Recipe theorem for the Tutte polynomial for matroids, renormalization group-like approach

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Abstract

Using a quantum field theory renormalization group-like differential equation, we give a new proof of the recipe theorem for the Tutte polynomial for matroids. The solution of such an equation is in fact given by some appropriate characters of the Hopf algebra of isomorphic classes of matroids, characters which are then related to the Tutte polynomial for matroids. This Hopf algebraic approach also allows to prove, in a new way, a matroid Tutte polynomial convolution formula appearing in W. Kook *et. al.*, *J. Comb. Series* **B 76** (1999).

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1 Introduction and motivation

Combinatorial physics is a growing field, defining itself as the interdisciplinary domain overlapping between combinatorics and physics.

One can clearly identify nowadays several aspects of this larger and larger overlapping. Thus, the interference between combinatorics and quantum mechanics has been investigated in [BF11], [BDS+10] (and references within). An important interplay area can be noticed between combinatorics (bijective, enumerative and so on) on one hand and statistical physics and integrable combinatorics on another hand (see, for example, [DF12] for a recent review on this topic).

Our paper deals with yet another aspect of this interdisciplinary field of research, namely the interference between combinatorics and quantum field theory (commutative or non-commutative). This type of interference has already appeared in the pioneering work of Alain Connes and Dirk Kreimer, who have defined a Hopf algebra encoding in an elegant way the combinatorics of the process of perturbative renormalization in commutative quantum field theory. This structure has then been generalized for noncommutative quantum field theory [TVT08], [TK12] and for spin-foam quantum gravity models [Mar03], [Tan10]. For general reviews of various interferences between algebraic and analytic (and not only) combinatorics and quantum field theory in general, we invite the interested reader to consult [Tan12a] and [Tan12b].

In this paper we consider appropriate characters of the the Hopf algebra of isomorphic classes of matroids, algebra defined (as a particularization of a more general construction of incidence Hopf algebra) in [Sch94] (and then extensively studied in [CS05]). We also show that these characters are related to the Tutte polynomial for matroids and we then use a quantum field theory renormalization group-like differential equation to prove the universality of the Tutte polynomial for matroids. More precisely, we show that a solution of such an equation is given by the characters that we have defined. As a by-product of our Hopf algebraic approach, we give a new proof of a convolution formula for the Tutte polynomial for matroids, formula exhibited in [KRS99].

The paper is structured as follows. In the following section we introduce the renormalization group equation in quantum field theory and we then recall some useful notions related to matroids and to the Tutte polynomial for matroids. Finally, we give the definition of the Hopf algebra of isomorphic classes of matroids. In the third section we define the Hopf algebra characters that will be used in the sequel. In the following section, we use this construction to give our new proof of the convolution formula for the Tutte polynomial for matroids mentioned above. The fifth section is dedicated to our main result, the proof of the universality of the Tutte polynomial for matroids. The last section presents some concluding remarks and perspectives for future work.

2 Quantum field theory and matroid reminders

In this section we first briefly recall some quantum field theory notions, namely the renormalization group differential equation. We then give the definition of matroids, of the associated

Tutte polynomial as well as some further properties which will be useful to prove the results of this paper. Finally, the Hopf algebra of isomorphic classes of matroids is given.

2.1 Quantum field theory, the renormalization group

A QFT model (for a general introduction to QFT see for example the books [ZJ02] or [KSF01]) is defined by means of a functional integral of the exponential of an *action S* which, from a mathematical point of view, is a functional of the *fields* of the model. For the Φ^4 scalar model - the simplest QFT model - there is only one type of field, which we denote by $\Phi(x)$. From a mathematical point of view, for an Euclidean QFT scalar model, one has $\Phi : \mathbb{R}^D \to \mathbb{K}$, where D is usually taken equal to 4 (the dimension of the space) and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (real, respectively complex fields).

The quantities computed in QFT are generally divergent. One thus has to consider a real, positive, *cut-off* Λ - the flowing parameter. This leads to a family of cut-off dependent actions, family denoted by S_Λ. The derivation $\Lambda \frac{\partial S_\Lambda}{\partial \Lambda}$ gives the *renormalization group equation.*

The quadratic part of the action - the *propagator* of the model - can be written in the following way

$$C_{\Lambda,\Lambda_0}(p,q) = \delta(p-q) \int_{\frac{1}{\Lambda_0}}^{\frac{1}{\Lambda}} d\alpha e^{-\alpha p^2}, \tag{2.1}$$

with p and q living in the Fourier transformed space $\mathbb{R}^D$ and Λ_0 a second real, positive cut-off. In perturbative QFT, one has to consider *Feynman graphs*, and to associate to each such a graph a *Feynman integral* (further related to quantities actually measured in physical experiments). The contribution of an edge of such a Feynman graph to its associated Feynman integral is given by an integral such as (2.1).

One can then get (see [Pol84]) the Polchinski flow equation

$$\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} = \int_{\mathbb{R}^{2D}} \frac{1}{2} d^D p d^D q \Lambda \frac{\partial C_{\Lambda,\Lambda_0}}{\partial \Lambda} \left(\frac{\delta^2 S}{\delta\tilde{\Phi}(p)\delta\tilde{\Phi}(q)} - \frac{\delta S}{\delta\tilde{\Phi}(p)} \frac{\delta S}{\delta\tilde{\Phi}(q)} \right), \tag{2.2}$$

where $\tilde{\Phi}$ represents the Fourier transform of the function Φ. The first term in the right hand side (rhs) of the equation above corresponds to the derivation of a propagator associated to a bridge in the respective Feynman graph. The second term corresponds to an edge which is not a bridge and is part of some circuit in the graph. One can see this diagrammatically in Fig. 1.
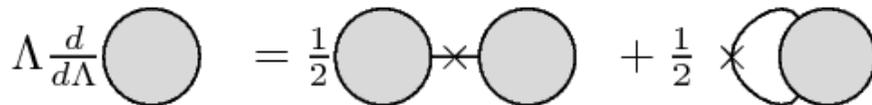

Figure 1: Diagrammatic representation of the flow equation.

This equation can then be used to prove perturbative renormalizability in QFT. Let us also stress here, that an equation of this type is also used to prove a result of E. M. Wright

which expresses the generating function of connected graphs under certain conditions (fixed excess). To get this generating functional (see, for example, Proposition $II.6$ the book [FS08]), one needs to consider contributions of two types of edges (first contribution when the edge is a bridge and a second one when not - see again Fig. 1).

2.2   Matroids and the Tutte polynomial for matroids

In this subsection we recall the definition and some properties of the Tutte polynomial for matroids as well as of the matroid Hopf algebra defined in [Sch94].

Following the book [Oxl92], one has the following definitions:

Definition 2.1 *A **matroid** M is a pair $(E, \mathcal{I})$ consisting of a finite set E and a collection of subsets of E satisfying the following set of axioms: $\mathcal{I}$ is non-empty, every subset of every member of $\mathcal{I}$ is also in $\mathcal{I}$ and, finally, if X and Y are in $\mathcal{I}$ and $|X| = |Y| + 1$, then there is an element x in $X - Y$ such that $Y \cup \{x\}$ is in $\mathcal{I}$. The set E is the ground set of the matroid and the members of $\mathcal{I}$ are the independent sets of the matroid.*

One can define a matroid on the edge set of any graph - **graphic matroid**. The reciproque does not hold (not every matroid is a graphic matroid).

Let E be an $n-$element set and let $\mathcal{I}$ be the collection of subsets of E with at most r elements, $0 \leq r \leq n$. One can check that $(E, \mathcal{I})$ is a matroid; it is called the **uniform matroid** $U_{r,n}$.

Remark 2.2 *If one takes $n = 1$, there are only two matroids, namely $U_{0,1}$ and $U_{1,1}$ and both of these matroids are graphic matroids. The graphs these two matroids correspond to are the graphs with one edge of Fig. 2 and Fig. 3.   In the first case, the edge is a loop*

Figure 2: The graph corresponding to the matroid $U_{0,1}$.

Figure 3: The graph corresponding to the matroid $U_{1,1}$.

(in graph theoretical terminology) or a tadpole (in QFT language). In the second case, the edge represents a bridge (in graph theoretical terminology) or a 1-particle-reducible line (in QFT terminology) - the number of connected components of the graphs increases by 1 if one deletes the respective edge.

Definition 2.3 *Maximal independent sets of a matroid are called bases. The collection of minimal dependent sets of a matroid are called circuits.*

Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{B} = \{B\}$ be the collection of bases of M. Let $\mathcal{B}^\star = \{E - B : B \in \mathcal{B}\}$. Then $\mathcal{B}^\star$ is the collection of bases of a matroid $M^\star$ on E. The matroid M^* is called the dual of M.

Definition 2.4 *Let $M = (E, \mathcal{I})$ be a matroid. The **rank** $r(A)$ of $A \subset E$ is defined as the cardinal of a maximal independent set in A.*

$$r(A) = max\{|B| \ s.t. \ B \in \mathcal{I}, B \subset A\} . \tag{2.3}$$

Definition 2.5 *Let $M = (E, \mathcal{I})$ be a matroid with a ground set E. The **nullity** function is given by*

$$n(M) = |E| - r(M) . \tag{2.4}$$

Definition 2.6 *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$. is a **loop** iff $\{e\}$ is the circuit.*

Definition 2.7 *Let $M = (E, \mathcal{I})$ be a matroid. The element $e \in E$ is a **coloop** iff, for any basis B, $e \in B$.*

Let us now define two basic operations on matroids. Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. Let $\mathcal{I}' = \{I \subseteq E - T : I \in \mathcal{I}\}$. One can check that $(E - T, \mathcal{I}')$ is a matroid. We denote this matroid by $M \backslash T$ - the **deletion** of T from M. The **contraction** of T from M, M/T, is given by the formula: $M/T = (M^\star \backslash T)^\star$.

Let us also recall the following results:

Lemma 2.8 *Let M be a matroid $(E, \mathcal{I})$ and T be a subset of E. One has*

$$M|_T = M \backslash_{E-T} . \tag{2.5}$$

Lemma 2.9 *If e is either a coloop or a loop of a matroid $M = (E, \mathcal{I})$, then $M/e = M \backslash e$.*

Lemma 2.10 *Let $M = (E, \mathcal{I})$ be a matroid and $T \subseteq E$, then, for all $X \subseteq E - T$,*

$$r_{M/T}(X) = r_M(X \cup T) - r_M(T) . \tag{2.6}$$

Let us now define the Tutte polynomial for matroids:

Definition 2.11 *Let $M = (E, \mathcal{I})$ be a matroid. The **Tutte polynomial** is given by the following formula:*

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{n(A)}. \tag{2.7}$$

The sum is computed over all subset of the matroid's ground set.

Example 2.12 *Let $U_{k,n}$ be a uniform matroid, $0 \leq k \leq n$. The Tutte polynomial of this matroid is given by*

$$T_{U_{k,n}}(x, y) = \sum_{i=0}^{k} \binom{n}{i} (x - 1)^{k-i} + \sum_{i=k+1}^{n} \binom{n}{i} (y - 1)^{i-k}. \tag{2.8}$$

It is worth stressing here that one can define the dual of any matroid; this is not the case for graphs, where only the dual of planar graph can be defined.

Let us recall, from [BO92] that

$$T_M(x, y) = T_{M^\star}(y, x). \tag{2.9}$$

2.3 Hopf algebra

Definition 2.13 *Let M_1 and M_2 be the matroids $(E_1, \mathcal{I}_1)$ and $(E_2, \mathcal{I}_2)$ where E_1 and E_2 are disjoint. Let*

$$M_1 \oplus M_2 = (E_1 \cup E_2, \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1, I_2 \in \mathcal{I}_2\}) \ .$$

Then $M_1 \oplus M_2$ is a matroid. This matroid is called the direct sum of M_1 and M_2.

In [Sch94], as a particularization of a more general construction of incidence Hopf algebras, the following result was proved:

Proposition 2.14 *If $\mathcal{M}$ is a minor-closed family of matroids (and if we denote by $\widetilde{\mathcal{M}}$ the set of isomorphism classes of it), then $k(\widetilde{\mathcal{M}})$ is a coalgebra, with coproduct Δ and counit ϵ determined by*

$$\Delta(M) = \sum_{A \subseteq E} M|A \otimes M/A \tag{2.10}$$

and respectively by $\epsilon(M) = \begin{cases} 1, & \text{if } E = \emptyset, \\ 0 & \text{otherwise }, \end{cases}$ for all $M = (E, \mathcal{I}) \in \mathcal{M}$. If, furthermore, the family $\mathcal{M}$ is closed under formation of direct sums, then $k(\widetilde{\mathcal{M}})$ is a Hopf algebra, with product induced by direct sum.

We refer to this Hopf algebra as to the matroid Hopf algebra. We follow [CS05] and, by a slight abuse of notation, we denote in the same way a matroid and its isomorphic class, since the distinction will be clear from the context (as it is already in Proposition 2.14).

We denote the unit of this Hopf algebra by **1** (the empty matroid, or $U_{0,0}$).

Example 2.15 *(Example 2.4 of [CS05]) Let $M = U_{k,n}$ be a uniform matroid with rank k. Its coproduct is given by*

$$\Delta(U_{k,n}) = \sum_{i=0}^{k} \binom{n}{i} U_{i,i} \otimes U_{k-i,n-i} + \sum_{i=k+1}^{n} \binom{n}{i} U_{k,i} \otimes U_{0,n-i} \ .$$

3 Hopf algebra characters

Let us give the following definitions:

Definition 3.1 *Let f, g be two mappings in $\mathrm{Hom}(\mathcal{M}, \mathcal{M})$. The convolution product of f and g is given by the following formula:*

$$f * g = (f \otimes g) \circ \Delta \ . \tag{3.1}$$

Definition 3.2 *A matroid Hopf algebra* **character** *f is an algebra morphism from the matroid Hopf algebra into a fixed commutative ring $\mathbb{K}$, i.e. such that*

$$f(M_1 \oplus M_2) = f(M_1)f(M_2), \quad f(\mathbf{1}) = 1_{\mathbb{K}}. \tag{3.2}$$

Definition 3.3 *A matroid Hopf algebra* **infinitesimal character** *g is a linear morphism from the matroid Hopf algebra into a fixed commutative ring $\mathbb{K}$, such that*

$$g(M_1 \oplus M_2) = g(M_1)\epsilon(M_2) + \epsilon(M_1)g(M_2). \tag{3.3}$$

Since we work in a Hopf algebra where the non-trivial part of the coproduct is nilpotent, we can also define an exponential map by the following expression

$$\exp_*(\delta) = \epsilon + \delta + \frac{1}{2}\delta * \delta + \dots \tag{3.4}$$

where δ is an infinitesimal character.

As already stated above (see Remark 2.2), there are only two matroids with unit cardinal ground set, $U_{0,1}$ and $U_{1,1}$. We now define two applications δ_{loop} and δ_{coloop}.

$$\delta_{\mathrm{loop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{0,1}, \\ 0_{\mathbb{K}} \text{ otherwise .} \end{cases} \tag{3.5}$$

$$\delta_{\mathrm{coloop}}(M) = \begin{cases} 1_{\mathbb{K}} \text{ if } M = U_{1,1}, \\ 0_{\mathbb{K}} \text{ otherwise .} \end{cases} \tag{3.6}$$

One can directly check that these applications are *infinitesimal characters* of the matroid Hopf algebra defined above.

We now define the following application:

$$\alpha(x, y, s, M) := \exp_* s\{\delta_{\mathrm{coloop}} + (y-1)\delta_{\mathrm{loop}}\} * \exp_* s\{(x-1)\delta_{\mathrm{coloop}} + \delta_{\mathrm{loop}}\}(M). \tag{3.7}$$

Example 3.4 *Let $U_{k,n}$ be a uniform matroid, $0 \le k \le n$. One has*

$$\alpha(x, y, s, U_{k,n}) = \sum_{i=0}^{k} \binom{n}{i} s^n (x-1)^{k-i} + \sum_{i=k+1}^{n} \binom{n}{i} s^n (y-1)^{i-k} = s^n T_{U_{k,n}}(x, y). \tag{3.8}$$

One then has:

Proposition 3.5 *The application* (3.7) *is a character.*

Proof – The proof can be done by a direct check. On a more general basis, this is a consequence of the fact that δ_{loop} and δ_{coloop} are infinitesimal characters and the space of infinitesimal characters is a vector space; thus $s\{\delta_{\mathrm{coloop}} + (y-1)\delta_{\mathrm{loop}}\}$ and $s\{(x-1)\delta_{\mathrm{coloop}} + \delta_{\mathrm{loop}}\}$ are infinitesimal characters. Since, $\exp_*(h)$ is a character when h is an infinitesimal character and since the convolution of two characters is a character, one gets that α is a character. $\square$

4 Proof of a Tutte polynomial convolution formula

Let $M = (E, \mathcal{I})$ be a matroid.

One then has:

Lemma 4.1 *Let $M = (E, \mathcal{I})$ be a matroid. One has*

$$exp_*\{a\delta_{\text{coloop}} + b\delta_{\text{loop}}\}(M) = a^{r(M)}b^{n(M)}. \tag{4.1}$$

Proof – Using the definition (3.4), the lhs of the identity (4.1) above reads:

$$\left(\sum_{k=0}^{\infty} \frac{(a\delta_{\text{coloop}} + b\delta_{\text{loop}})^{*k}}{k!} \right)(M). \tag{4.2}$$

All the terms in the sum above vanish, except the one for whom k is equal to $|E|$. Using the definition (3.1) of the convolution product, this terms writes

$$\frac{1}{k!} \left(\sum_{i=0}^{k} a^{k-i} b^i \sum_{\substack{i_1 + \cdots + i_n = k-i \\ j_1 + \cdots + j_m = i}} \delta_{\text{coloop}}^{\otimes(i_1)} \otimes \delta_{\text{loop}}^{\otimes(j_1)} \otimes \cdots \otimes \delta_{\text{coloop}}^{\otimes(i_n)} \otimes \delta_{\text{loop}}^{\otimes(j_m)} \right) \left(\sum_{(i)} M^{(1)} \otimes \cdots \otimes M^{(k)} \right) \tag{4.3}$$

where we have used the notation $\Delta^{(k-1)}(M) = \sum_{(i)} M^{(1)} \otimes \cdots \otimes M^{(k)}$. Using the definitions (3.5) and respectively (3.6) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, implies that the submatroids $M^{(j)}$ ($j = 1, \ldots, k$) are equal to $U_{0,1}$ or $U_{1,1}$.

Using the definition of the nullity and of the rank of a matroid concludes the proof. $\square$

Example 4.2 *Let us illustrate Lemma 4.1 for the uniform matroid $U_{k,n}$. One has $r(U_{k,n}) = k$ and $n(U_{n,k}) = n - k$. We now use the definitions 3.5 and 3.6 of δ_{loop} and δ_{coloop} to work out the lhs of identity 4.1.*

$$exp_*\{a\delta_{\text{coloop}} + b\delta_{\text{loop}}\}(U_{k,n}) = \frac{1}{n!} a^k b^{n-k} \delta_{\text{coloop}}^{\otimes k} \otimes \delta_{\text{loop}}^{\otimes(n-k)} \left(\binom{n}{n-1} \cdots \binom{2}{1} U_{1,1}^{\otimes k} \otimes U_{0,1}^{\otimes(n-k)} \right)$$
$$= a^k b^{n-k}. \tag{4.4}$$

One has:

$$\begin{aligned} \alpha(x, y, s, M) &= exp_* \left(s(\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}) \right) * exp_* \left(s(-\delta_{\text{coloop}} + \delta_{\text{loop}}) \right) \\ &* exp_* \left(s(\delta_{\text{coloop}} - \delta_{\text{loop}}) \right) * exp_* \left(s((x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}) \right). \end{aligned} \tag{4.5}$$

Proposition 4.3 *Let $M = (E, \mathcal{I})$ be a matroid. The character α is related to the Tutte polynomial of matroids by the following identity:*

$$\alpha(x, y, s, M) = s^{|E|} T_M(x, y). \tag{4.6}$$

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Proof – Using the definition (3.1) of the convolution product in the definition (3.7) of the character α, one has the following identity:

$$\alpha(x, y, s, M) = \sum_{A \subseteq E} \exp_* s\{\delta_{\text{coloop}} + (y-1)\delta_{\text{loop}}\}(M|_A) \exp_* s\{(x-1)\delta_{\text{coloop}} + \delta_{\text{loop}}\}(M/A).$$
(4.7)

We can now apply Lemma 4.1 on each of the two terms in the rhs of equation (4.7) above. This leads to the result. □

Using (2.9) and the Proposition 4.3, one has:

Corollary 4.4 *One has:*

$$\alpha(x, y, s, M) = \alpha(y, x, s, M^\star).$$
(4.8)

The Proposition 4.3 allows to give a different proof of a matroid Tutte polynomial convolution identity, which was shown in [KRS99]. One has:

Corollary 4.5 *(Theorem 1 of [KRS99]) The Tutte polynomial satisfies*

$$T_M(x, y) = \sum_{A \subset E} T_{M|A}(0, y) T_{M/A}(x, 0).$$
(4.9)

Proof – Taking $s = 1$, this is as a direct consequence of identity (4.5), and of Proposition 4.3. □

5   The recipe theorem

Let us define an application

$$\varphi_{a,b}(M) \longmapsto a^{r(M)} b^{n(M)} M .$$
(5.1)

Lemma 5.1 *The application $\varphi_{a,b}$ is a bialgebra automorphism.*

Proof – One can directly check that the application $\varphi_{a,b}$ is an algebra automorphism. Let us now check that this application is also a coalgebra automorphism. Using Lemma 2.8 and Lemma 2.10,

$$r(M|_T) + r(M/T) = r(M).$$
(5.2)

Thus, using the definitions of the application $\varphi_{a,b}$ of the matroid coproduct, one has:

$$\Delta \circ \varphi_{a,b}(M) = \sum_{T \subseteq E} (a^{r(M|_T)} b^{n(M|_T)} M|_T) \otimes (a^{r(M/T)} b^{n(M/T)} M/_T).$$
(5.3)

Using again the definition of the application $\varphi_{a,b}$ leads to

$$\Delta \circ \varphi_{a,b}(M) = (\varphi_{a,b} \otimes \varphi_{a,b}) \circ \Delta(M),$$
(5.4)

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which concludes the proof. □

Let us now define:

$$[f, g]_* := f * g - g * f. \tag{5.5}$$

Using the definition (3.7) of the Hopf algebra character α, one can directly prove the following result:

Proposition 5.2 *The character α is the solution of the differential equation:*

$$\frac{d\alpha}{ds}(M) = x\alpha * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \alpha + [\delta_{\text{coloop}}, \alpha]_* - [\delta_{\text{loop}}, \alpha]_* (M). \tag{5.6}$$

It is the fact that the matroid Tutte polynomial is a solution of the differential equation (5.6) that will be used now to prove the universality of the matroid Tutte polynomial. In order to do that, we take a four-variable matroid polynomial $Q_M(x, y, a, b)$ satisfying a multiplicative law and which has the following properties:

- if e is a coloop, then
$$Q_M(x, y, a, b) = xQ_{M\backslash e}(x, y, a, b) \ , \tag{5.7}$$

- if e is a loop, then
$$Q_M(x, y, a, b) = yQ_{M/e}(x, y, a, b) \tag{5.8}$$

- if e is a nonseparating point, then

$$Q_M(x, y, a, b) = aQ_{M\backslash e}(x, y, a, b) + bQ_{M/e}(x, y, a, b). \tag{5.9}$$

Remark 5.3 *Note that, when one deals with the same problem in the case of graphs, a supplementary multiplicative condition for the case of one-point joint of two graphs (i. e. identifying a vertex of the first graph and a vertex of the second graph into a single vertex of the resulting graph) is required (see, for example, [EMM10] or [Sok05]).*

We now define the application:

$$\beta(x, y, a, b, s, M) := s^{|E|}Q_M(x, y, a, b). \tag{5.10}$$

One then directly checks (using the definition (5.10) above and the multiplicative property of the polynomial Q) that this application is again a matroid Hopf algebra character.

Proposition 5.4 *The character (5.10) satisfies the following differential equation:*

$$\frac{d\beta}{ds}(M) = (x\beta * \delta_{\text{coloop}} + y\delta_{\text{loop}} * \beta + b[\delta_{\text{coloop}}, \beta]_* - a[\delta_{\text{loop}}, \beta]_*) (M). \tag{5.11}$$

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Proof – Applying the definition (3.1) of the convolution product, the rhs of equation (5.11) above writes

$$= (x - b) \sum_{A \subseteq E} \beta(M|_A)\delta_{\mathrm{coloop}}(M/_A) + (y - a) \sum_{A \subseteq E} \delta_{\mathrm{loop}}(M|_A)\beta(M/_A)$$

$$+ b \sum_{A \subseteq E} \delta_{\mathrm{coloop}}(M|_A)\beta(M/_A) + a \sum_{A \subseteq E} \beta(M|_A)\delta_{\mathrm{loop}}(M/_A). \tag{5.12}$$

Using the definitions (3.5) and respectively (3.6) of the infinitesimal characters δ_{loop} and respectively δ_{coloop}, constraints the sums on the subsets A above. The rhs of (5.11) becomes:

$$(x - b) \sum_{A, M/_A = U_{1,1}} \beta(M|_A) + (y - a) \sum_{A, M|_A = U_{0,1}} \beta(M/_A)$$

$$+ b \sum_{A, M|_A = U_{1,1}} \beta(M/_A) + a \sum_{A, M/_A = U_{0,1}} \beta(M|_A) \tag{5.13}$$

We now apply the definition of the Hopf algebra character β; one obtains:

$$s^{|E|-1}[(x - b) \sum_{A, M/_A = U_{1,1}} Q(x, y, a, b, M|_A) + (y - a) \sum_{A, M|_A = U_{0,1}} Q(x, y, a, b, M/_A)$$

$$+ b \sum_{A, M|_A = U_{1,1}} Q(x, y, a, b, M/_A) + a \sum_{A, M/_A = U_{0,1}} Q(x, y, a, b, M|_A)]. \tag{5.14}$$

We can now directly analyze the four particular cases $M/_A = U_{1,1}$, $M/_A = U_{0,1}$, $M|_A = U_{1,1}$ and $M|_A = U_{0,1}$:

- If $M/_A = U_{1,1}$, we can denote the ground set of $M/_A$ by $\{e\}$. Note that e is a coloop. From the Lemma 2.8, one has $M|_A = M\backslash_{E-A} = M\backslash e$. One then has $Q(x, y, a, b, M) = xQ(x, y, a, b, M|_A)$.

- If $M|_A = U_{0,1}$, then $A = \{e\}$ and e is a loop of M. Thus, one has $Q(x, y, a, b, M) = yQ(x, y, a, b, M/_A)$

- If $M|_A = U_{1,1}$, then $A = \{e\}$. One has to distinguish between two subcases:

  - e is a coloop of M. Then, by Lemma 2.9, $M/e = M\backslash e$. Thus, one has $Q(x, y, a, b, M) = xQ(x, y, a, b, M|_A)$.
  - e is a nonseparating point of M.

- If $M/_A = U_{0,1}$, one can denote the ground set of $M/_A$ by $\{e\}$. There are again two subcases to be considered:

  - e is a loop of M, one has that $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}} = M/e$. Then one has $Q(x, y, a, b, M) = yQ(x, y, a, b, M|_A)$.
  - e is a nonseparating point of M, then one has $M|_A = M\backslash_{(E-A)} = M\backslash_{\{e\}}$

We now insert all of this in equation (5.14); this leads to three types of sums over some element e of the ground set E, e being a loop, a coloop or a nonseparating point:

$$s^{|E|-1}[\sum_{e \in E: e \text{ is a coloop}} Q(x, y, a, b, M) + \sum_{e \in E: e \text{ is a loop}} Q(x, y, a, b, M) + \sum_{e \in E: e \text{ is a regular element}} Q(x, y, a, b, M)]$$

$$\tag{5.15}$$

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This rewrites as

$$|E|s^{|E|-1}Q(x,y,a,b,M) = \frac{d\beta}{ds}(M),\tag{5.16}$$

which completes the proof. $\qquad\square$

We can now state the *main result* of this paper, the recipe theorem specifying how to recover the matroid polynomial Q as an evaluation of the Tutte polynomial T_M:

Theorem 5.5 *One has:*

$$Q(x,y,a,b,M) = a^{n(M)}b^{r(M)}T_M(\frac{x}{b},\frac{y}{a}).\tag{5.17}$$

Proof – The proof is a direct consequence of Propositions 4.3, 5.2 and 5.4 and of Lemma 5.1. This comes from the fact that one can apply the automorphism ϕ defined in (5.1) to the differential equation (5.11). One then obtains the differential equation (5.6) with modified parameters x/b and y/a. Finally, the solution of this differential equation is (trivially) related to the matroid Tutte polynomial T_M (see Proposition 4.3) and this concludes the proof. $\quad\square$

Let us end this section by stating that all the results obtained in this paper naturally hold for graphs (instead of matroids), since graphs are a particular class of matroids (the graphic matroids, see subsection 2.2). We have thus given here the proofs of the graph results conjectured in [KM11].

6   Concluding remarks and perspectives

We have used in this paper a quantum field theory renormalization group-like equation to prove the universality of the Tutte polynomial for matroids. Moreover, we gave a new proof of the convolution identity established in [KRS99] by W. Kook *et. al.*

Let us emphasize that the Hopf algebra coproduct (2.10) used in this paper is a so-called type I coproduct, namely a coproduct using a *selection-quotient* rule. Examples of such coproducts are the Connes-Kreimer coproduct for commutative quantum field theory Feynman graphs [CK00], its generalization to non-commutative quantum field theory [TVT08, TK12] and so on. As it was already noticed in [Tan12a] or [HN12], this type of rule is fundamentally different of the one used to define a so-called type II coproduct, namely a *selection-deletion* rule. This latter rule has been extensively used in algebraic combinatorics (see, for example, [DHNT11] and references within). Let us also notice that for these type II coproducts, explicit polynomial realizations have been recently obtained (see [FNT10] and references within). These polynomial realizations are particularly interesting in order to give new, straightforward, proofs of the coassociativity of the respective coproducts (see, for example, J.-Y. Thibon's talk [Thi12]).

It thus appears to us as a particularly interesting perspective for future work the investigation of the existence of connections between these type I and II combinatorial Hopf algebra coproducts. Such connections could eventually be obtained by exhibiting explicit polynomial realizations for the type I coproduct combinatorial Hopf algebras, thus completing the picture of combinatorial Hopf algebra polynomial realizations.

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