e'_λ is any minimal idempotent of shape λ. In particular, there exists $h', h'' \in \mathcal{H}$ such that $e'_\lambda = \square_{\omega_\lambda} h' \nabla_{\omega_{\lambda'}} h''$.*

Proof. Of course, we have

$$\mathfrak{I}(I, J) = \bigoplus_{\lambda \vdash n} e_\lambda . \mathfrak{I}(I, J) = \bigoplus_{\substack{\lambda \vdash n \\ (S_\lambda, \Lambda_J) \neq 0}} \square_{\omega_I} . \mathcal{H}.e_\lambda.\nabla_{\omega_J}.\mathcal{H}$$

as, if $(S_\lambda, \Lambda^J) = 0$, then $\mathcal{H}.e_\lambda.\nabla_{\omega_J}.\mathcal{H} = \{0\}$ and $\mathcal{H}.e_\lambda.\mathfrak{I}(I, J) = \square_{\omega_I}.\mathcal{H}.e_\lambda.\nabla_{\omega_J}.\mathcal{H} = \{0\}$ (indeed, this is the case also if $(S_\lambda, S^J) = 0$). Otherwise, there exists a minimal idempotent e'_λ of shape λ such that $\nabla_{\omega_J}.\mathcal{H} \supset e'_\lambda.\mathcal{H}$, and hence

$$\mathcal{H}.e_\lambda.\mathcal{H} \supset \mathcal{H}.e_\lambda \nabla_{\omega_J}.\mathcal{H} \supset \mathcal{H}.e'_\lambda.\mathcal{H} = \mathcal{H}.e_\lambda.\mathcal{H}.$$

This proves the claim. ∎

Remark 11. Let $\mathfrak{J}$ be an ideal of $\mathcal{H}$. Then, with the previous notation, we have

$$\mathfrak{J} e_\lambda = \mathfrak{J} e'_\lambda . \mathcal{H}.$$

This ideal is the sum of the components of "type λ" in $\mathfrak{J}$. We will denote this ideal by $\mathfrak{J}_\lambda$. One has

$$\dim(\mathfrak{J}_\lambda) = m_\lambda . n_\lambda$$

where m_λ is the multiplicity of the representation associated with λ and n_λ — its dimension.

Here, we have

$$e_\lambda.\mathfrak{I}(I,J) = \Box_{\omega_I}.\mathcal{H}.e'_\lambda.\mathcal{H} = (\Box_{\omega_I}.\mathcal{H})_\lambda.$$

II.2. *Character and basis of* $\mathfrak{I}(I,J)$. Let, as above, m_λ be the multiplicity of $e'_\lambda.\mathcal{H}$ in $\mathfrak{I}(I,J)$. From Remark 11 we deduce that, if $(S_\lambda, \Lambda_J) \neq 0$, then

$$m_\lambda = \dim(\mathrm{Hom}_{\mathcal{H}}(\Box_{\omega_I}.\mathcal{H}, e'_\lambda.\mathcal{H})) = \dim(\Box_{\omega_I}.\mathcal{H}.e'_\lambda).$$

This allows us to construct bases of $\mathfrak{I}(I,J)$.

PROPOSITION 12. *Let* $(\Box_{\omega_I}.h_i e'_\lambda)_{i\in I_\lambda}$ *be a basis in each non-vanishing* $\Box_{\omega_I}.\mathcal{H}.e'_\lambda$ *and* $(e'_\lambda.h'_j)_{j\in J_\lambda}$ *a basis of* $e'_\lambda.\mathcal{H}$. *Then*

$$(\Box_{\omega_I} h_i e'_\lambda h'_j)_{(i,j)\in I_\lambda\times J_\lambda}$$

is a basis of $\mathfrak{I}_\lambda(I,J)$.

Proof. Let $(\Box_{\omega_I}.h_i e'_\lambda)_{i\in I_\lambda}$ (resp. $(e'_\lambda.h'_j)_{j\in J_\lambda}$) be a basis of $\Box_{\omega_I}.\mathcal{H}.e'_\lambda$ (resp. $e'_\lambda.\mathcal{H}$). It is obvious that $(\Box_{\omega_I} h_i e'_\lambda h'_j)_{(i,j)\in I_\lambda\times J_\lambda}$ generates $\mathfrak{I}_\lambda(I,J)$. But Remark 11 states that $\dim(\mathfrak{I}_\lambda(I,J)) = \mathrm{card}\,(I_\lambda \times J_\lambda)$ and, then, this family is a basis of $\mathfrak{I}_\lambda(I,J)$. ∎

Remark 13. Since

$$\mathfrak{I}(I,J) = \bigoplus_{\lambda\vdash n} \mathfrak{I}_\lambda(I,J),$$

the sum being taken over all λ such that $(S_\lambda, S^I)(S_\lambda, \Lambda^J) \neq 0$ (otherwise $\mathfrak{I}_\lambda(I,J) = \{0\}$), we get a basis of $\mathfrak{I}(I,J)$ by union of bases of Proposition 12.

EXAMPLE. With $I = 211$, $J = 22$, we have $S^{211} = S_4 + 2S_{13} + S_{22} + S_{112}$ and $\Lambda^{22} = S_{22} + S_{112} + S_{1111}$ then $\mathfrak{I}_\lambda(I,J) \neq 0$ for $\lambda = 211$ and $\lambda = 22$. We get a basis of $\mathfrak{I}_{211}$: $\Box_{2341}\nabla_{3214}$ and a basis of $\mathfrak{I}_{22}$: $\Box_{2413}\nabla_{2143}$. Then the desired basis of $\mathfrak{I}(211,22)$ is

$$\{\Box_{2341}\nabla_{3214}, \Box_{2413}\nabla_{2143}\}.$$

III. Conclusion. Bases and characters of the ideal $\mathfrak{I}(I,J)$ have been described. In general, bases for an isotypic component are indexed by two indices. For the second range, we have a nice description for a basis of $e'_\lambda.\mathcal{H}$ in case e'_λ is chosen to be of the form $\Box_{\omega_\lambda} h' \nabla_{\omega_{\lambda'}} h''$ [4], [3].

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