Degrees of entanglement for multipartite systems

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We propose a unified mathematical scheme, based on a classical tensor isomorphism, for characterizing entanglement that works for pure states of multipartite systems of any number of particles. The degree of entanglement is indicated by a set of absolute values of the determinants for each subspace of the multipartite systems. Our scheme provides a characterization of the degrees of entanglement when the qubits are measured or lost successively, and leads naturally to necessary and sufficient conditions for multipartite pure states to be separable. For systems with a large number of particles, a rougher indication of the degree of entanglement is provided by the set of mean values of the determinantal values for each subspace of the multipartite systems.

I. INTRODUCTION

As Schrödinger once said, "...entanglement is *the* characteristic trait of quantum mechanics, the one that forces its entire departure from classical lines of thought. " [1]. Entanglement now lies at the heart of quantum information. As such, characterizing entanglement is currently one of the main tasks in quantum information theory.

For bipartite states, it is convenient that a single determinantal condition is enough to discriminate between separability and entanglement. This is the so-called concurrence [2]. Unfortunately, for multi-partite systems it seems impossible to classify the degree of entanglement with a single quantity like the concurrence.

In the literature, most people have been interested in defining measures of entanglement which are invariant under local unitary transformations. This is motivated by the belief that one has the freedom to measure the system in any direction, and that entanglement should be invariant under such freedom of measurement (for a general discussion of various entanglement measures, see e.g.: [3]).

For tripartite states, a measure of entanglement invariant under local unitary transformations was given in [4]. This measure, called the 3-tangle in [4], turns out to be just the Cayley hyperdeterminant (Det) of the corresponding system. Such tangle type formulation of entanglement measure has been generalised to N-qubit systems in [5].

Unfortunately, the 3-tangle cannot be a complete measure of entanglement for tripartite states. One needs only mention the fact that the Cayley hyperdeterminant for the well-known GHZ-state [6] and the W-state [7] are one and zero, respectively. However, the W-state *is* a genuinely entangled tripartite state; so Det = 0 does not provide a criterion for separability as the simple concurrence does in the bipartite case. Further, one knows that the W-state

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is in fact more robust under measurement-collapse than the GHZ-state [8]. For example, if Alice measures the (first) qubit of the GHZ-state to be 0, then this leaves the separable state $|00\rangle$. And similarly for any measurement of any qubit in any of the three subspaces for the tripartite GHZ-state. On the other hand, the determination (*in the same basis*) of the value "0" of any qubit in any space for the W-state still leaves the state (maximally) entangled, and only if the value "1" is measured will the collapsed state be separable. Again this difference is not reflected in the values of the Cayley hyperdeterminant for these two states. So one needs additional indicators to reflect this difference in entanglement properties.

Motivated by such observation, we propose to consider entanglement properties of multipartite systems as each one of its qubits is successively measured in the same basis, in order to supplement other measures which concern mainly the invariant properties of entanglement under local unitary transformations without making any measurement, i.e., without losing any qubit.

One such scheme was presented in [9], where six additional indicators (sub-determinants) were introduced to supplement the Cayley hyperdeterminant. Together these 7 numbers distinguish the GHZ-state, the W-state and other tripartite states, and - more significantly - provide a necessary and sufficient criterion for the separability of a tripartite pure state. Of these 7 numbers, the Cayley hyperdeterminant is linked to the tripartite system, and the six sub-determinants indicate degrees of entanglement of the 6 possible bipartite systems when one of the three qubits is measured.

One would like to extend the scheme in [9] to multipartite cases. Unfortunately, it is not easy to generalize the Cayley hyperdeterminant to higher dimensional systems (for a generalization to four qubits, see e.g., [10]). It is therefore advantageous to find a classification scheme of entanglement that applies to any number of particles. There have been some attempts to characterize entanglement based on certain invariants under local unitary transformations (see e.g., [11]).

In this note we propose a unified mathematical scheme that works for multipartite systems of any number of particles. This scheme is based on a classical tensor isomorphism, as expounded - for example - by Bourbaki [12]. It provides an indication of the degrees of entanglement when the qubits are measured or lost successively.

II. BIPARTITE SPACES, CONCURRENCE

We first consider the property of the *separability* of *bipartite pure states*.

Definition 1 (Separable state) (see e.g.: [13]) *An element $v \in V_1 \otimes V_2$ is said to be* separable *(equivalently, non-entangled) if v can be written as a direct product*

$$v = v_1 \otimes v_2 \quad v_1 \in V_1, v_2 \in V_2.$$

Note that this statement is basis-dependent. For example, if V_1 and V_2 are qubit spaces (that is, their elements are complex 2-vectors of norm 1), then by a suitable non-local unitary transformation U ($U \in U(4)$) given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{1}$$

the separable state $(\alpha|0\rangle + \beta|1\rangle)|0\rangle$ is transformed to

$$U(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|11\rangle, \tag{2}$$

which is non-separable insofar as $\alpha, \beta \neq 0$. The transformation U in Eq. (1), commonly called the CNOT-gate, is non-local, and is used to generate entangled states in quantum computation.

However, the property of being separable is obviously conserved under a local unitary transformation (and even by local transformations) defined as follows:

Definition 2 (Local unitary transformation) *The transformation U on $V_1 \otimes V_2$ is said to be a local unitary transformation if $U = U_1 \otimes U_2$ where U_i is a unitary operator acting on V_i.*

Clearly the above considerations hold for general direct product spaces.

For bipartite pure qubit states a single determinantal condition is enough to discriminate between separability and entanglement.

Let $v \in V = V_1 \otimes V_2$, where V_1 and V_2 are two-dimensional (qubit) vector spaces with basis

$$\{e_0 \equiv |0\rangle \equiv [1,0]^T, e_1 \equiv |1\rangle \equiv [0,1]^T\}. \tag{3}$$

In general we may write $v \in V$ as

$$v \in V_1 \otimes V_2 = \sum_{i,j=0}^{1} c_{ij} \; e_i \otimes e_j \,. \tag{4}$$

If v is separable, then

$$v = (x_0 e_0 + x_1 e_1) \otimes (y_0 e_0 + y_1 e_1), \tag{5}$$

and so

$$c_{ij} = x_i y_j \quad \{i,j = 0,1\} \tag{6}$$

from which we deduce that the matrix c of coefficients c_{ij} has determinant zero, $\det c = 0$ or, equivalently, is of rank 1. This condition is clearly necessary and sufficient.

In fact, by suitably normalizing, we may use this determinant to provide a *measure* of entanglement for pure states called the *concurrence* $\mathcal{C}$, with

$$\mathcal{C} = 2|\det c|. \tag{7}$$

This measure of entanglement varies between 0 (separable) and 1 (maximally entangled) and may be conveniently extended to mixed states [2]. It may be shown by direct calculation that this measure of entanglement is invariant under local unitary transformations (see for example [9]).

For tripartite (and higher) spaces we shall show that a single number is *not* sufficient to describe separability (or the measure of entanglement).

III.   FINITE VECTOR SPACES

Since we shall be regarding pure-state entanglement as essentially a property of the elements of direct product vector spaces, we briefly review some definitions.

A.   Direct Product Spaces

We consider $\mathcal{V}$ a (finite) direct product of m (finite) vector spaces:

$$\mathcal{V} = V_1 \otimes V_2 \otimes \ldots \otimes V_m \tag{8}$$

where V_r $(r = 1, 2, \ldots, m)$ has basis

$$\{e_0^{(r)}, e_1^{(r)}, \ldots, e_{n_r-1}^{(r)}\}$$

and the underlying field will be taken to be $\mathbb{C}$. The basis elements of the dual $V^{*(r)}$ are given by

$$\{e_0^{*(r)}, e_1^{*(r)}, \ldots, e_{n_r-1}^{*(r)}\}$$

whose elements are defined by

$$\langle e_i^{*(r)}, e_j^{(r)} \rangle = \delta_{ij}, \quad i,j = 0, 1, \ldots, n_r - 1$$

using a standard notation for the action of a dual element on a vector. The space V has product basis

$$\{e_{s_1}^{(1)} \otimes e_{s_2}^{(2)} \otimes \ldots \otimes e_{s_m}^{(m)} | s_i = 0 \ldots n_i - 1\} \tag{9}$$

The dual space $\mathcal{V}^*$ also has dimension $\Pi_{r=1}^m n_r$, with the standard dual basis

$$\{e_{s_1}^{*(1)} \otimes e_{s_2}^{*(2)} \otimes \ldots \otimes e_{s_m}^{*(m)} | s_i = 0 \ldots n_i - 1\}. \tag{10}$$

B.   Rank of an element of a direct product space

As a preliminary we consider the direct product of *two* spaces, $E = \mathbb{C}^2$, $F = \mathbb{C}^2$. For $u \in E \otimes F$ we have $u = \Sigma_i\, x_i \otimes y_i$ $(x_i \in E, y_i \in F)$. Following [12], we may define a linear map u_1 corresponding to u by

$$u_1 : E^* \rightarrow F \tag{11}$$
$$x^* \mapsto \Sigma_i\langle x^*, x_i\rangle y_i.$$

Definition 3 (Rank of an element of a product space) [12] *The Rank of $u \in E \otimes F$ is defined as the rank of u_1 (as a linear map).*

Example III.1 (Bipartite qubit state) Let $u \in V \otimes V$ where V is the qubit space $\mathbb{C}^2$. Then $u = \Sigma_i x_i \otimes y_i$, and the corresponding linear map $u_1 : V^* \rightarrow V$ is given by

$$u_1(v^*) = \Sigma_i\langle v^*, x_i\rangle y_i \quad (x_i, y_i \in V) \tag{12}$$

If u consists of a single product, then choosing a basis,

$$u = x \otimes y \quad x = \begin{bmatrix} a \\ b \end{bmatrix} \quad y = \begin{bmatrix} c \\ d \end{bmatrix}. \tag{13}$$

Now for any $v \in V$ given by

$$v = \begin{bmatrix} v_1 \\ v_1 \end{bmatrix},$$

the action of the map u_1 in (12) is

$$
\begin{aligned}
u_1(v^*) &= \left([v_1, v_2] \begin{bmatrix} a \\ b \end{bmatrix} \right) \begin{bmatrix} c \\ d \end{bmatrix} \\
&= \begin{bmatrix} ac & bc \\ ad & bd \end{bmatrix} \begin{bmatrix} v_1 \\ v_1 \end{bmatrix},
\end{aligned}
$$

Hence

$$u_1 = \begin{bmatrix} ac & bc \\ ad & bd \end{bmatrix}, \tag{14}$$

which has Rank 1. Conversely, if $u = \Sigma_i(x_i \otimes y_i)$, $\quad x_i = a_i e_0 + b_i e_1$, $y_i = c_i e_0 + d_i e_1$, then

$$u = ACe_0 \otimes e_0 + ADe_0 \otimes e_1 + BCe_1 \otimes e_0 + BDe_1 \otimes e_1, \tag{15}$$

with

$$AC \equiv \Sigma_i a_i c_i \ \ AD \equiv \Sigma_i a_i d_i \ \ BC \equiv \Sigma_i b_i c_i \ \ BD \equiv \Sigma_i b_i d_i$$

and

$$u_1 = \begin{bmatrix} AC & BC \\ AD & BD \end{bmatrix}, \tag{16}$$

where not all AC, AD, BC, BD are zero. Without loss of generality we may choose $AC \neq 0$.
Thus $\det(u_1) = 0 \Rightarrow u = (ACe_0 + BCe_1) \otimes (e_0 + \frac{AD}{AC}e_1)$ and u is separable.

We therefore have the following theorem:

Theorem III.2 *A necessary and sufficient condition for the bipartite vector $u \in V \otimes V$ to be separable is that the corresponding linear transformation u_1 be of Rank 1.*

C. Rank of an element of a multi-direct product space

We now extend the result of the previous section to a multi-direct product space. We consider the multipartite vector space $\mathcal{V}$ of Eq.(8):

$$\mathcal{V} = V_1 \otimes V_2 \otimes \ldots \otimes V_m$$

and remark that the space of homomorphisms from $V_1 \otimes V_2 \otimes \ldots \otimes V_{m-1}$ to V_m is isomorphic to $(V_1 \otimes V_2 \otimes \ldots \otimes V_{m-1})^* \otimes V_m$; that is,

$$
\begin{aligned}
\mathcal{H}om(V_1 \otimes V_2 \otimes \ldots \otimes V_{m-1}, V_m) &\simeq (V_1 \otimes V_2 \otimes \ldots \otimes V_{m-1})^* \otimes V_m \\
&\simeq V_1 \otimes V_2 \otimes \ldots \otimes V_{m-1} \otimes V_m.
\end{aligned}
\tag{17}
$$

Corresponding to an element

$$u \in \mathcal{V}, \quad u = \Sigma_i w_i \otimes z_i, \quad (w_i \in V_1 \otimes V_2 \otimes \ldots \otimes V_{m-1}, z_i \in V_m)$$

we define the linear map

$$
\begin{aligned}
u_1 : V_1^* \otimes V_2^* \otimes \ldots \otimes V_{m-1}^* &\rightarrow V_m \\
v^* &\mapsto \Sigma_i \langle v^*, w_i \rangle z_i.
\end{aligned}
\tag{18}
$$

We may therefore define the *Rank* of the direct product u by the rank of the linear map u_1 in Eq.(18).

For other than bipartite states there are of course varying degrees of separability. We propose the following definition:

Definition 4 (Partial separability of an element of a product space) *We say that the vector u*

$$u \in \mathcal{V} = V_1 \otimes V_2 \otimes \ldots \otimes V_m$$

is partially separable with respect to V_s $(1 \leq s \leq m)$ if

$$u = v \otimes z \otimes w \quad (v \in V_1 \otimes V_2 \otimes \ldots \otimes V_{s-1}, \ z \in V_s, w \in V_{s+1} \otimes V_{s+2} \otimes \ldots \otimes V_m).$$

We may then also define:

Definition 5 (Complete separability of an element of a product space) *The vector u*

$$u \in \mathcal{V} = V_1 \otimes V_2 \otimes \ldots \otimes V_m$$

is completely separable if it is partially separable with respect to $V_s, s = 1 \ldots m$.

We shall often simply say in the foregoing case that u is separable.

It therefore follows from the preceding discussion that a necessary and sufficient condition for u to be partially separable with respect to V_m is that u_1 be of rank one. When this condition is fulfilled we may write $u = w \otimes z$ (z is taken in the dimension 1 image of u_1 and the decomposition is then unique up to scalars).

For *complete separability* of u as defined above we must demand that w be completely separable, and so on recursively. This procedure then provides not only a necessary and sufficient condition for u to be (completely) separable but gives an algorithm for its factorization.

IV. TRIPARTITE STATES

We now show how the ideas discussed in Sect. III provide indicators of degrees of entanglement in tripartite systems.

A. Separable cases

As an important example, we consider a (completely) separable element of a tripartite qubit space. We take a simple direct product tripartite state (separable pure state)

$$u \in V_1 \otimes V_2 \otimes V_3 \quad (V_i \equiv \mathbb{C}^2) \tag{19}$$

with

$$u = w \otimes z, \quad (w = x \otimes y \in V_1 \otimes V_2, \quad z \in V_3). \tag{20}$$

According to the foregoing discussion, to u corresponds the linear map

$$\begin{aligned} u_1 : V_1^* \otimes V_2^* &\to V_3 \\ v* &\mapsto \langle v^*, w \rangle z \end{aligned} \quad . \tag{21}$$

Choosing a basis,

$$w = x \otimes y \quad x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \quad y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \quad z = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \tag{22}$$

and so

$$u_1 = \begin{bmatrix} x_0 y_0 z_0 & x_0 y_1 z_0 & x_1 y_0 z_0 & x_1 y_1 z_0 \\ x_0 y_0 z_1 & x_0 y_1 z_1 & x_1 y_0 z_1 & x_1 y_1 z_1 \end{bmatrix} \tag{23}$$

which has Rank 1. Eq. (23) is a direct extension of (14). Note that if we replace the term $w \otimes z$ by $\Sigma_i w_i \otimes z$ the Rank is still 1. This shows that Rank=1 is necessary but not sufficient for (complete) separability. This preliminary condition on u_1 guarantees partial separability with respect to V_3, as in Definition 4. For complete separability as defined in Definition 5 we require the further condition that the Rank of the element of $V_1 \otimes V_2$ also be 1.

B. General cases

We now discuss the important example of a general tripartite qubit state. We follow the procedure above in III C for the general multipartite case, which it illustrates.

Consider a general element u of a tripartite qubit space

$$u \in V_1 \otimes V_2 \otimes V_3 \quad (V_i \equiv \mathbb{C}^2). \tag{24}$$

We choose the standard basis as in Eq.(3), to write

$$\begin{aligned} u &= \sum_{i,j,k=0}^{1} x_{ijk} |ijk\rangle \\ &= \sum_{i,j,k=0}^{1} x_{ijk} \; e_i \otimes e_j \otimes e_k \\ &= \sum_{k=0,1} w_k \otimes z_k \quad (w_k \in V_1 \otimes V_2, \quad z_k \in V_3) \end{aligned} \tag{25}$$

with $w_0 = \sum_{ij} x_{ij0} \; e_i \otimes e_j$, $w_1 = \sum_{ij} x_{ij1} \; e_i \otimes e_j$, $z_k = e_k$. As in Eq.(18) above, we define the linear map

$$\begin{aligned} u_1 : V_1^* \otimes V_2^* &\to V_3 \\ v^* &\mapsto \sum_k \langle v^*, w_k \rangle z_k. \end{aligned} \tag{26}$$

Writing in the standard basis

$$v^* = \sum_{m,n=0}^{1} v_{mn} \; e_m^* \otimes e_n^* = (v_{00}, v_{01}, v_{10}, v_{11})$$

the relevant linear transformation in matrix form is

$$(v_{00}, v_{01}, v_{10}, v_{11}) \mapsto \begin{bmatrix} x_{000} & x_{010} & x_{100} & x_{110} \\ x_{001} & x_{011} & x_{101} & x_{111} \end{bmatrix} (v_{00}, v_{01}, v_{10}, v_{11})^T. \tag{27}$$

The condition for separability between the $V_1 \otimes V_2$ space and V_3 is that the 2×4 matrix in Eq.(27) should be of rank 1. This means, from Eq. (27), that the following six determinants

$$\begin{aligned}
\det(1)_3 &= x_{000}x_{011} - x_{001}x_{010}, \\
\det(2)_3 &= x_{000}x_{101} - x_{001}x_{100}, \\
\det(3)_3 &= x_{000}x_{111} - x_{001}x_{110}, \\
\det(4)_3 &= x_{010}x_{101} - x_{011}x_{100}, \\
\det(5)_3 &= x_{010}x_{111} - x_{011}x_{110}, \\
\text{and} \quad \det(6)_3 &= x_{100}x_{111} - x_{101}x_{110}
\end{aligned} \tag{28}$$

are identically zero. Here the subscript "3" indicates that these determinants are related to the tripartite system. The vanishing of these 6 determinants essentially means that the vectors x_{ij0} and x_{ij1} are parallel, or either one of them is a null vector. Complete separability is then attained by applying the Rank 1 condition to the 4-vector $[x_{000}, x_{010}, x_{100}, x_{110}]$ (or to $[x_{001}, x_{011}, x_{101}, x_{111}]$), namely that the determinant (subscript "2" indicates that the determinant is related to the bipartite system)

$$\det(1)_2 = \begin{vmatrix} x_{000} & x_{100} \\ x_{010} & x_{110} \end{vmatrix} \quad \text{or} \quad \det(2)_2 = \begin{vmatrix} x_{001} & x_{101} \\ x_{011} & x_{111} \end{vmatrix} \tag{29}$$

vanishes.

In the case where the 6 determinants in Eq. (28) are *not* identically zero, the third qubit is entangled with the other two. Then the two determinants in Eq. (29) provide an indicator as to whether the remaining two qubits are entangled when the third qubit is lost: $\det(1)_2 = 0$ ($\det(2)_2 = 0$) means the two qubits are separable if the third qubit is measured to be "0" ("1").

The above discussion then suggests a classification scheme of entanglement by a list of the 8 determinants given by

$$[|\det(1)_3|, |\det(2)_3|, |\det(3)_3|, |\det(4)_3|, |\det(5)_3|, |\det(6)_3|; |\det(1)_2|, |\det(2)_2|].$$

One may define a set of coarse-grained indicators of the tripartite entanglement by

$$[\mathcal{C}_3, \mathcal{C}_2], \quad \mathcal{C}_m = \frac{1}{l_m} \sum_{k=1}^{l_m} |\det(k)_m|, \tag{30}$$

where $l_3 = 6, l_2 = 2$ are the numbers of determinants for the 3- and 2-partite spaces.

In Table I we present this classification scheme for some tripartite states based on the methods of this note. As in [9], the determinants are all normalized to 1 by applying a normalization factor $1/|\det A|$ for all non-vanishing determinants.

TABLE I: Classification of some tripartite qubit states.

	State	$[|\det(k)_3|, \ k = 1, 2, \ldots, 6]$	$[|\det(k)_2|, \ k = 1, 2]$	$[\mathcal{C}_3; \mathcal{C}_2]$
General Separable State	$\Sigma a_i e_i \otimes \Sigma b_j e_j \otimes \Sigma c_k e_k$	[0,0,0,0,0,0]	[0,0]	[0;0]
GHZ-state	$(1/\sqrt{2})(|000\rangle + |111\rangle)$	[0,0,1,0,0,0]	[0,0]	$[\frac{1}{6}; 0]$
W-state	$(1/\sqrt{3})(|001\rangle + |010\rangle + |100\rangle)$	[1,1,0,0,0,0]	[1,0]	$[\frac{1}{3}; \frac{1}{2}]$
Cluster state	$(1/\sqrt{8})(|000\rangle + |001\rangle + |010\rangle - |011\rangle$	[1,0,1,1,0,1]	[1,1]	$[\frac{2}{3}; 1]$
	$+|100\rangle + |101\rangle - |110\rangle + |111\rangle)$			
ψ-state [9]	$(1/2)(|001\rangle + |010\rangle + |100\rangle + |111\rangle)$	[1,1,0,0,1,1]	[1,1]	$[\frac{2}{3}; 1]$
ϕ-State [9]	$(1/2)(|000\rangle + |011\rangle + |101\rangle + |110\rangle$	[1,1,0,0,1,1]	[1,1]	$[\frac{2}{3}; 1]$

From the table it is seen that $\det(k)_3$ are not identically zero for the GHZ, W, ψ, ϕ and the cluster state, meaning that the three qubits are entangled for these states. Yet from the values of $\det(k)_2$, it is clear that the GHZ state is completely separable when it loses its third qubits, while the W-state is only separable if the third qubit is measured

to be "1". The ψ and ϕ states are obtained from the GHZ state by local unitary transformations [9], yet unlike the GHZ state, they remain entangled whatever the measured value (in the $\{|0\rangle, |1\rangle\}$ basis) is for the third qubit. Note that these two states are in fact identical, related simply by a redefinition of $|1\rangle$ and $|0\rangle$, which the classification scheme exposes.

These eight determinants thus give us a complete picture of the degrees of entanglement of the tripartite systems, while the two coarse-grained indicators $\mathcal{C}_3$ and $\mathcal{C}_2$ provide a rough idea of the degrees of entanglement of the tripartite systems and the reduced bipartite systems when the 3rd qubit is lost.

C.   Cayley Hyperdeterminant

In [9] it was also shown that 7 parameters were necessary in order to determine the separability of a tripartite qubit state. The analysis involved showing that 6 submatrices had Rank 1, as well as the vanishing of the Cayley hyperdeterminant. This was shown to give a necessary andy sufficient condition for separability. For the state in Eq.(25), the six submatrices of [9] were given by

$$
\begin{aligned}
A_{x_0} &= (x_{0ij}), \quad A_{x_1} = (x_{1ij}) \\
A_{y_0} &= (x_{i0j}), \quad A_{y_1} = (x_{i1j}) \\
A_{z_0} &= (x_{ij0}), \quad A_{z_1} = (x_{ij1})
\end{aligned}
\tag{31}
$$

and the Cayley Hyperdeterminant $\operatorname{Det} A$ is given by[1]

$$
\begin{aligned}
\operatorname{Det} A = {}& x_{000}^2 x_{111}^2 + x_{001}^2 x_{110}^2 + x_{010}^2 x_{101}^2 + x_{100}^2 x_{011}^2 \\
& - 2\left[x_{000} x_{001} x_{110} x_{111} + x_{000} x_{010} x_{101} x_{111} + x_{000} x_{011} x_{100} x_{111} \right. \\
& \left. + \; x_{001} x_{010} x_{101} x_{110} + x_{001} x_{011} x_{101} x_{100} + x_{010} x_{011} x_{101} x_{100} \right] \\
& + \; 4\left[x_{000} x_{011} x_{101} x_{110} + x_{001} x_{010} x_{100} x_{111} \right].
\end{aligned}
\tag{32}
$$

Of these 7 numbers, the Cayley hyperdeterminant is linked to the tripartite system, and the six sub-determinants indicate degrees of entanglement of the 6 possible bipartite systems when one of the three qubits is measured.

The submatrices Eq.(31) are *not* the same as the submatrices considered in this paper, namely, Eq. (28). Specifically, the Cayley hyperdeterminant is in fact a function of the determinants Eq.(28) considered here, namely,

$$
\operatorname{Det} A = \det(3)^2 + \det(4)^2 - 2 \det(2) \det(5) - 2 \det(1) \det(6).
\tag{33}
$$

One advantage of the previous approach in [9] is that the determinants of the submatrices (31) have a physical significance, being the subconcurrences, as has the Cayley Hyperdeterminant which in [4] was considered as the 3-tangle. However, it is not easy to see how the Cayley Hyperdeterminant may be generalized to higher multipartite systems, and the current approach used here appears more direct.

V.   THE N-QUBIT CASE

It is now clear how the recipe elucidated in the previous section can be directly extended to N-qubit systems.

To determine the separability of the N-qubit state

$$
u \in V_1 \otimes V_2 \otimes \ldots \otimes V_N \quad (V_i \equiv \mathbb{C}^2).
\tag{34}
$$

the outlined procedure involves determining first the rank of u_1

$$
u_1 : V_1^* \otimes V_2^* \otimes \ldots V_{N-1}^* \to V_N.
\tag{35}
$$

In the standard basis, u_1 is represented by a $2 \times 2^{N-1}$ matrix.

Partial separability with respect to V_N is guaranteed by the rank of u_1 being 1; i.e. the $\binom{2^{N-1}}{2}$ 2×2 submatrices of u_1 must have determinant zero, $\det(k)_N = 0, \quad k = 1, 2, \ldots, \binom{2^{N-1}}{2}$.

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[1] We take this opportunity to correct a typographic error in [9], where the third term in Eq. (32) was erroneously left out.

Proceeding recursively, one sees that the degrees of entanglement of a N-qubit system are indicated by the list of determinants

$$[\{\det(k)_m, \; k = 1, 2, \ldots, l_m\}; \; m = N, N-1, \ldots, 2] \,. \tag{36}$$

Here the number l_m of determinants for $\{\det(k)_m\}$ is

$$l_m = 2^{N-m} \times \binom{2^{m-1}}{2}, \;\; m = N, N-1, \ldots, 2. \tag{37}$$

Clearly, we have to examine

$$\sum_{m=2}^{N} \frac{l_m}{2^{N-m}} = \binom{2^{N-1}}{2} + \binom{2^{N-2}}{2} + \cdots + \binom{2}{2} \tag{38}$$

2×2 submatrices to determine complete separability. The factor $1/2^{N-m}$ for l_m comes from the fact that when the $(m+1)$-th qubit is factorizable from the rest of the m qubits, then the remaining m-qubit is only of dimension $l_m/2$, because either its coefficients associated with the $|0\rangle_m$ and $|1\rangle_m$ are proportional , or one set of them is identically zero. This set of numbers gives rise to the combinatorial sequence $1, 7, 35, 155, 651, 2667, 10795, 43435, 174251, 698027 \ldots$ which is, *inter alia*, the Gaussian Binomial Coefficient $[N, 2]_{q=2}$ [14] but in any case diverges exponentially.

Just as for the tripartite cases, one may define a set of coarse-grained indicators for multipartite entanglement by

$$[\{\mathcal{C}_m\}; \; m = N, N-1, \ldots, 2] \,, \tag{39}$$

where $\mathcal{C}_m$ are as defined by Eq. (30). We emphasize that $\mathcal{C}_m = 0$ implies that the multipartite system, after losing $(N-m)$ of its qubits, is separable in one of its remaining m bits. Hence $\mathcal{C}_m = 0$ for all $m = N, N-1, \ldots, 2$ means the multipartite system is separable.

We present in Table II this classification scheme for some representative 4-qubit systems [15], which are the generalization of the corresponding tripartite states in Table I. From (37) one sees that there are 28, 12 and 4 (normalized) determinants to compute at the 4-, 3- and 2-qubit level, respectively. For simplicity of presentation we shall use the notation $[0_k, 1_{28-k}]$ to indicate the values of the 28 determinants at the 4-qubit level, i.e.: k "0" and (28-k) "1". We note that the coarse-grained indicators $[\mathcal{C}_4; \mathcal{C}_3; \mathcal{C}_2]$ for the generalized GHZ-state $(|0000\rangle + |1111\rangle)/\sqrt{2}$ and the W-state $(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)/\sqrt{4}$ are, respectively, $[1/28; 0; 0]$ and $[3/28; 1/6; 1/4]$. This means the W-state is more robust than the GHZ-state, and the fact that $\mathcal{C}_3 = \mathcal{C}_2 = 0$ for the GHZ-state indicates that the GHZ-state is completely separable when it loses one of its qubits. Again, as in the tripartite cases, the generalized ψ and ϕ states, though related to the GHZ state by local unitary transformations, remain entangled whenever one qubit is measured (in the $\{|0\rangle, |1\rangle\}$ basis).

TABLE II: Classification of some 4-qubit states.

State		$[|\det(k)_4|]$	$[|\det(k)_3|]$	$[|\det(k)_2|]$	$[\mathcal{C}_4; \mathcal{C}_3; \mathcal{C}_2]$
Separable	$\Sigma a_i e_i \otimes \Sigma b_j e_j \otimes \Sigma c_k e_k \otimes \Sigma d_l e_l$	$[0_{28}]$	$[0,0,0,0,0,0, \; 0,0,0,0,0,0]$	$[0,0, \; 0,0]$	$[0;0;0]$
GHZ	$(1/\sqrt{2})(|0000\rangle + |1111\rangle)$	$[0_{27}, 1]$	$[0,0,0,0,0,0,0,0,0,0,0,0]$	$[0,0,0,0]$	$[\frac{1}{28}; 0; 0]$
W	$(1/\sqrt{4})(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$	$[0_{25}, 1_3]$	$[1,1,0,0,0,0,0,0,0,0,0,0]$	$[1,0,0,0]$	$[\frac{3}{28}; \frac{1}{6}; \frac{1}{4}]$
Cluster [16]	$(1/\sqrt{4})(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)$	$[0_{24}, 1_4]$	$[0,0,0,0,0,0,0,0,0,0,0,0]$	$[1,0,0,1]$	$[\frac{1}{12}; 0; 1]$
ψ	$(1/\sqrt{8})(|0001\rangle + |0010\rangle + |0100\rangle + |0111\rangle)$	$[0_{12}, 1_{16}]$	$[1,1,0,0,1,1,1,1,0,0,1,1]$	$[1,1,1,1]$	$[\frac{4}{7}; \frac{2}{3}; 1]$
	$+|1000\rangle + |1011\rangle - |1101\rangle + |1110\rangle)$				
ϕ	$(1/\sqrt{8})(|0000\rangle + |0011\rangle + |0101\rangle + |0110\rangle)$	$[0_{12}, 1_{16}]$	$[1,1,0,0,1,1,1,1,0,0,1,1]$	$[1,1,1,1]$	$[\frac{4}{7}; \frac{2}{3}; 1]$
	$+|1001\rangle + |1010\rangle - |1100\rangle + |1111\rangle)$				

VI. GENERALIZATION TO HIGHER SPIN CASES

Our result in Sect. III.C may be generalized to multipartite pure states of arbitrary dimension, i.e., to multi-qudit case with higher spins. One simply replaces the qubit space V_k in $\mathcal{V} = V_1 \otimes V_2 \otimes \ldots \otimes V_m$ by the corresponding qudit space.

Suppose we denote by M the dimension of the qudit space with M levels, $M \times N$ the bipartite space where the first and the second particle have spin M and N respectively, and so on. Hence $M = 2$ for a qubit, and 2×2 for the state space of a two-partite qubit states, etc.

Consider, for instance, a bipartite qutrit states (i.e., 3×3 case)

$$u = \sum_{i,j=0}^{2} x_{ij}|ij\rangle. \tag{40}$$

The necessary and sufficient condition for its separability is that the matrix

$$\begin{bmatrix} x_{00} & x_{10} & x_{20} \\ x_{01} & x_{11} & x_{21} \\ x_{02} & x_{12} & x_{22} \end{bmatrix} \tag{41}$$

has rank one. Similarly, for a tripartite qutrit $(3 \times 3 \times 3)$ states, the condition for partial separability of the third qutrit with the first two qutrits is given by the rank=1 condition of the matrix

$$\begin{bmatrix} x_{000} & x_{010} & \cdots & \cdots & x_{210} & x_{220} \\ x_{001} & x_{011} & \cdots & \cdots & x_{211} & x_{221} \\ x_{002} & x_{012} & \cdots & \cdots & x_{212} & x_{222} \end{bmatrix}. \tag{42}$$

When the third qutrit collapses to one of its three states after a measurement is made on it, the separability of the remaining bipartite qutrits is then determined as in bipartite case, with the matrix (41) formed from the first, the second, or the third row of Eq. (42), according to wether the measured value of the third qutrit is 0, 1, or 2, respectively.

As another example, consider the case with $M \times N \times L$ space. The condition for partial separability of the third qudit with the first two qudits is given by the rank=1 condition of the matrix

$$\begin{bmatrix} x_{000} & \cdots & \cdots & x_{M-1,N-1,0} \\ x_{001} & \cdots & \cdots & x_{M-1,N-1,1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{0,0,L-1} & \cdots & \cdots & x_{M-1,N-1,L-1} \end{bmatrix}. \tag{43}$$

And the separability of the remaining qudits is determined recursively as discussed before.

The procedure can be easily extended to all multi-qudit states. Certainly, the number of sub-determinants to be computed increases as the number of particles and the dimension increase. This is unavoidable in any scheme of measure of entanglement. Our scheme has the advantage that the same prescription applies to the determination of separability of pure states with any number of particles and spin .

VII.   DISCUSSION

In this note we have proposed a unified mathematical scheme for characterizing entanglement that works for pure states (vectors) of multipartite systems of any number of particles. This scheme is based on Bourbaki's approach of defining the rank of a vector in terms of the associated linear mapping.

Our scheme provides an indication of the degrees of entanglement when the qubits are measured or lost successively. A rougher characterization of the degree of entanglement is provided by a set of coarse-grained indicators defined by the mean values of the absolute values of the determinants for each subspace of the multipartite systems.

For an N-qubit system, the number of parameters required to distinguish separability of the state is equal to the Gaussian Binomial Coefficient. This number is, unfortunately, exponentially large for large N. Thus the corresponding set of indicators cannot be taken as a practical measure of multipartite entanglement. However, it should be noted that as yet there is no simple scheme to quantify multipartite entanglement, and we believe that the present scheme is the most systematic and mathematically direct one.

In the tripartite case, the present scheme requires 7 2×2 determinants to determine complete separability. These 7 numbers are different from those given in [9]. We have shown that the Cayley hyperdeterminant is expressible in terms of 6 of the 7 determinants presented here.

It is well known that the GHZ, ψ and ϕ states are related by local unitary transformations, and hence it is often said that they are *equivalent* because of this fact. Yet, there is no denying that they behave differently when one qubit is lost, as we pointed out in the Introduction. When, say, the first qubit is measured in the z-basis, GHZ becomes separable, while the other two states remain entangled. We believe this has indeed been noted by many people, though not always mentioned in the literature. Some people then argue that, when the first qubit is measured in the x-basis, say, the remaining two qubits of the GHZ are still entangled, since when the first qubit is expressed in terms of the eigenstates $|0\rangle_x$ and $|1\rangle_x$ of the spin operator S_x, the GHZ state is expressed as

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_x\left(|00\rangle_z + |11\rangle_z\right) + \left[|1\rangle_x\left(|00\rangle_z - |11\rangle_z\right)\right]\right]. \tag{44}$$

But then it is also easily verified that, under the same measurement in the x-basis, the remaining two qubits of the ψ and ϕ states are separable. For example, the ϕ state is given by

$$|\phi\rangle = \frac{1}{2\sqrt{2}}\left[|0\rangle_x\left(|0\rangle_z + |1\rangle_z\right)\left(|0\rangle_z + |1\rangle_z\right) + |1\rangle_x\left(|0\rangle_z - |1\rangle_z\right)\left(|0\rangle_z - |1\rangle_z\right)\right].$$

All in all, the ψ and ϕ states behave differently from the GHZ state when a qubit is measured in the *same* basis. Actually, the very fact that the GHZ state behaves differently when a qubit is measured in different bases is already an indication that certain properties of entanglement cannot be invariant under local transformations. We believe that one must also look at such non-invariant properties in order to have a better understanding of entanglement.

We emphasize here that our scheme differs from other measures of entanglement [3] in that we consider entanglement properties of multipartite systems as each one of its qubits is successively measured in the same basis, while other measures concern mainly the invariant properties of entanglement under local unitary transformations without making any measurement, i.e., without losing any qubit.

We are still far from having a definitive way to quantify multipartite entanglement. In this regard, we believe that an interesting direction is that of relating entanglement to the link structures of knot theory [17, 18].

Acknowledgments

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