Stepped planes, Stepped surfaces and Generalized Substitutions

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Sturmian words and Sturmian morphisms
A substitution on words is a non-erasing morphism of the free monoid: it maps letters of a finite alphabet $\mathcal{A}$ to non-empty finite words of $\mathcal{A}^*$, and maps concatenations of letters onto concatenations of the images of these letters. A *Sturmian word* is a two-letter infinite word which can be defined as the digitization of an irrational straight line of the plane. Then, a *Sturmian morphism* is a substitution on words which maps Sturmian words to Sturmian words. We are here interested in bidimensional extensions of these notions.

Stepped planes
It is natural to define a bidimensional word over $\mathcal{A}$ as a bidimensional array with coefficients in $\mathcal{A}$. Then *Sturmian bidimensional words* are defined in [5] as digitizations of real planes whose normal vectors have entries linearly independent over $\mathbb{Q}$. This extends the definition of Sturmian words as digitizations of real lines whose slopes are irrational. More precisely, consider the union of the unit cubes with vertices in $\mathbb{Z}^3$ which intersect the half-space below a given real plane. The boundary of this union of cubes is called a *stepped plane*- it is a union of unit squares called *faces*. Stepped planes can then be digitized by bidimensional three-letter words (see [3]). Fig. 1 illustrates this.

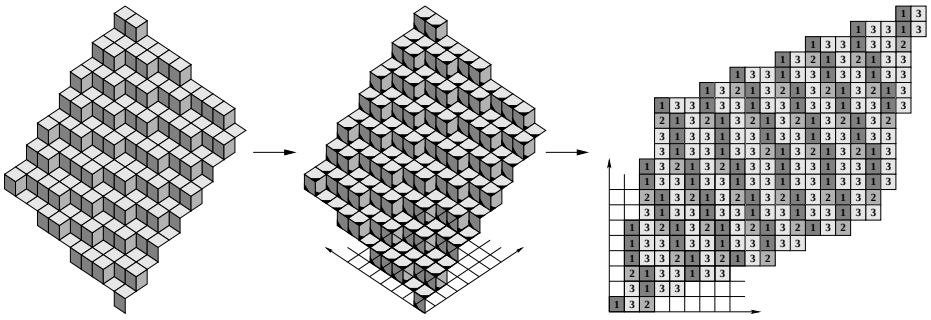

Fig. 1. Digitization of a stepped plane: specific vertices (highlighted by black corners) are associated with faces and form a bidimensional lattice, what leads to a natural coding by a bidimensional three-letter word (to each type of face corresponds a letter).

Stepped surfaces

A natural operation on the union of unit cubes intersecting the half-space below a real plane consists in adding or removing a cube on the boundary. This operation corresponds, in terms of stepped planes, to a local exchange of three faces - called *flip* - and can be iterated. The obtained unions of faces are not, generally, stepped planes. In fact, it is proven in [1] that these unions exactly correspond to the *stepped surfaces* introduced in [4]. Moreover, stepped surfaces can be digitized by bidimensional words over a 3-letter alphabet as well as stepped planes. Fig. 2 illustrates this.

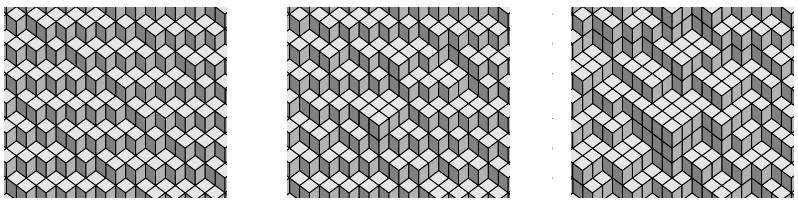

Fig. 2. Some flips are performed on a stepped plane (from left to right).

Generalized substitutions

Duality has been used in [2] to associate with classic substitutions on words so-called *generalized substitutions*. It is proven in [3] that generalized substitutions map aperiodic stepped planes to aperiodic stepped planes, as Sturmian morphisms map Sturmian words to Sturmian words. Moreover, it is proven in [1] that generalized substitutions also map stepped surfaces to stepped surfaces, as Sturmian morphisms also map two-letter words to two-letter words. Thus, it could be natural to consider generalized substitutions and digitizations of stepped surfaces as multi-dimensional extensions of, respectively, Sturmian morphisms and two-letter words. Fig. 3 and 4 illustrate this.

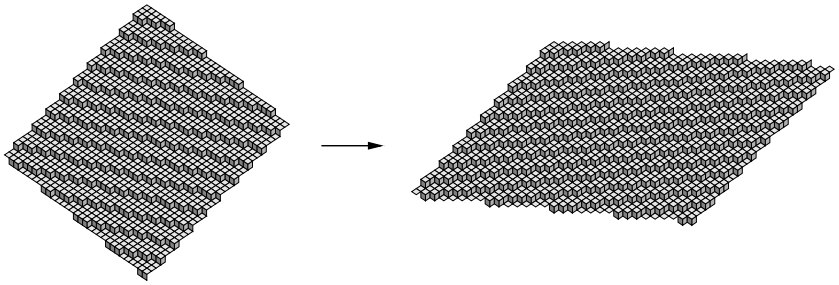

Fig. 3. Generalized substitutions map stepped planes to stepped planes.

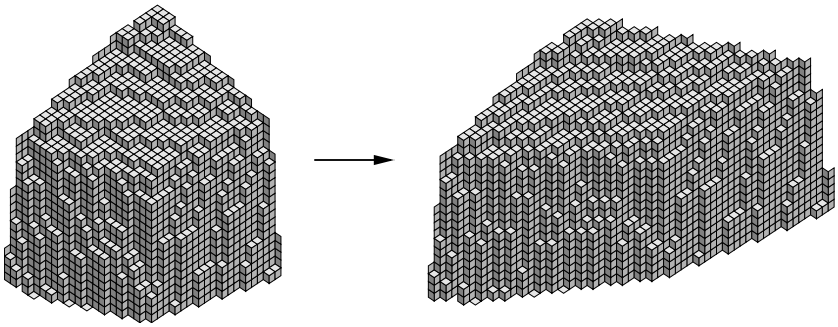

Fig. 4. Generalized substitutions map stepped surfaces to stepped surfaces.

In short...
We can sum up the previous bidimensional extensions as follows:

two-letter words	$\longrightarrow$	stepped surfaces
Sturmian words	$\longrightarrow$	aperiodic stepped planes
Sturmian morphisms	$\longrightarrow$	generalized substitutions

Note that these extensions are also suitable in a multi-dimensional framework, more precisely in codimension one, by considering hyperplanes instead of planes.

References

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