Substitutions on Multidimensional Sequences

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Abstract. We provide in this paper a multidimensional generalization of substitutions on words, which is defined as the action on multidimensional sequences of a *non-pointed substitution* endowed with *local rules*. The non-pointed substitutions and the local rules have in the multidimensional case respectively the roles played by the substitutions defined on letters and by the concatenation on words. This definition then allows us to provide a (yet partial) multidimensional generalization of an algebraic characterization of Sturmian words which are fixed-point or morphic image of a fixed-point of a non-trivial substitution on words.

Introduction

A substitution acts on a word in this way: the image of each letter is a word, and the image of the whole word is then just the concatenation of the images of its letters. Substitutions are powerful combinatorical tools, and have natural interactions with language theory, geometry of tilings, automata theory, and many others (see e.g. [14] and the references inside). It thus would be useful to define a similar tool in the more general framework of multidimensonal sequences, that are sequences of letters indexed by $\mathbb{Z}^n$ (whereas words are sequences of letters indexed by $\mathbb{N}$). It is however a difficult problem, mainly for lack of a natural "multidimensional concatenation".

Such a generalization has already been introduced in [15]: for $p_1, \ldots, p_n$ fixed in $\mathbb{N}$, a letter u indexed by $(i_1, \ldots, i_n)$ is mapped to a set $\sigma(u)$ of letters indexed by $\{(j_1, \ldots, j_n) \mid \forall k,\ p_k i_k \leq j_k < p_k(i_k + 1)\}$ (that is, a $p_1 \times \ldots \times p_n$-rectangle). But it generalizes in fact only *constant-length* substitutions on words (which map letters to words all of the same length). An algebraic characterization of all the multidimensional sequences which are fixed point of such substitutions is also proved (see again [15]), what generalizes a similar result for words which are fixed-point of a constant-length substitution (see e.g. [1]).

A first aim of this paper is to introduce a notion of multidimensional substitution which generalizes any type of substitutions on words, and not only the constant-length ones (or any other particular type). Second, we would like to give an algebraic characterization of the multidimensional sequences which are fixed-point of such a multidimensional substitution. More precisely, Theorem 2 generalizes the following result (see e.g. [6, 9]):

Let α be an irrational number in $[0,1]$. One defines the Sturmian sequence *$u_\alpha = (u_n)$ over the alphabet $\{1,2\}$ by:*

$$\forall n \geq 1, \quad u_n = 1 \;\Leftrightarrow\; (n\alpha) \; mod \; 1 \in I_\alpha,$$

where $I_\alpha = (0, 1-\alpha]$ or $I_\alpha = [0, 1-\alpha)$. Then u_α is a fixed point (resp. the morphic image of a fixed point) of a substitution on words if and only if α has a purely periodic (resp. eventually periodic) continued fraction expansion.

Notice that this characterization concerns only Sturmian sequences, that is, a subset of the set of all the sequences. Thus, generalizing this result also requires to define a notion of "multidimensional Sturmian sequence".

The paper is organized as follows. In the first section, we define *non-pointed substitutions* and *local rules*, that are our multidimensional equivalents of the "classic" substitutions defined on letters, and of the concatenation product used to make such substitutions act on sequences. It allows us, under conditions on the local rules, to define our notion of multidimensional substitution. In Section 2, we describe a type of local rules which satisfy the conditions required to define a multidimensional substitution: the local rules derived from a *global rule*. In Section 3, we resume the notion of *generalized substitutions*, define *Sturmian hyperplane sequences* and then we show that these generalized substitutions provide global rules from which we can derive local rules as described in Section 2. It yields multidimensional substitutions on Sturmian hyperplane sequences, and allows us to give (Theorem 2) a partial generalization of the algebraic characterization of fixed-points stated above.

1 Non-pointed substitutions and local rules

Let $\mathcal{A}$ be a finite alphabet. A *pointed letter* is an element $L = (x, l)$ of $\mathbb{Z}^n \times \mathcal{A}$, where x is the *location* of the letter l. We denote by $\mathcal{L}$ the set of pointed letters.

A *pointed pattern* is a set of pointed letters with distinct locations. The *support* of a pointed pattern is defined as the set of the locations of its letters. Two pointed patterns are said *consistent* if two letters with the same location are identical. The notions of union, intersection and inclusion are then defined for consistent patterns as for usual sets. We denote by $\mathcal{P}$ the set of pointed patterns.

The lattice $\mathbb{Z}^n$ acts on pointed letters (resp. pointed patterns) by translation on the locations (resp. supports): the classes of equivalence of this action are called *non-pointed letters* and denoted by $\overline{\mathcal{L}}$ (resp. *non-pointed patterns*, denoted by $\overline{\mathcal{P}}$).

Thus, to each pointed pattern P corresponds a unique non-pointed pattern, called its *underlying non-pointed pattern* and denoted $\overline{P}$. Conversely, to each non-pointed pattern $\overline{P}$ corresponds all the *congruent* pointed patterns, called *realizations* of $\overline{P}$, that have $\overline{P}$ as underlying non-pointed pattern. If P and P' are congruent pointed patterns, one denotes $v(P,P') \in \mathbb{Z}^n$ the vector that maps P onto P' by translation.

We are now in a position to give our multidimensional generalization of the definition on letters of a substitution on words:

Definition 1. *A* non-pointed substitution *is a map from* $\overline{\mathcal{L}}$ *to* $\overline{\mathcal{P}}$.

In what follows, $\overline{\sigma}$ denote a non-pointed substitution. We now define *local rules*, which are the main ingredient of our "multidimensional concatenation".

Definition 2. *We define two types of* local rules *for* $\overline{\sigma}$*:*

- *an* initial rule λ^* *is defined on a set* $I(\lambda^*) = \{L\}$ *of* ***one*** *pointed letter, and maps* L *to a realization of* $\overline{\sigma}(\overline{L})$*;*
- *an* extension rule λ *is defined on a set* $E(\lambda) = \{L, L'\}$ *of* ***two*** *pointed letters with distinct locations, and maps* L *and* L' *to disjoint realizations of respectively* $\overline{\sigma}(\overline{L})$ *and* $\overline{\sigma}(\overline{L'})$.

Roughly speaking, an initial rule tells us how to position $\overline{\sigma}(\overline{L})$ for a particular pointed letter L, while an extension rule λ such that $E(\lambda) = \{L, L'\}$ is used, for a pointed pattern $\{A, A'\}$ congruent to $\{L, L'\}$, to position $\overline{\sigma}(\overline{A'})$ relatively to $\overline{\sigma}(\overline{A})$ in the same way $\lambda(L')$ is positioned relatively to $\lambda(L)$. We first define the action of $\overline{\sigma}$ on Λ-*paths*:

Definition 3. *Let* U *be a pointed pattern and* Λ *be a set of local rules for* $\overline{\sigma}$*. A* Λ-path *of* U *is a sequence* $R = (R_1, \ldots, R_k)$ *of pointed letters of* U *such that:*

- *there exists an initial rule* $\lambda^* \in \Lambda$ *such that* $I(\lambda^*) = \{R_1\}$*;*
- *for* $i = 1 \ldots k-1$*, there exist an extension rule* $\lambda_i \in \Lambda$ *and* $x_i \in \mathbb{Z}^n$ *such that* $E(\lambda_i) = \{L_i, L_i'\}$ *with* $R_i = L_i + x_i$ *and* $R_{i+1} = L_i' + x_i$*.*

One then defines by induction a map denoted by $(\overline{\sigma}, \Lambda, R)$ *on the letters of* R *(see Fig. 1):*

- $(\overline{\sigma}, \Lambda, R)(R_1) = \lambda^*(R_1)$*;*
- *for* $i = 1 \ldots k-1$*,* $(\overline{\sigma}, \Lambda, R)(R_{i+1}) = \lambda_i(L_i') + v(\lambda_i(L_i), (\overline{\sigma}, \Lambda, R)(R_i))$*.*

Notice that, when computing the action of a substitution σ on a word, we proceed in the same way: the image by σ of the first letter of the word (here seen as a path) has a specified position (here given by an initial rule), while the position of the image of a letter follows, by induction, from the position of the concatenation of the images of the previous letters (here, we use extension rules to do that). We then define the action of $\overline{\sigma}$ on pointed patterns:

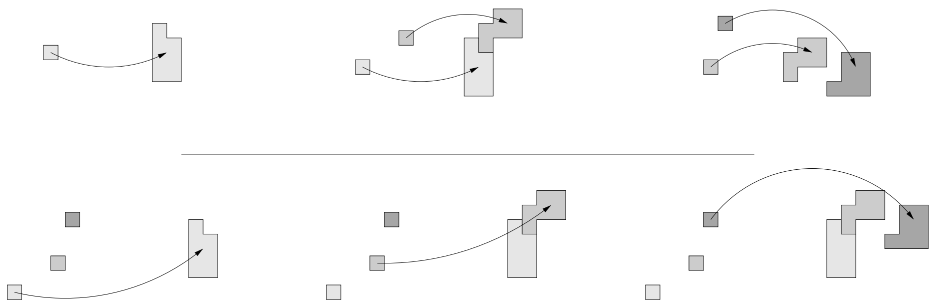

Fig. 1. Top: from left to right, an initial rule and two extension rules; bottom: computation of the image of a path using successively the three previous local rules.

Definition 4. *Let Λ be a set of local rules for $\overline{\sigma}$ and U be a pointed pattern. The set Λ is said to* cover *U if any pointed letter of U belongs to a Λ-path of U and is said to be* consistent *on U if for any two Λ-paths R and R' of U which both contain a pointed letter L, $(\overline{\sigma}, \Lambda, R)(L) = (\overline{\sigma}, \Lambda, R')(L)$.*
If Λ covers U and is consistent on U, one then defines the action of $\overline{\sigma}$ endowed with the set of local rules Λ, denoted by $(\overline{\sigma}, \Lambda)$, as follows:

$$(\overline{\sigma}, \Lambda)(U) = \bigcup \{(\overline{\sigma}, \Lambda, R)(L) \mid R \textit{ is a } \Lambda\textit{-path of } U \textit{ and } L \in R\}\,.$$

Thus, $(\overline{\sigma}, \Lambda)$ is our notion of multidimensional substitution on pointed patterns. It can be shown that it generalizes the substitutions on words as well as the multidimensional substitutions described in [15]. The possibilities are much larger, but it is in general not easy to obtain sets of local rules that are consistent on a set of pointed patterns and cover this set: the next section presents a way to obtain such sets of local rules.

2 Local rules derived from a global rule

Let $\overline{\sigma}$ be a non-pointed substitution and $\mathcal{H}$ be a set of pointed patterns. We are here interested in a generic way to obtain sets of local rules for $\overline{\sigma}$ that cover $\mathcal{H}$ and are consistent on it (that is, that cover any pointed pattern of $\mathcal{H}$ and are consistent on any of them). We derive such sets of local rules from *global rules*:

Definition 5. *A* global rule *on $\mathcal{H}$ for $\overline{\sigma}$ is a map Γ defined on the set of pointed letters $\{L \in U \mid U \in \mathcal{H}\}$ such that:*

- *a pointed letter L is mapped to a realization of $\overline{\sigma}(\overline{L})$;*
- *pointed letters with distinct locations are mapped to disjoint pointed patterns.*

Let us denote by $d(L, L')$ the distance $\sum |x_i - x'_i|$ between the locations (x_i) and (x'_i) of L and L'. We introduce a notion of weak connexity:

Definition 6. *The* span *between two pointed letters* L *and* L' *of* $U \in \mathcal{H}$*, denoted by* $sp(L, L')$*, is the smallest integer* D *such that there exists a sequence* $(L_1 = L, L_2, \ldots, L_k = L')$ *of pointed letters of* U *which verifies:* $\forall j,\ d(L_j, L_{j+1}) \leq D$*. The spans of* U *and* $\mathcal{H}$ *are then defined by:*

$$sp(U) = \sup_{L,L' \in U} sp(L, L') \qquad \textit{and} \qquad sp(\mathcal{H}) = \sup_{U \in \mathcal{H}} sp(U).$$

For example, $\mathrm{sp}(U) = 1$ if and only if U is 4-connected. Let us now derive a set of local rules from a global rule:

Definition 7. *Let* H_0 *be a pointed pattern and* Γ *a global rule on* $\mathcal{H}$ *for* $\overline{\sigma}$*. A set* Λ *of local rules for* $\overline{\sigma}$ *is said to be* derived *from* $(\mathcal{H}, H_0, \Gamma)$ *if it verifies:*

1. *if* λ^* *is an initial rule of* Λ *with* $I(\lambda^*) = \{L\}$*, then* $L \in H_0$ *and* $\lambda^*(L) = \Gamma(L)$*;*
2. *if* λ *is an extension rule of* Λ *with* $E(\lambda) = \{L, L'\}$*, then* $d(L, L') \leq sp(\mathcal{H})$*,* $\lambda(L) = \Gamma(L)$ *and* $\lambda(L') = \Gamma(L')$*;*
3. *if* λ *and* λ' *are extension rules of* Λ*, then* $E(\lambda)$ *and* $E(\lambda')$ *are not congruent.*

Such derived sets of locals rules have interesting properties:

Proposition 1. *If* H_0 *is finite and* $sp(\mathcal{H})$ *is bounded, then any set of local rules derived from* $(\mathcal{H}, H_0, \Gamma)$ *is finite.*

Proof. Let Λ be derived from $(\mathcal{H}, H_0, \Gamma)$. There is no more than $|H_0|$ initial rules in Λ. There are $|\mathcal{A}|^{|(\mathrm{sp}(\mathcal{H})+1)^n/\mathbb{Z}^n|}$ non-congruent pointed patterns $\{L, L'\}$ that verify $d(L, L') \leq \mathrm{sp}(\mathcal{H})$: it follows that there is a finite number of extension rules in Λ. Thus, Λ is finite. □

Definition 8. *A global rule* Γ *on* $\mathcal{H}$ *is said* context-free *if, for* $U \in \mathcal{H}$*,* $L, L' \in U$ *and* $x \in \mathbb{Z}^n$ *such that* $L + x, L' + x \in U$*, one has:*

$$v(\Gamma(L), \Gamma(L + x)) = v(\Gamma(L'), \Gamma(L' + x)).$$

We present examples of such global rules in Section 3.

Proposition 2. *If* Γ *is a context-free global rule on* $\mathcal{H}$*, then any set of local rules derived from* $(\mathcal{H}, H_0, \Gamma)$ *is consistent on* $\mathcal{H}$*.*

Proof. Suppose that Γ is context-free, and let Λ be a set of local rules derived from $(\mathcal{H}, H_0, \Gamma)$. Let $R = (R_1, \ldots, R_k)$ be a Λ-path of $U \in \mathcal{H}$. Let us prove by induction that for all i, $(\overline{\sigma}, \Lambda, R)(R_i) = \Gamma(R_i)$. Since R is a Λ-path, there exists an initial rule $\lambda^* \in \Lambda$ such that $I(\lambda^*) = \{R_1\}$, and since Λ is derived from $(\mathcal{H}, H_0, \Gamma)$, $(\overline{\sigma}, \Lambda_k, R)(R_1) = \lambda^*(R_1) = \Gamma(R_1)$. Suppose now that $(\overline{\sigma}, \Lambda_k, R)(R_i) = \Gamma(R_i)$. According to Definition 3, there exists an extension rule $\lambda_i \in \Lambda$ and $x_i \in \mathbb{Z}^n$ such that $E(\lambda_i) = \{L_i, L'_i\}$ with $R_i = L_i + x_i$ and $R_{i+1} = L'_i + x_i$, and $(\overline{\sigma}, \Lambda_k, R)(R_{i+1}) = \lambda(L'_i) + v(\lambda(L_i), (\overline{\sigma}, \Lambda_k, R)(R_i))$. But Λ is derived from $(\mathcal{H}, H_0, \Gamma)$, hence $\lambda(L_i) = \Gamma(L_i)$ and $\lambda(L'_i) = \Gamma(L'_i)$. Moreover, $(\overline{\sigma}, \Lambda_k, R)(R_i) = \Gamma(R_i) = \Gamma(L_i + x_i)$. Thus, $(\overline{\sigma}, \Lambda_k, R)(R_{i+1}) = \Gamma(L'_i) + v(\Gamma(L_i), \Gamma(L_i + x_i)$. Finally, since Γ is context-free, $(\overline{\sigma}, \Lambda_k, R)(R_{i+1}) = \Gamma(L'_i) + v(\Gamma(L'_i), \Gamma(L'_i + x_i)) = \Gamma(L'_i + x_i) = \Gamma(R_{i+1})$. It yields that Λ is consistent on $\mathcal{H}$. □

Proposition 3. *If H_0 intersects any pointed pattern of $\mathcal{H}$, then there exist sets of local rules derived from $(\mathcal{H}, H_0, \Gamma)$ that cover $\mathcal{H}$.*

Proof. Let us define $\mathcal{E} = \{\{L, L'\} \mid L, L' \in U,\ U \in \mathcal{H} \text{ and } d(L, L') \leq \mathrm{sp}(\mathcal{H})\}$, and let $\mathcal{E}'$ be a maximal subset of $\mathcal{E}$ that does not contain congruent pointed patterns. Let Λ be the set of the following local rules:

- for each $L \in H_0$, the initial rule λ^* defined on $I(\lambda^*) = \{L\}$ by $\lambda^*(L) = \Gamma(L)$;
- for each $\{L, L'\} \in \mathcal{E}'$, the extension rule λ defined on $E(\lambda) = \{L, L'\}$ by $\lambda(L) = \Gamma(L)$ and $\lambda(L') = \Gamma(L')$.

One easily checks that Λ is derived from $(\mathcal{H}, H_0, \Gamma)$. Let us prove that Λ covers $\mathcal{H}$. Let $U \in \mathcal{H}$ and $L' \in U$. Since H_0 intersects any pointed pattern of $\mathcal{H}$, there exists $L \in U \cup H_0$. By definition, there also exists a sequence of pointed letters $(L_1 = L, L_2, \ldots, L_k = L')$ such that $\forall i$, $d(L_i, L_{i+1}) \leq \mathrm{sp}(\mathcal{H})$. Then, for all i there exists $x_i \in \mathbb{Z}^n$ such that $\{L_i, L_{i+1}\} + x_i \in \mathcal{E}'$, and there exists an initial rule of Λ defined on $\{L_1\}$. It yields that $(L_1, \ldots, L_k)$ is a Λ-path which contains L'. Thus, Λ covers $\mathcal{H}$. $\square$

We can resume the previous propositions in the following theorem:

Theorem 1. *Let Γ be a context-free global rule on $\mathcal{H}$ for $\overline{\sigma}$. If $sp(\mathcal{H})$ is bounded and if $H_0 \in \mathcal{P}$ is a finite pointed pattern intersecting any pointed pattern of $\mathcal{H}$, then one can derive from $(\mathcal{H}, H_0, \Gamma)$ a finite set of local rules that is consistent on $\mathcal{H}$ and covers it.*

We thus have a way to derive, from a context-free global rule, local rules consistent on a given set of pointed pattern and covering this set. This result is applied in the next section to a particular type of context-free global rule.

3 Sturmian hyperplane sequences and algebraicity

We first briefly resume the notion of *generalized substitution* (see e.g. [4, 5, 14]). Let $\boldsymbol{e_1}, \ldots, \boldsymbol{e_n}$ denote the canonical basis of $\mathbb{R}^n$ and let $\langle ., . \rangle$ denote the canonical scalar product on $\mathbb{R}^n$.

A *face* (x, i^*), for $x \in \mathbb{Z}^n$ and $i \in \{1, \ldots, n\}$ is defined by:

$$(x, i^*) = \{x + \sum_{j \neq i} r_j \boldsymbol{e_j} \mid 0 \leq r_j \leq 1\}.$$

Such faces generate the $\mathbb{Z}$-module of the formal sums of weighted faces $\mathcal{G} = \{\sum m_{x,i}(x, i^*) \mid m_{x,i} \in \mathbb{Z}\}$, on which the lattice $\mathbb{Z}^n$ acts by translation: $y + (x, i^*) = (y + x, i^*)$. Faces are used to approximate hyperplanes of $\mathbb{R}^n$:

Definition 9. *Let $\boldsymbol{\alpha} \in \mathbb{R}^n_+$, $\boldsymbol{\alpha} \neq 0$. The hyperplane $\mathcal{P}_{\boldsymbol{\alpha}}$ of $\mathbb{R}^n$ is defined by:*

$$\mathcal{P}_{\boldsymbol{\alpha}} = \{x \in \mathbb{R}^n \mid \langle x, \boldsymbol{\alpha} \rangle = 0\}.$$

The stepped hyperplane $\mathcal{S}_{\boldsymbol{\alpha}}$ *associated to* $\mathcal{P}_{\boldsymbol{\alpha}}$ *is defined by:*

$$\mathcal{S}_{\boldsymbol{\alpha}} = \{(x, i^*) \mid \langle x, \boldsymbol{\alpha} \rangle > 0 \text{ and } \langle x - \boldsymbol{e_i}, \boldsymbol{\alpha} \rangle \leq 0\},$$

and a patch *of* $\mathcal{S}_{\boldsymbol{\alpha}}$ *is a finite subset of the set of faces of* $\mathcal{S}_{\boldsymbol{\alpha}}$*.*

Notice that a patch of $\mathcal{S}_{\boldsymbol{\alpha}}$ belongs to the $\mathbb{Z}$-module $\mathcal{G}$, but is geometric, that is, without multiple faces. Let us recall that the *incidence matrix* M_σ of a substitution on words σ gives at position (i, j) the number of occurences of the letter i in the word $\sigma(j)$. If $det M_\sigma = \pm 1$, then σ is said *unimodular*.

Definition 10. *The* generalized substitution *associated to the unimodular substitution* σ *is the endomorphism* Θ_σ *of* $\mathcal{G}$ *defined by:*

$$\begin{cases} \forall i \in \mathcal{A}, & \Theta_\sigma(0, i^*) = \sum_{j=1}^{3} \sum_{s:\sigma(j)=p\cdot i\cdot s} \left(M_\sigma^{-1}(f(s)), j^*\right), \\ \forall x \in \mathbb{Z}^3,\ \forall i \in \mathcal{A}, & \Theta_\sigma(x, i^*) = M_\sigma^{-1} x + \Theta_\sigma(0, i^*), \\ \forall \sum m_{x,i}(x, i^*) \in \mathcal{G}, & \Theta_\sigma\left(\sum m_{x,i}(x, i^*)\right) = \sum m_{x,i} \Theta_\sigma(x, i^*), \end{cases}$$

where $f(w) = (|w|_1, |w|_2, |w|_3)$ *and* $|w|_i$ *is the number of occurences of the letter* i *in* w*.*

The following type of substitution is particularly interesting:

Definition 11. *A substitution* σ *is of* Pisot *type if its incidence matrix* M_σ *has eigenvalues* $\lambda, \mu_1, \ldots, \mu_{n-1}$ *satisfying* $0 < |\mu_i| < 1 < \lambda$*. The generalized substitution* Θ_σ *is then also said of Pisot type.*

Indeed, the following result is proved in [4, 5]:

Proposition 4 ([4, 5]). *If* σ *is of Pisot type and if* $\boldsymbol{\alpha}$ *is a* left *eigenvector of* M_σ *for the dominant eigenvalue* λ*, then* $\Theta_\sigma(\mathcal{S}_{\boldsymbol{\alpha}}) \subset \mathcal{S}_{\boldsymbol{\alpha}}$ *and* Θ_σ *maps distinct faces of the stepped hyperplane* $\mathcal{S}_{\boldsymbol{\alpha}}$ *to disjoint patches of* $\mathcal{S}_{\boldsymbol{\alpha}}$*.*

The stepped hyperplane $\mathcal{S}_{\boldsymbol{\alpha}}$ is called the *invariant hyperplane* of Θ_σ. It is also proved in [11]:

Proposition 5 ([11]). *If the modified Jacobi-Perron algorithm ([8]) yields a purely periodic (resp. eventually periodic) continued fraction expansion for* $\boldsymbol{\alpha} \in \mathbb{R}^n$*, then the stepped hyperplane* $\mathcal{S}_{\boldsymbol{\alpha}}$ *is a fixed point (resp. the image by a generalized substitution of a fixed point) of a generalized substitution of Pisot type.*

We then define hyperplane sequences, mapping stepped hyperplanes of $\mathbb{R}^n$ to $(n-1)$-dimensional sequences over the alphabet $\{1, \ldots, n\}$. The following proposition (proved in Appendix) resumes a result given in [2, 3]:

Proposition 6. *Let* $\mathcal{V}_{\boldsymbol{\alpha}} \subset \mathbb{Z}^n$ *be the set of the vertices that belong to the faces of* $\mathcal{S}_{\boldsymbol{\alpha}}$*. Let* $v_{\boldsymbol{\alpha}}$ *and* $\pi_{\boldsymbol{\alpha}}$ *be the maps defined respectively on* $\mathcal{S}_{\boldsymbol{\alpha}}$ *and* $\mathcal{V}_{\boldsymbol{\alpha}}$ *by:*

$$v_{\boldsymbol{\alpha}}(x, i^*) = x + \boldsymbol{e_1} + \ldots + \boldsymbol{e_{i-1}} \quad \text{and} \quad \pi_{\boldsymbol{\alpha}}(x_1, \ldots, x_n) = (x_1 - x_n, \ldots, x_{n-1} - x_n).$$

Then, $v_{\boldsymbol{\alpha}}$ *(resp.* $\pi_{\boldsymbol{\alpha}}$*) is a bijection from* $\mathcal{S}_{\boldsymbol{\alpha}}$ *onto* $\mathcal{V}_{\boldsymbol{\alpha}}$ *(resp. from* $\mathcal{V}_{\boldsymbol{\alpha}}$ *onto* $\mathbb{Z}^{n-1}$*).*

Let $\phi_{\boldsymbol\alpha}$ be defined on $\mathcal{S}_{\boldsymbol\alpha}$ by $\phi_{\boldsymbol\alpha}(x,i^*)=(\pi_{\boldsymbol\alpha}(v_{\boldsymbol\alpha}(x,i^*)),i)$: it maps bijectively the faces of $\mathcal{S}_{\boldsymbol\alpha}$ to the letters of a $(n-1)$-dimensional sequence over $\{1,\ldots,n\}$. Notice that **not all** these $(n-1)$-dimensional sequences over $\{1,\ldots,n\}$ correspond to a stepped hyperplane. We thus introduce the following definition:

Definition 12. *An* hyperplane sequence *is an $(n-1)$-dimensional sequence over $\{1,\ldots,n\}$ defined, for $\boldsymbol\alpha\in\mathbb{R}^n$, by $\phi_{\boldsymbol\alpha}(\mathcal{S}_{\boldsymbol\alpha})$. One denotes by $\mathcal{H}_{\boldsymbol\alpha}$ such an hyperplane sequence. Moreover, if $\boldsymbol\alpha=(\alpha_1,\ldots,\alpha_n)$ is such that $1,\alpha_1,\ldots,\alpha_n$ are linearly independent over $\mathbb{Q}$, then $\mathcal{H}_{\boldsymbol\alpha}$ is called a* Sturmian hyperplane sequence*.*

For $n=2$, Sturmian hyperplane sequences are nothing but Sturmian sequences over $\{1,2\}$ (see [12]), and for $n=3$, one retrieves the notion of two-dimensional Sturmian sequence of [7]. Notice that an hyperplane sequence $\mathcal{H}_{\boldsymbol\alpha}$ is defined on the whole $\mathbb{Z}^{n-1}$: it yields $\mathrm{sp}(\mathcal{H}_{\boldsymbol\alpha})=1$. Let us now derive, from generalized substitution, context-free global rules on hyperplane sequences:

Proposition 7. *Let σ be a Pisot unimodular substitution on words over $\{1,\ldots,n\}$. Let Θ_σ be the associated generalized substitution, and $\mathcal{S}_{\boldsymbol\alpha}$ its invariant stepped hyperplane. Let $\mathcal{H}_{\boldsymbol\alpha}=\phi_{\boldsymbol\alpha}(\mathcal{S}_{\boldsymbol\alpha})$. We set $\mathcal{L}=\mathbb{Z}^{n-1}\times\{1,\ldots,n\}$ and define:*

$$\Gamma_\sigma=\phi_{\boldsymbol\alpha}\circ\Theta_\sigma\circ\phi_{\boldsymbol\alpha}^{-1}\qquad \textit{and}\qquad \overline{\sigma^*}\ :\ \overline{(0,i)}\in\overline{\mathcal{L}}\mapsto\overline{\Gamma_\sigma(0,i)}\in\overline{\mathcal{P}}.$$

Then, Γ_σ is a context-free global rule on $\mathcal{H}_{\boldsymbol\alpha}$ for the non-pointed substitution $\overline{\sigma^*}$.

Proof. For $(x,i)\in\mathcal{H}_{\boldsymbol\alpha}$ and $y\in\mathbb{Z}^{n-1}$, one computes:

$$\Gamma_\sigma((x,i)+y)=\Gamma_\sigma(x,i)+\pi_{\boldsymbol\alpha}(M_\sigma^{-1}\pi_{\boldsymbol\alpha}^{-1}(y)).$$

It follows that $\overline{\Gamma_\sigma(x,i)}=\overline{\Gamma_\sigma(0,i)}=\overline{\sigma^*}\left(\overline{(0,i)}\right)$. Moreover, since Θ_σ maps distinct faces of $\mathcal{S}_{\boldsymbol\alpha}$ to disjoint patches of $\mathcal{S}_{\boldsymbol\alpha}$ (see Proposition 4) and since $\phi_{\boldsymbol\alpha}$ maps bijectively the faces of $\mathcal{S}_{\boldsymbol\alpha}$ to the letters of $\mathcal{H}_{\boldsymbol\alpha}$, $\Gamma_\sigma=\phi_{\boldsymbol\alpha}\circ\Theta_\sigma\circ\phi_{\boldsymbol\alpha}^{-1}$ maps letters with distinct locations to disjoint pointed patterns. Thus, Γ_σ is a global rule on $\mathcal{H}_{\boldsymbol\alpha}$ for $\overline{\sigma^*}$.
Then, if $(x,i)\in\mathcal{H}_{\boldsymbol\alpha}$, $(x',i)\in\mathcal{H}_{\boldsymbol\alpha}$ and $y\in\mathbb{Z}^{n-1}$, one has:

$$v(\Gamma_\sigma(x,i),\Gamma_\sigma((x,i)+y))=\pi_{\boldsymbol\alpha}(M_\sigma^{-1}\pi_{\boldsymbol\alpha}^{-1}(y))=v(\Gamma_\sigma(x',i),\Gamma_\sigma((x',i)+y)).$$

Hence Γ_σ is context-free, according to Definition 8. □

Finally, combining Theorem 1 and Proposition 5 and 7, we obtain:

Theorem 2. *If the modified Jacobi-Perron algorithm ([8]) yields a purely periodic (resp. eventually periodic) continued fraction expansion for $\boldsymbol\alpha\in\mathbb{R}^n$, then the Sturmian hyperplane sequence $\mathcal{H}_{\boldsymbol\alpha}$ is a fixed point (resp. the image by a multidimensional substitution of a fixed point) of a multidimensional substitution.*

This result can thus be seen as a multidimensional generalization of the algebraic characterization resumed in the introduction, though it provides only a sufficient condition for a Sturmian hyperplane sequence to be a fixed point of a multidimensional substitution or the image by a multidimensional substitution of such a fixed point. In fact, the proof of the algebraic characterization resumed in the introduction uses the notion of *return words* of [10]. This notion has already been generalized, in terms of tilings, in [13]: it thus gives us a possible way to achieve the characterization of Theorem 2.

Example 1. Let σ be the classic substitution defined on $\{1,2,3\}$ by $\sigma(1)=13$, $\sigma(2)=1$ and $\sigma(3)=2$. One computes:

$$M_\sigma^{-1}=\begin{pmatrix}0&0&1\\1&0&-1\\0&1&0\end{pmatrix},\qquad\text{and}\qquad \Theta_\sigma\ :\ \begin{array}{l}(0,1^*)\mapsto((1,-1,0),1^*)+(0,2^*)\\(0,2^*)\mapsto(0,1^*)\\(0,3^*)\mapsto(0,2^*)\end{array},$$

which yields the non-pointed substitution:

$$\overline{\sigma^*}\ :\ \overline{1_{0,0}}\mapsto\overline{\{1_{0,0},2_{0,1}\}},\qquad \overline{2_{0,0}}\mapsto\overline{\{3_{0,0}\}},\qquad \overline{3_{0,0}}\mapsto\overline{\{1_{0,0}\}},$$

which one can also represent as follows:

$$\overline{\sigma^*}\ :\ 1\mapsto\begin{array}{c}2\\1\end{array},\quad 2\mapsto 3,\quad 3\mapsto 1.$$

Let us define $\mathcal{H}=\{\Gamma_\sigma^n((0,0),1),n\geq 1\}$. One can prove in this particular case that $\mathrm{sp}(\mathcal{H})=1$. Thus, one can compute (Theorem 1) a finite set of local rules that covers $\mathcal{H}$ and is consistent on it. One obtains for example the initial rule defined by:

$$\lambda^*\ :\ ((0,0),1)\mapsto\{((0,0),1),((0,1),2)\},$$

and five extension rules, represented as follows (the bolded letters are mapped to the bolded letters, so the information about *relative* locations is still conserved):

$$\lambda_1\ :\ \begin{array}{c}2\\\mathbf{1}\end{array}\mapsto\begin{array}{cc}&\mathbf{2}\\3&\mathbf{1}\end{array},\qquad \lambda_2\ :\ 3\,\mathbf{1}\mapsto\begin{array}{c}\mathbf{2}\\\mathbf{1}\\1\end{array},\qquad \lambda_3\ :\ \begin{array}{c}1\\\mathbf{1}\end{array}\mapsto\begin{array}{cc}&\mathbf{2}\\2&\mathbf{1}\\1&\end{array},$$

$$\lambda_4\ :\ 2\,\mathbf{1}\mapsto\begin{array}{c}\mathbf{2}\\\mathbf{1}\\3\end{array},\qquad \lambda_5\ \begin{array}{c}\mathbf{1}\\3\end{array}\mapsto\begin{array}{cc}\mathbf{1}&1\\\mathbf{2}&\end{array}.$$

For example, computing the sequence $(\overline{\sigma^*},\{\lambda^*,\lambda_1,\ldots,\lambda_5\})^n((0,0),1)$ for $n=1,\ldots,7$ gives (the letter with location $(0,0)$ is bolded):

$$\mathbf{1}\mapsto\begin{array}{c}2\\\mathbf{1}\end{array}\mapsto\begin{array}{cc}&2\\3&\mathbf{1}\end{array}\mapsto\begin{array}{cc}&2\\3&\mathbf{1}\\&1\end{array}\mapsto\begin{array}{ccc}&&2\\&2&1\\3&\mathbf{1}&\\&1&\end{array}\mapsto\begin{array}{cccc}&2&&2\\3&1&2&1\\&3&\mathbf{1}&\\&&1&\end{array}\mapsto\begin{array}{cccc}&2&&2\\3&1&2&1\\&3&\mathbf{1}&\\&2&1&\\3&1&&\\&1&&\end{array}\mapsto\ldots$$

We can in this way generate arbitrarely large patches of the hyperplane sequence $\mathcal{H}_{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$ is a left eigenvector of M_σ. Moreover, $\mathcal{H}_{\boldsymbol{\alpha}}$ is a fixed-point of this multidimensional substitution.

Acknowledgements. We would like to thank Valérie Berthé and Pierre Arnoux for many useful suggestions.

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Appendix

Proof of Proposition 6:
Let (x,i^*) and (y,j^*) be two faces of $\mathcal{S}_{\boldsymbol{\alpha}}$ such that $v_{\boldsymbol{\alpha}}(x,i^*) = v_{\boldsymbol{\alpha}}(y,j^*)$. If $i<j$, then $x = y + \boldsymbol{e_i} + \ldots + \boldsymbol{e_{j-1}}$, and $\langle x - \boldsymbol{e_i}, \boldsymbol{\alpha}\rangle = \langle (y + \boldsymbol{e_{i+1}} + \ldots + \boldsymbol{e_{j-1}}, \boldsymbol{\alpha}\rangle = \langle y, \boldsymbol{\alpha}\rangle + \langle \boldsymbol{e_{i+1}} + \ldots + \boldsymbol{e_{j-1}}, \boldsymbol{\alpha}\rangle$. Since $(y,j^*) \in \mathcal{S}_{\boldsymbol{\alpha}}$, $\langle y, \boldsymbol{\alpha}\rangle > 0$. Moreover, $\langle \boldsymbol{e_{i+1}} + \ldots + \boldsymbol{e_{j-1}}, \boldsymbol{\alpha}\rangle \geq 0$. Thus, $i<j$ would yield $\langle x - \boldsymbol{e_i}, \boldsymbol{\alpha}\rangle > 0$, what would contradict $(x,i^*) \in \mathcal{S}_{\boldsymbol{\alpha}}$. Similarly, $i>j$ is impossible. Hence $i=j$, and $x=y$ follows. It proves that $v_{\boldsymbol{\alpha}}$ is one-to-one from $\mathcal{S}_{\boldsymbol{\alpha}}$ to $\mathcal{V}_{\boldsymbol{\alpha}}$.

If $y \in \mathcal{V}_{\boldsymbol{\alpha}}$, then there exist $(x,i^*) \in \mathcal{S}_{\boldsymbol{\alpha}}$ and $I \subset \{1,\ldots,n\}$, $i \notin I$, such that $y = x + \sum_{j\in I} \boldsymbol{e_j}$. Let us denote $f : k \mapsto \langle x + \sum_{j \in I} \boldsymbol{e_j} - \boldsymbol{e_1} - \ldots - \boldsymbol{e_k}, \boldsymbol{\alpha}\rangle$. One has:

$$f(0) = \langle x, \boldsymbol{\alpha}\rangle + \sum_{j\in I} \langle \boldsymbol{e_j}, \boldsymbol{\alpha}\rangle > 0, \qquad f(n) = \langle x - \boldsymbol{e_i}, \boldsymbol{\alpha}\rangle - \sum_{j \notin I, j \neq i} \langle \boldsymbol{e_j}, \boldsymbol{\alpha}\rangle \leq 0,$$

and f is decreasing. Let k_0 such that $f(k_0 - 1) > 0$ and $f(k_0) \leq 0$. Let $y_0 = y - \boldsymbol{e_1} - \ldots - \boldsymbol{e_{k_0-1}}$. Then, $\langle y_0, \boldsymbol{\alpha}\rangle = f(k_0-1) > 0$, and $\langle y_0 - \boldsymbol{e_{k_0}}, \boldsymbol{\alpha}\rangle = f(k_0) \leq 0$. Thus, $(y_0, k_0^*) \in \mathcal{S}_{\boldsymbol{\alpha}}$. Since $v_{\boldsymbol{\alpha}}(y_0, k_0^*) = y$, it proves that $v_{\boldsymbol{\alpha}}$ is onto from $\mathcal{S}_{\boldsymbol{\alpha}}$ on $\mathcal{V}_{\boldsymbol{\alpha}}$.

Let us denote $\boldsymbol{\alpha}$ by $(\alpha_1, \ldots, \alpha_n)$. Recall that the α_i are positive and not all equal to zero. Let then $x = (x_1, \ldots, x_n) \in \mathcal{V}_{\boldsymbol{\alpha}}$ and $(x', i^*) = v_{\boldsymbol{\alpha}}^{-1}(x)$. One has $0 < \langle x', \boldsymbol{\alpha}\rangle \leq \langle \boldsymbol{e_i}, \boldsymbol{\alpha}\rangle = \alpha_i$. Thus:

$$0 < \sum_{j=1}^{n} x_j \alpha_j - \sum_{j=1}^{i-1} \alpha_j \leq \alpha_i.$$

Suppose now $\pi_{\boldsymbol{\alpha}}(x) = (y_1, \ldots, y_{n-1})$. The previous formula yields:

$$0 < \sum_{j=1}^{n-1} y_j \alpha_j + x_n \sum_{j=1}^{n} \alpha_j \leq \sum_{j=1}^{i-1} \alpha_j + \alpha_i \leq \sum_{j=1}^{n} \alpha_j,$$

and performing the division by $\sum_{j=1}^{n} \alpha_j > 0$, it then gives:

$$0 < \frac{\sum_{j=1}^{n-1} y_j \alpha_j}{\sum_{j=1}^{n} \alpha_j} + x_n \leq 1,$$

that is, since $x_n \in \mathbb{Z}$:

$$x_n = 1 - \left\lceil \frac{\sum_{j=1}^{n-1} y_j \alpha_j}{\sum_{j=1}^{n} \alpha_j} \right\rceil.$$

Conversely, given $(y_1, \ldots, y_{n-1}) \in \mathbb{Z}^{n-1}$, setting $x_n \in \mathbb{Z}$ as above and then, for $i = 1 \ldots n-1$, $x_i = y_i + x_n$ yields $\pi_{\boldsymbol{\alpha}}(x_1, \ldots, x_n) = (y_1, \ldots, y_{n-1})$. Thus, $\pi_{\boldsymbol{\alpha}}$ is a bijection from $\mathcal{V}_{\boldsymbol{\alpha}}$ to $\mathbb{Z}^{n-1}$ (and the proof provides an explicit formula for $\pi_{\boldsymbol{\alpha}}^{-1}$).