Stochastic Flips on Dimer Tilings

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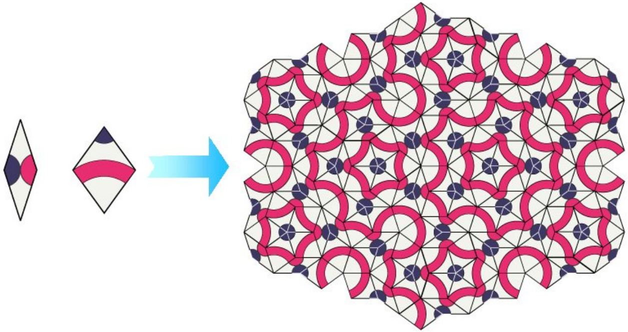
Moscow, June 21, 2013

Context

Quasicrystal: non-periodic ordered material modeled by tilings.



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However, non-periodicity yields “frequent” deadlocks.

Alternative: first allow mismatches to facilitate self-assembly, then perform random locally-defined corrections. Convergence?.

1 Tilings and flips

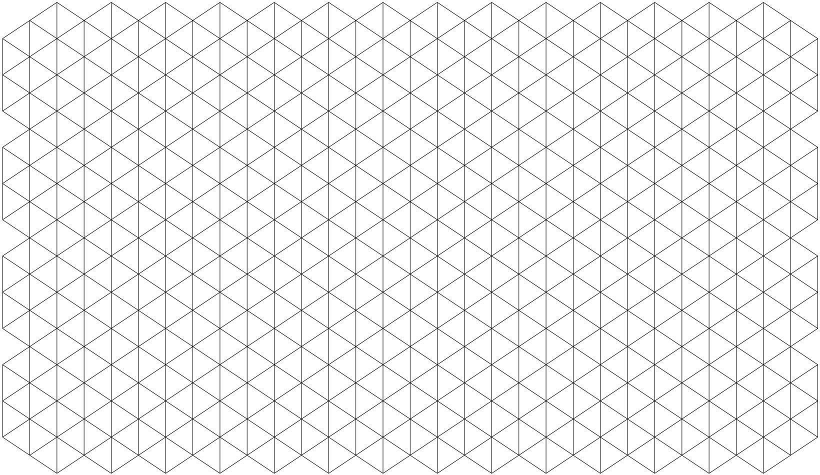
2 Cooling process

3 An upper bound

4 A second upper bound

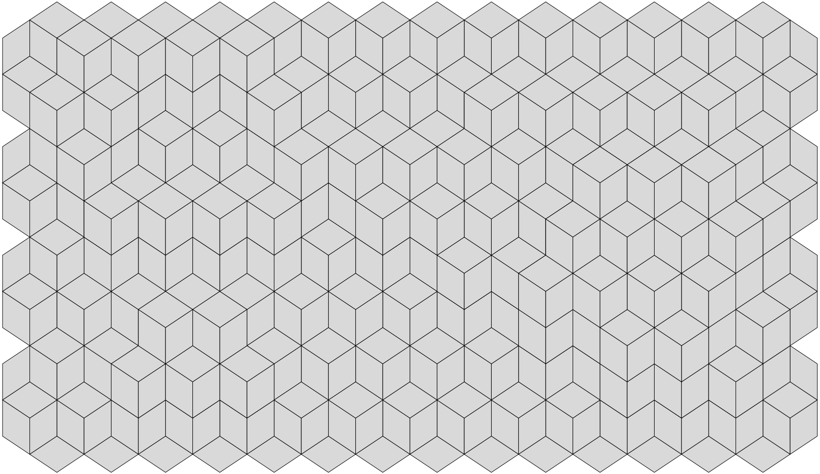
Dimer tilings

Bounded (simply) connected subset of the triangular grid.



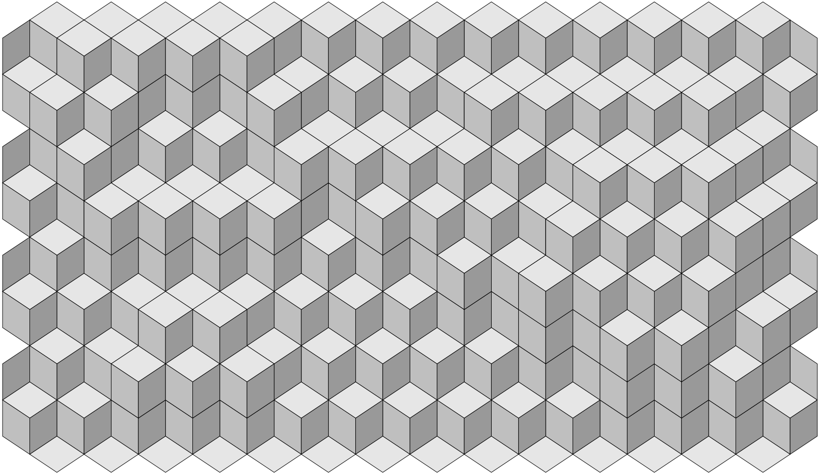
Dimer tilings

Dimer tiling: perfect matching of adjacent triangles.



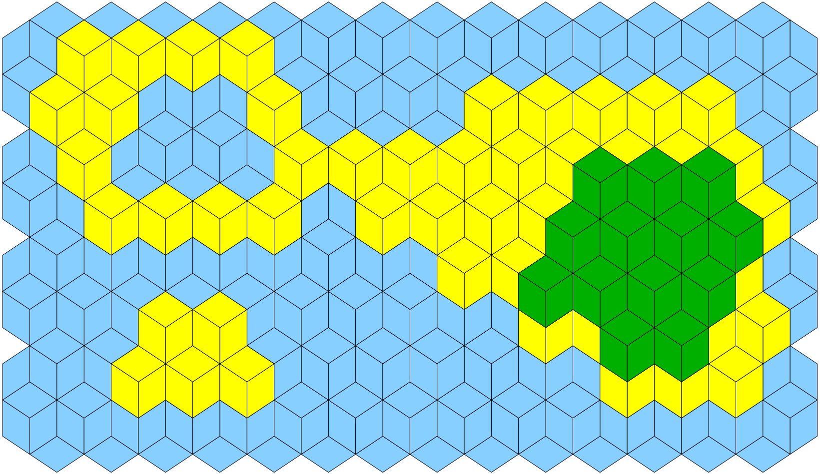
Dimer tilings

Shading dimers $\rightsquigarrow$ 3D-viewpoint.



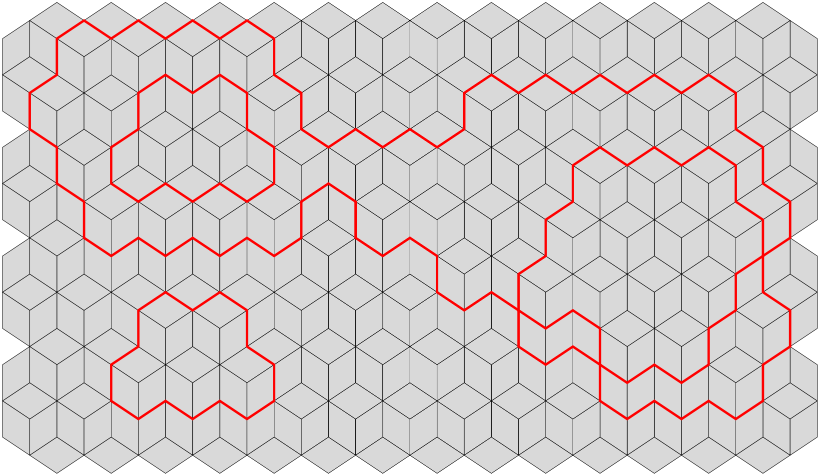
Dimer tilings

3D-viewpoint $\rightsquigarrow$ distance to the plane $x + y + z = 0$ (height).



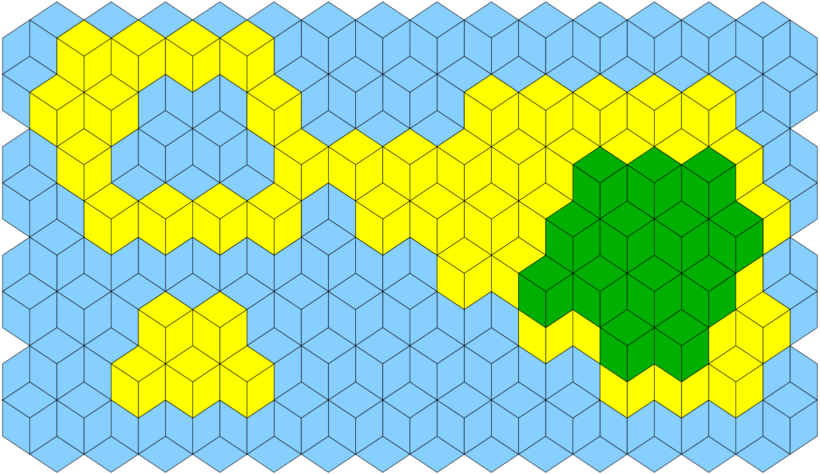
Errors

Error: edge between two dimers equal up to a translation.



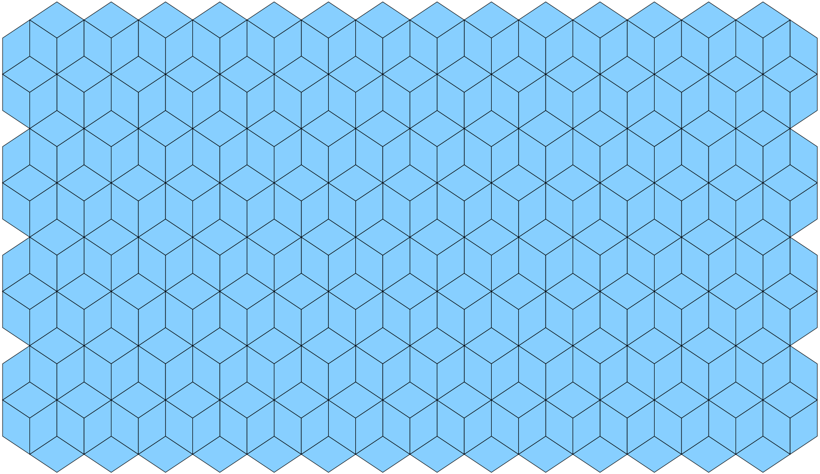
Errors

Errors form contours lines between dimers of different heights.



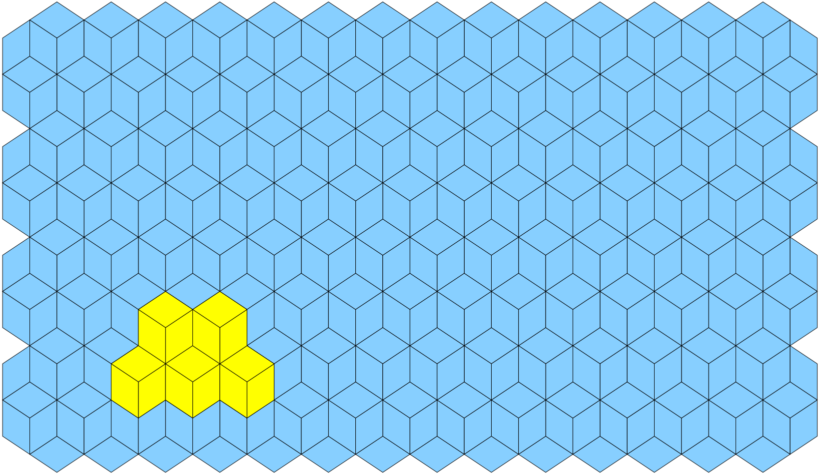
Errors

Error-free tiling plays the quasicrystal role.



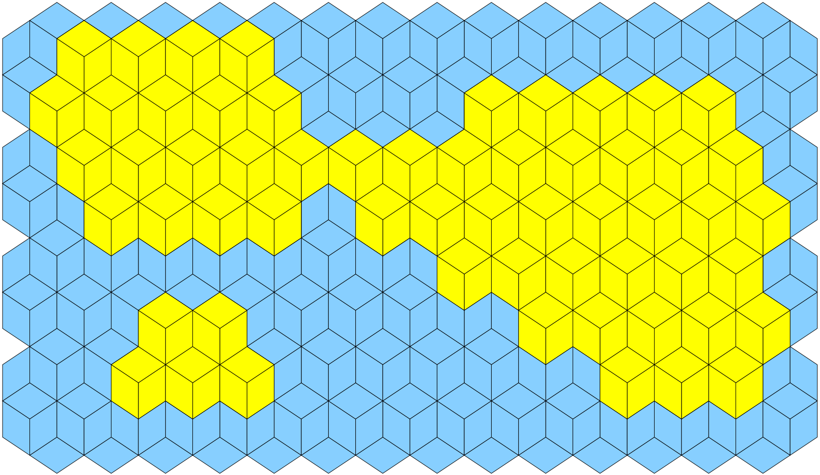
Islands and holes

An island with height 1, area 5 and perimeter 16.



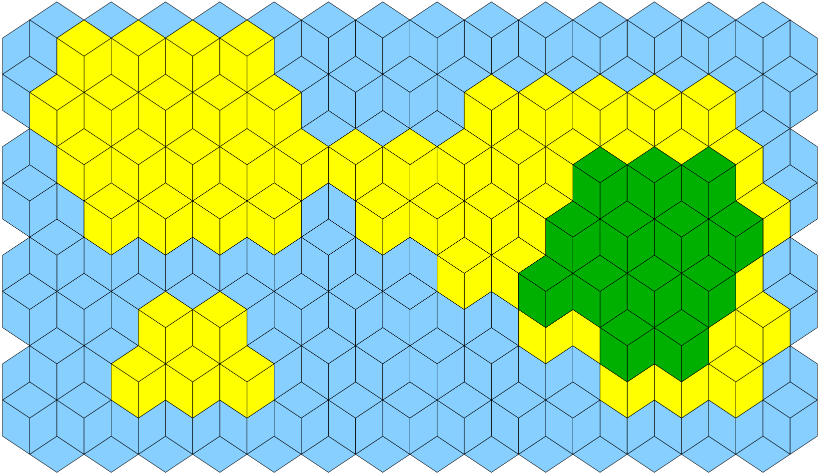
Islands and holes

A larger island of height 1.



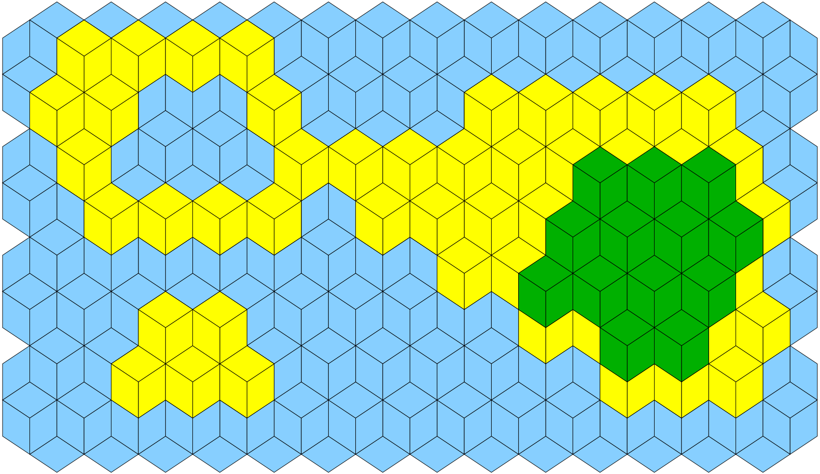
Islands and holes

Another one, of height 2 this time.



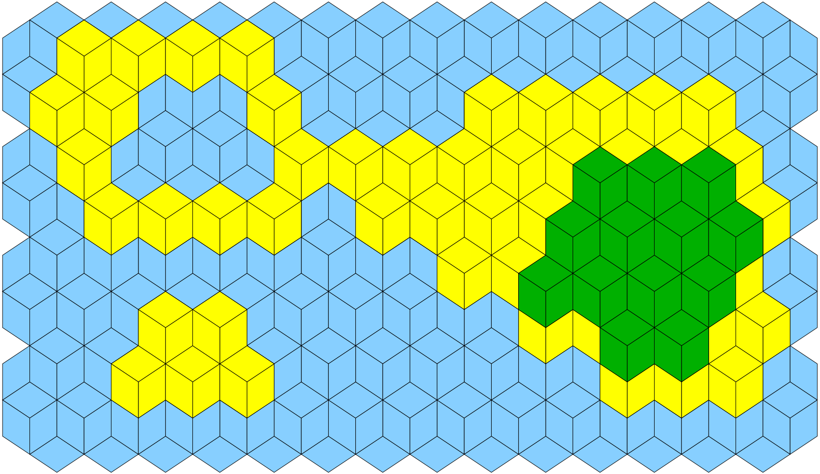
Islands and holes

A hole in the largest island.



Flips

Flip: exchange of thee dimers $\simeq$ add/remove a cube.



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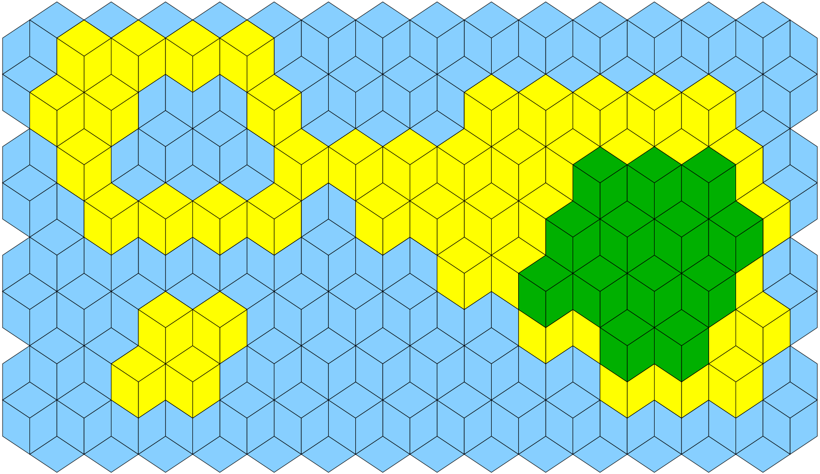
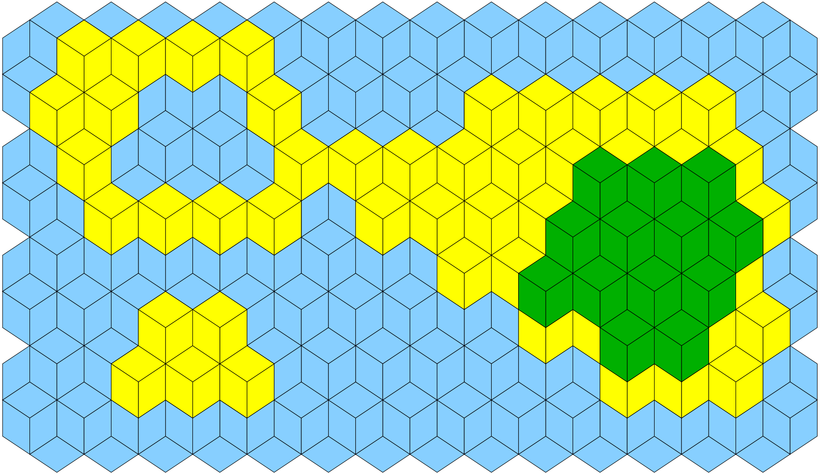


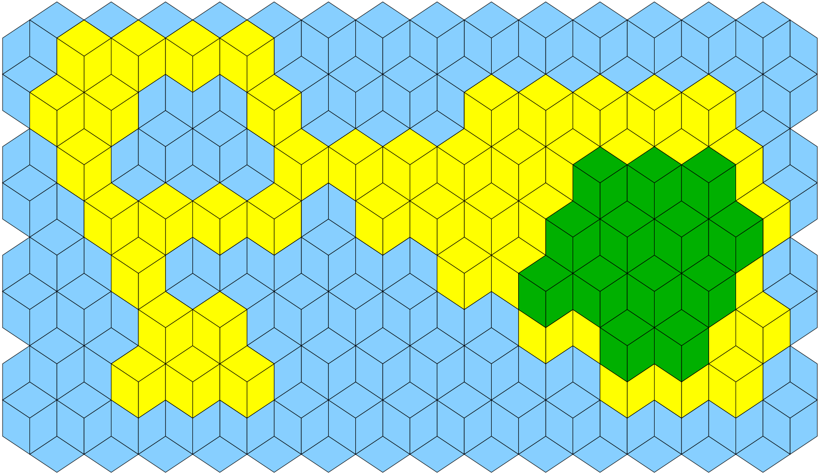
Flips

Flip change the area of an island or its perimeter (error number).



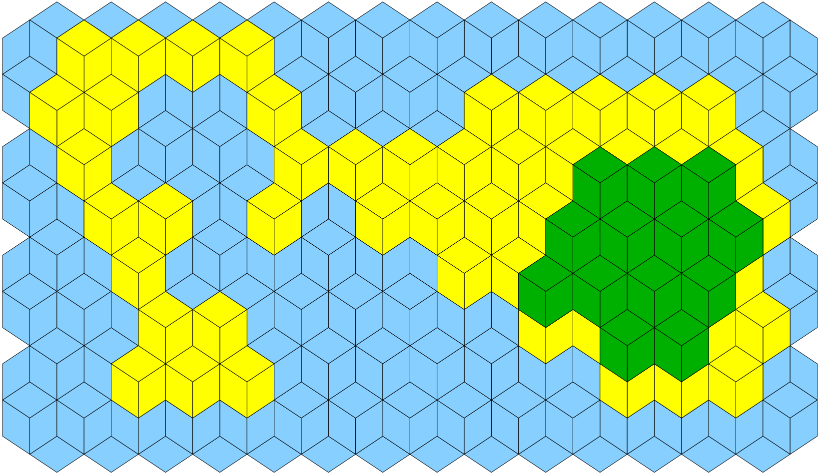
Flips

Topologically, flip can merge or split islands. . .



Flips

. . . and destroy or create holes.



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Cooling

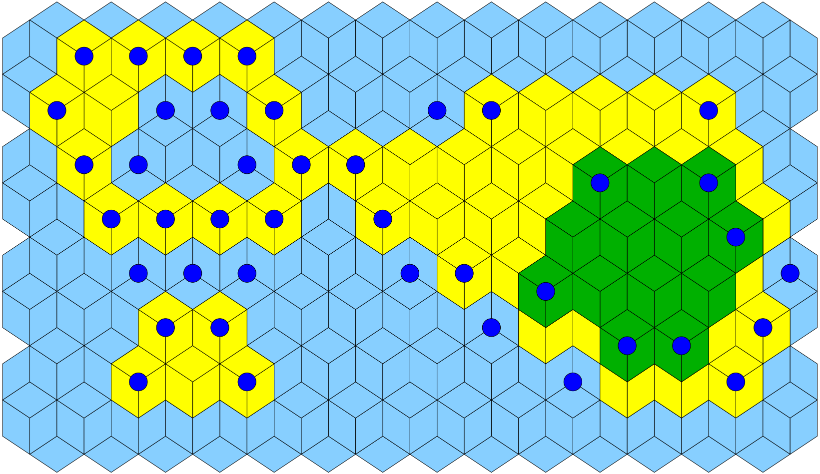
Cooling: Markov chain $(\omega_t)_{t\geq 0}$ defined by

- an initial tiling ω_0;
- $\omega_t \to \omega_{t+1}$: perform unif. at random a flip s.t. $\Delta E \leq 0$;
- stop if no flip s.t. $\Delta E \leq 0$.

The cooling stops only on error-free tilings (our "quasicrystals").

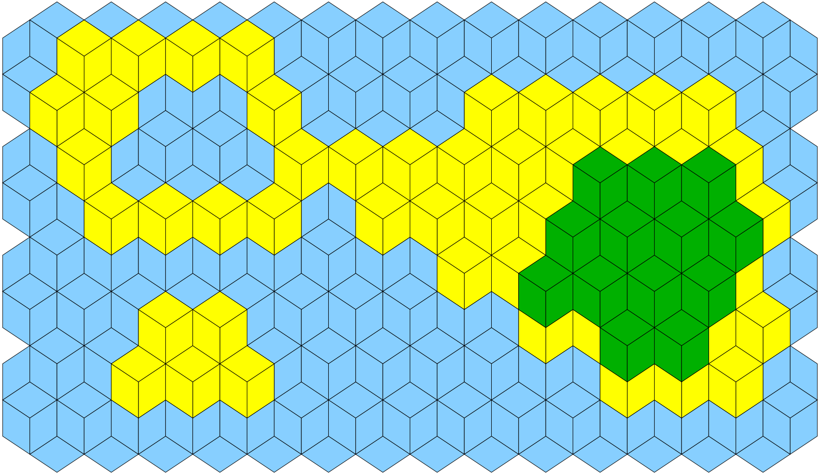
Example

Flips such that $\Delta E \leq 0$: around blue points.



Example

Cooling: random flips such that $\Delta E \leq 0$ (one at a time).



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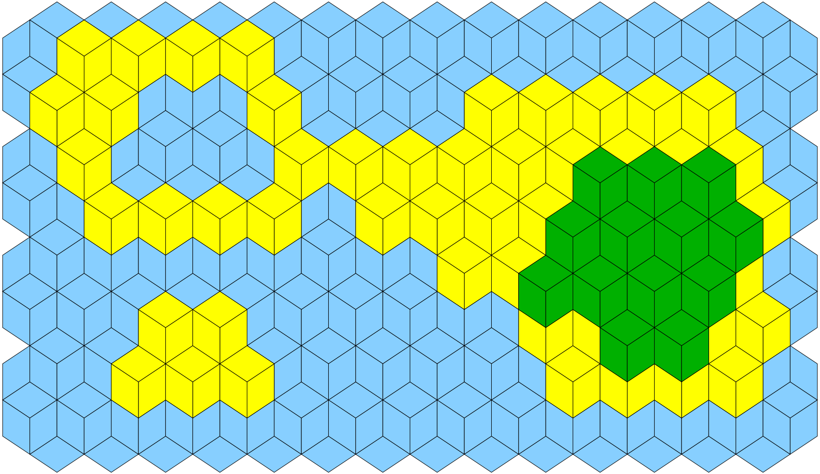

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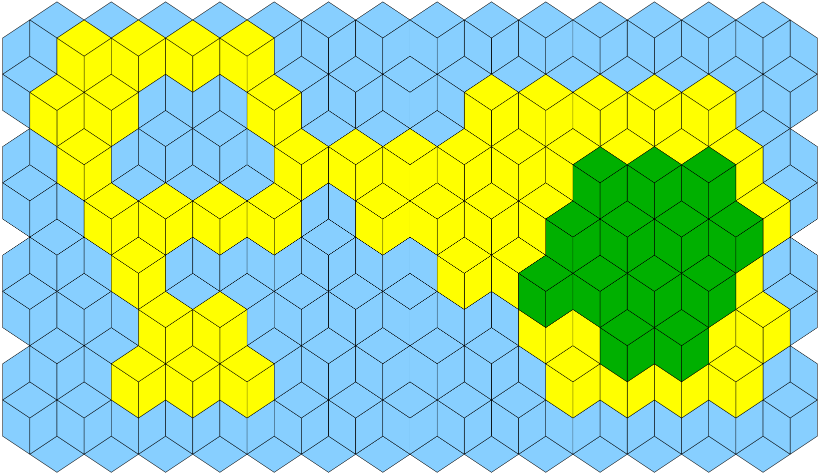

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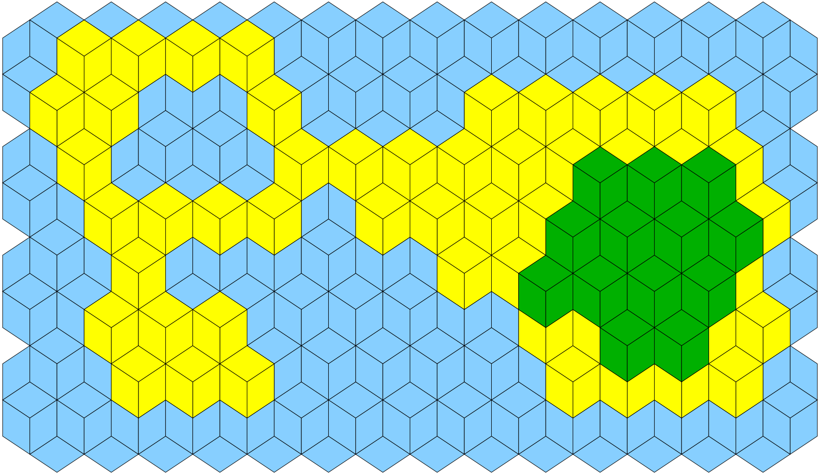

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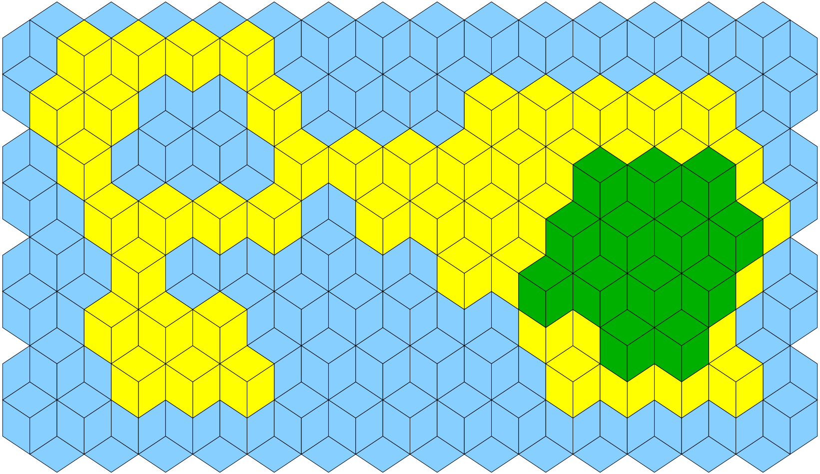

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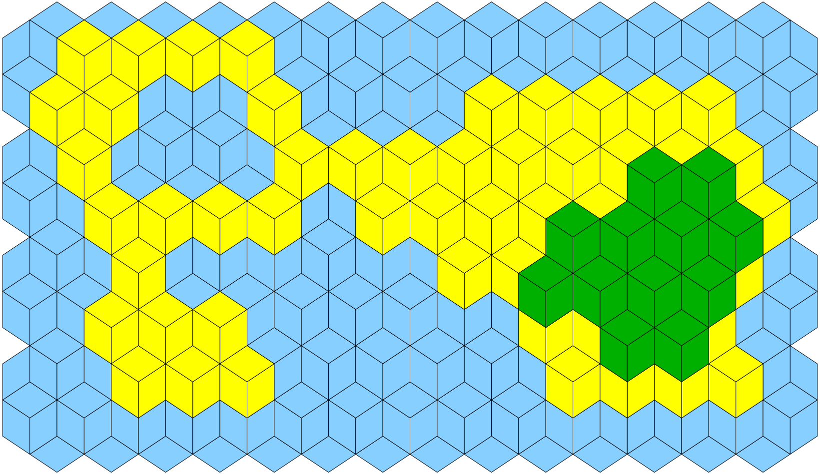

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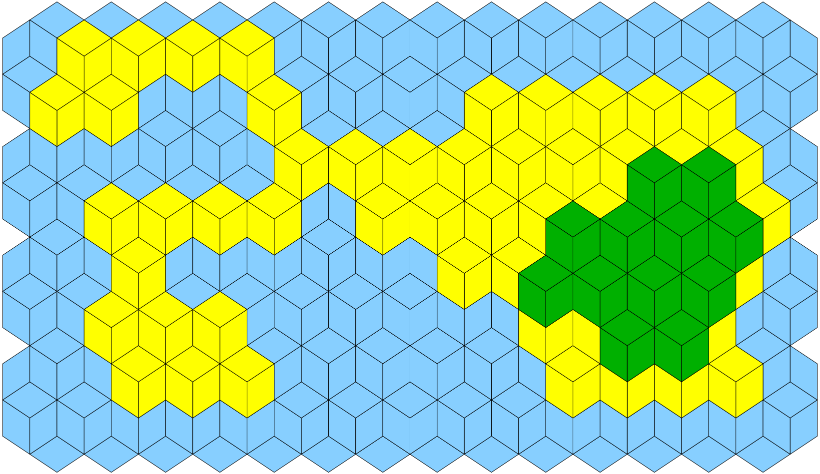

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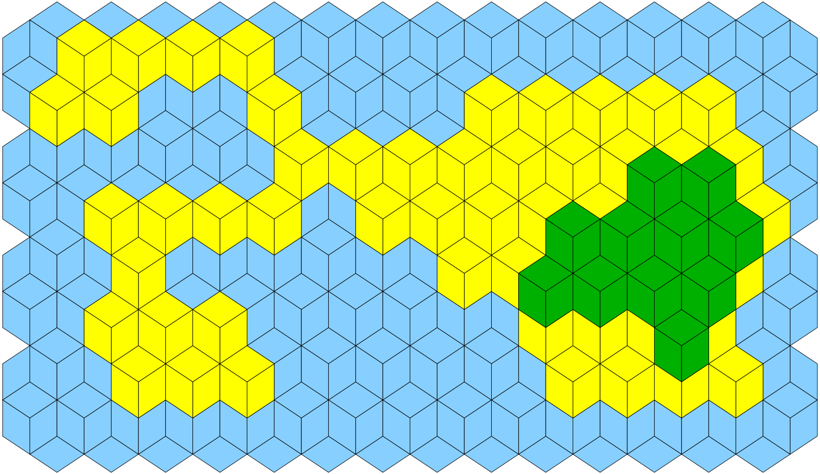

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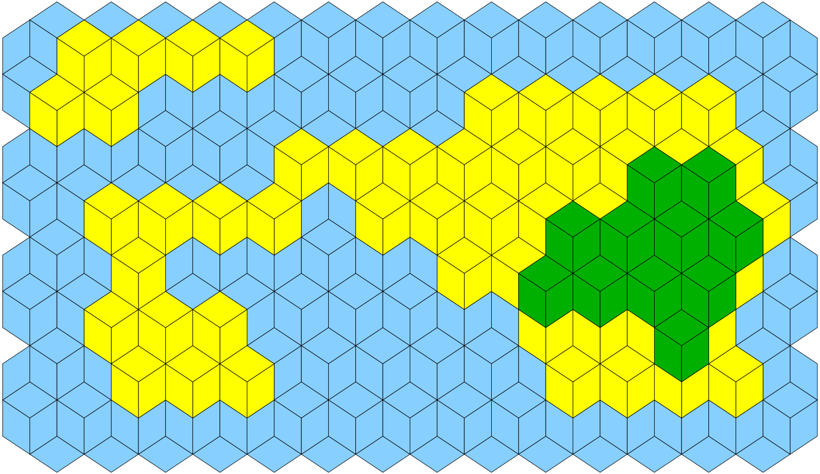

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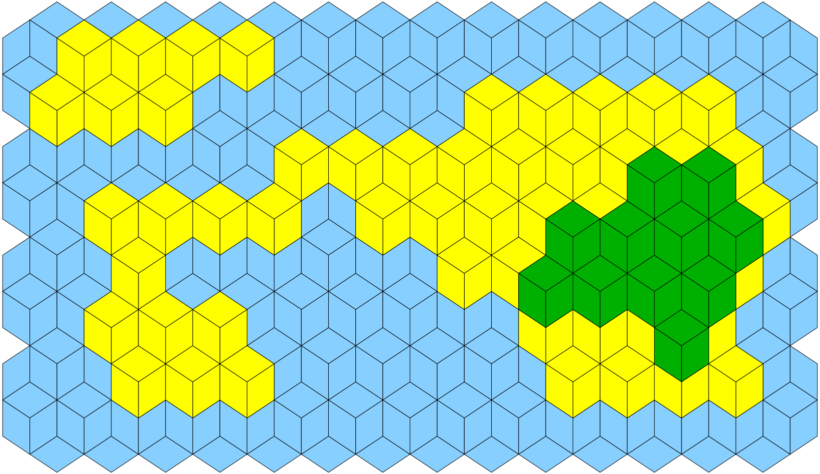
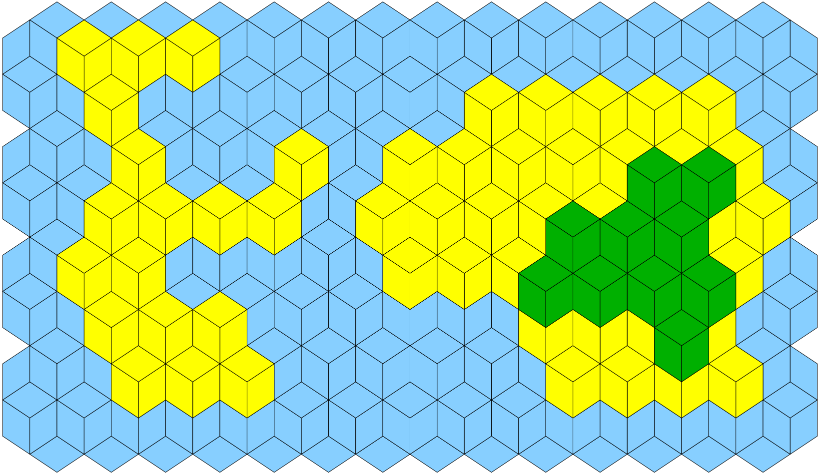


Example

Cooling: random flips such that $\Delta E \leq 0$ (10 at a time).



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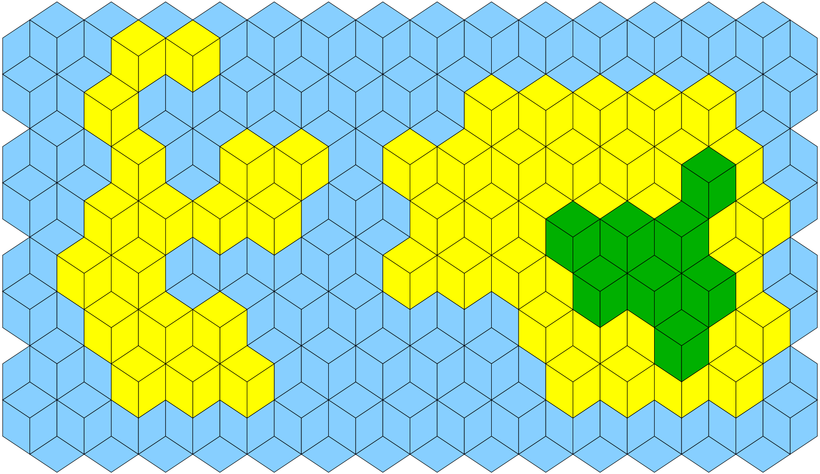

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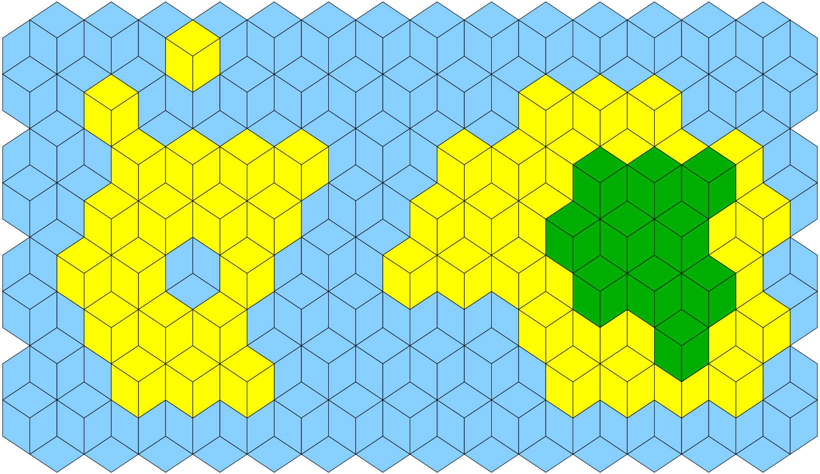

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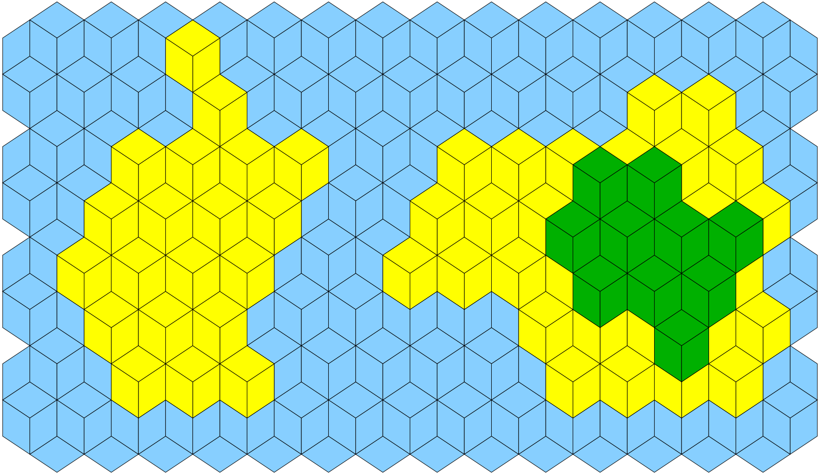

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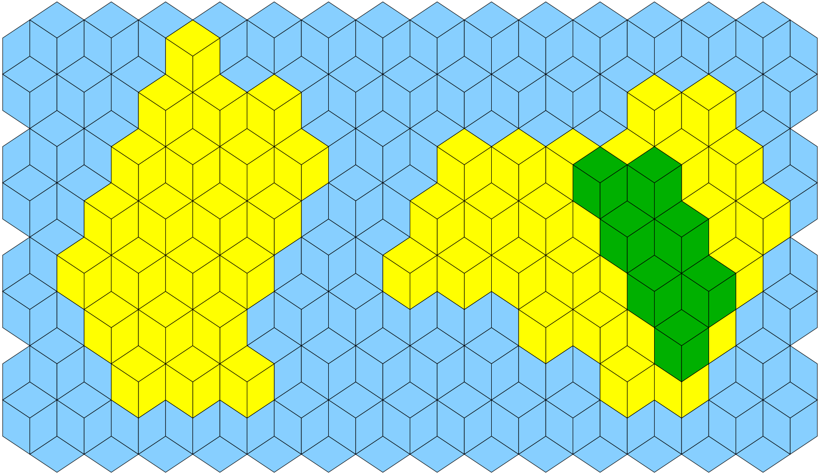

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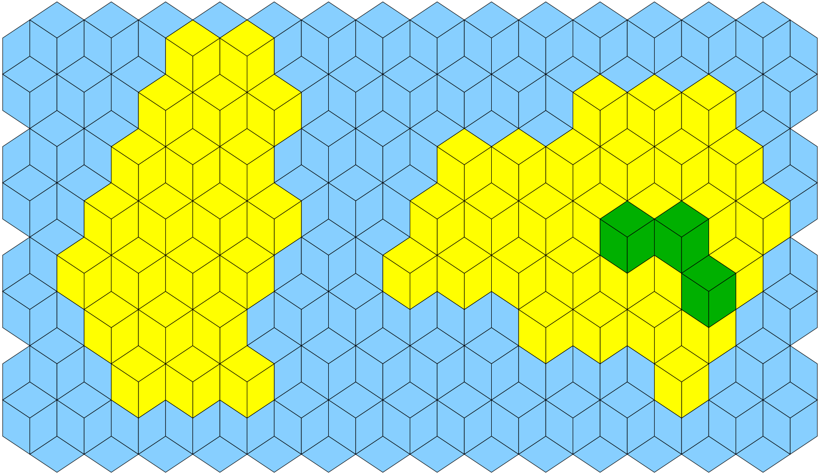

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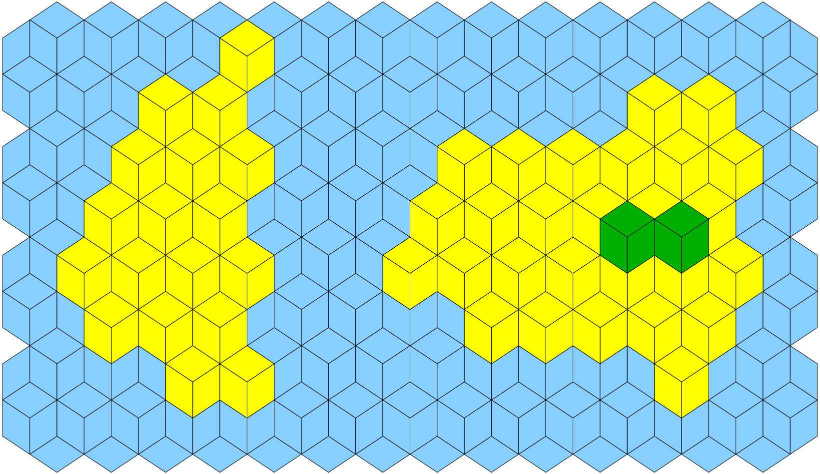

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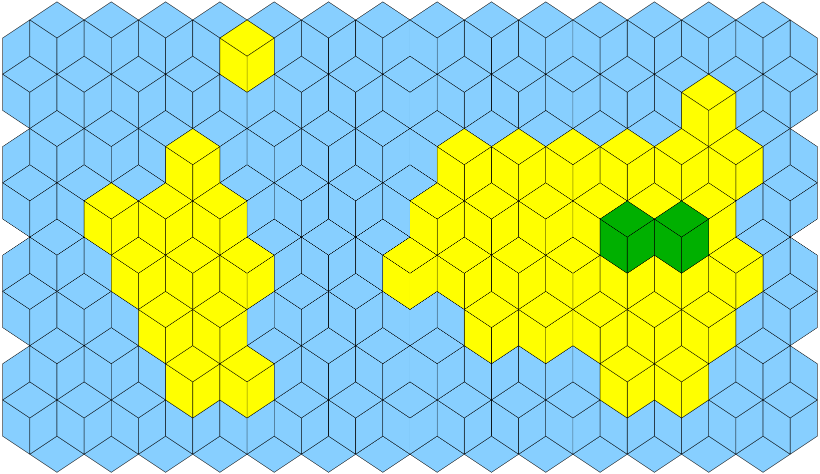

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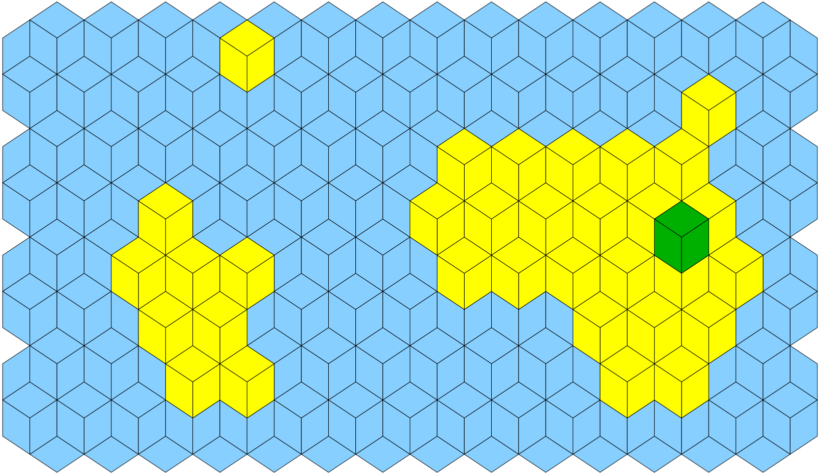

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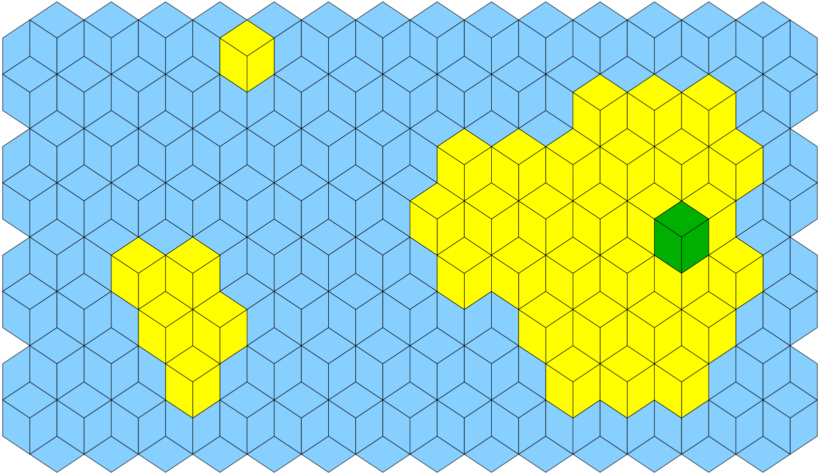

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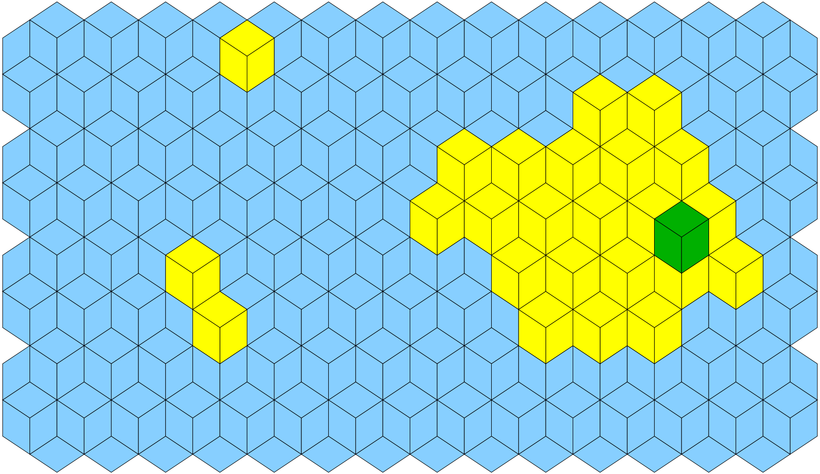

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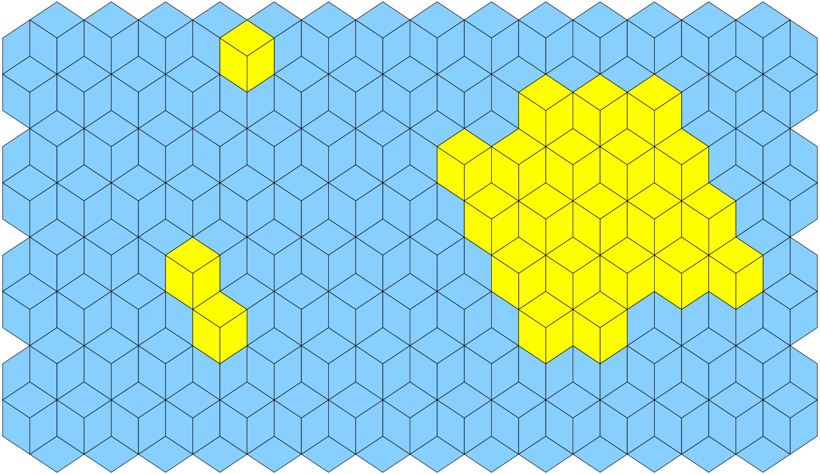

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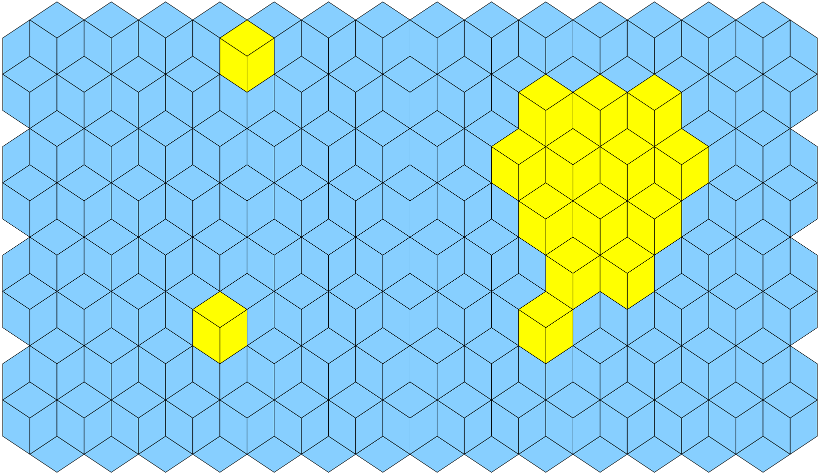

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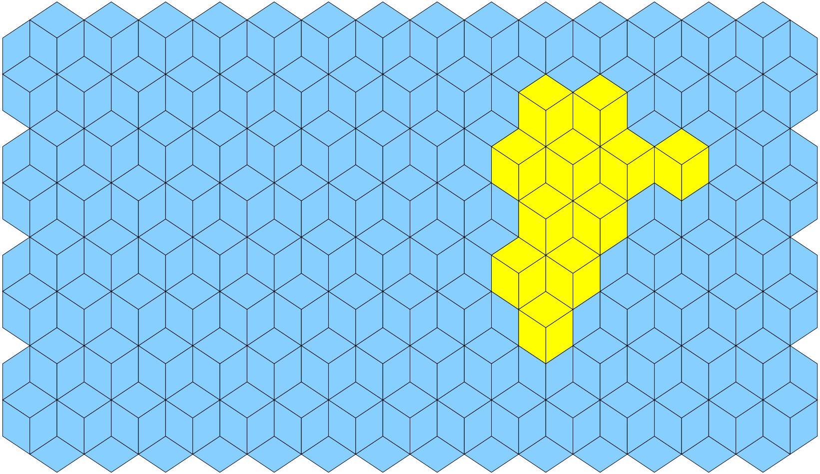

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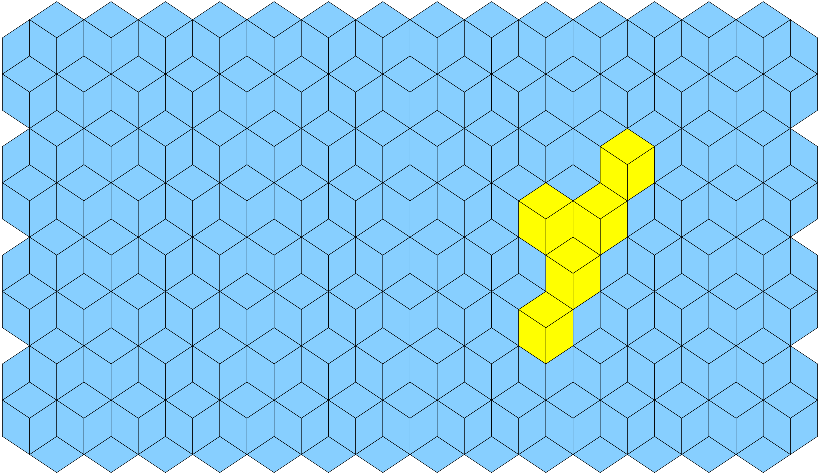
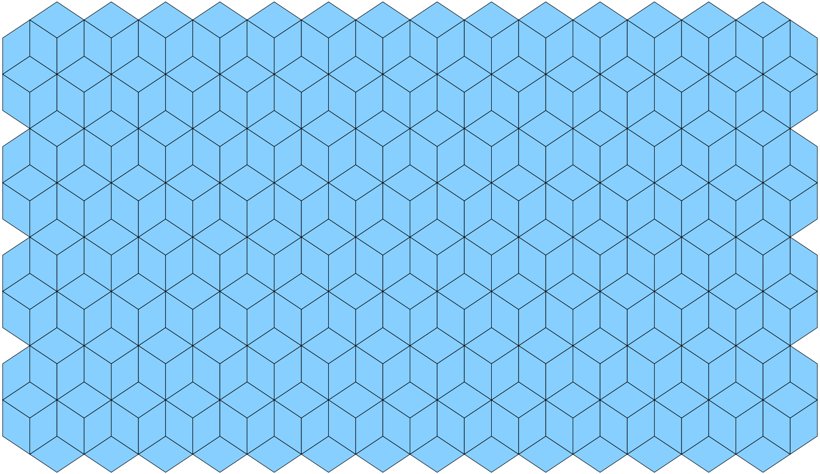


Example

Until no more flips are allowed (169 flips perfomed).



Convergence time

random variable T: number of performed flips.

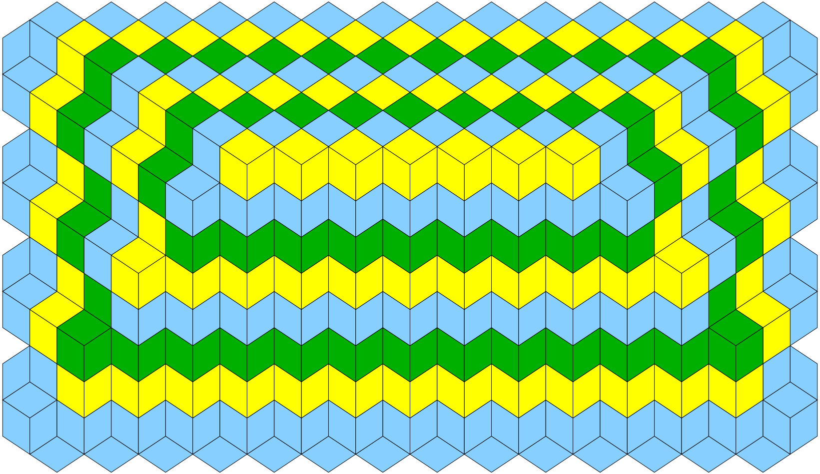
Worst average convergence time on a given region D:

$$\widehat{T} = \max_{\omega_0 \in \mathcal{P}(D)} \mathbb{E}(T \mid \omega = \omega_0).$$

Asymptotic behavior of $\widehat{T}$ when $n := |D| \to \infty$?

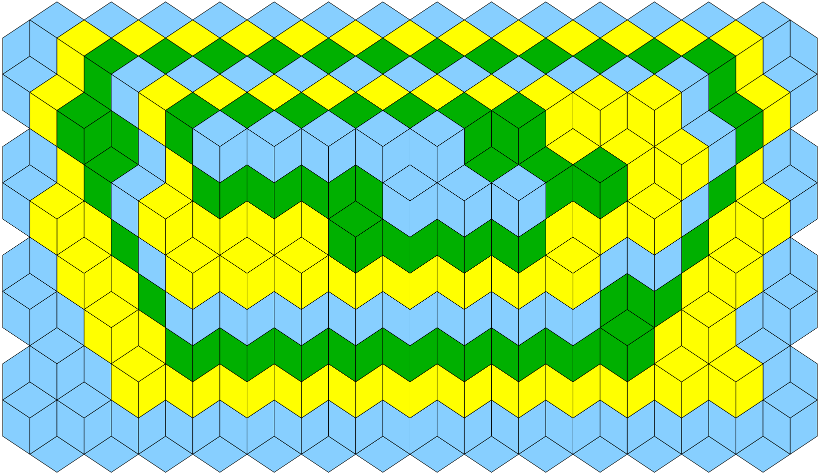
Numerical simulations

Worst case = maximal volume tiling? (colors: height modulo 3).



Numerical simulations

$\Theta(\sqrt{n})$ stacked islands of area $\Theta(n)$ to clear (100 flips at a time).



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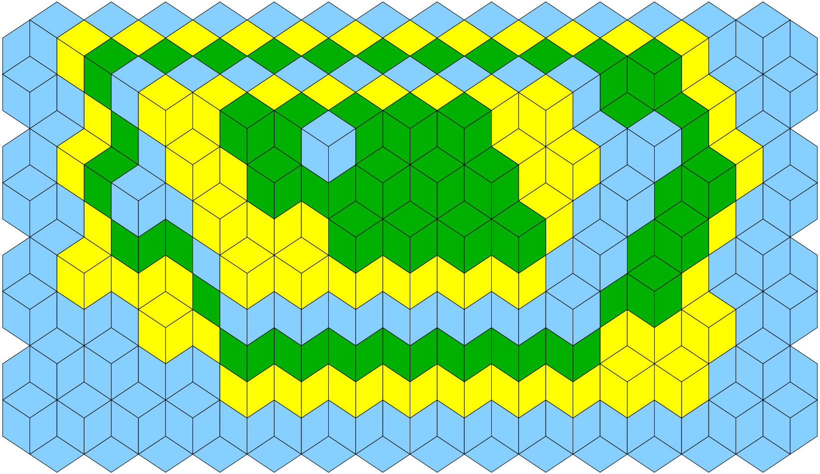

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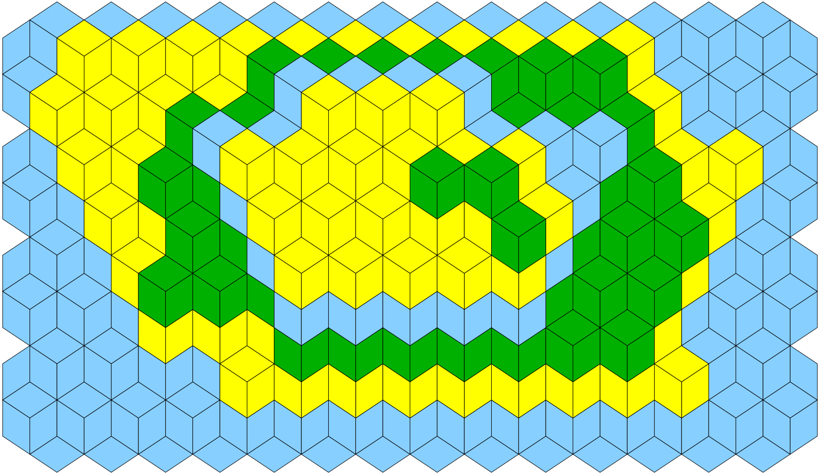

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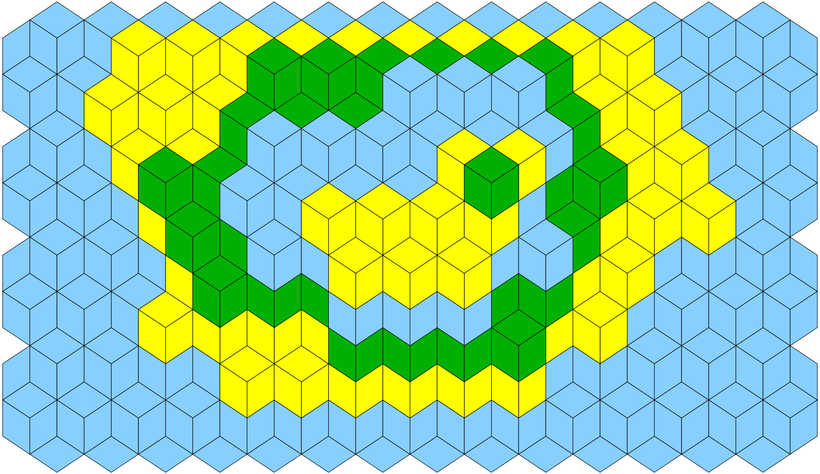

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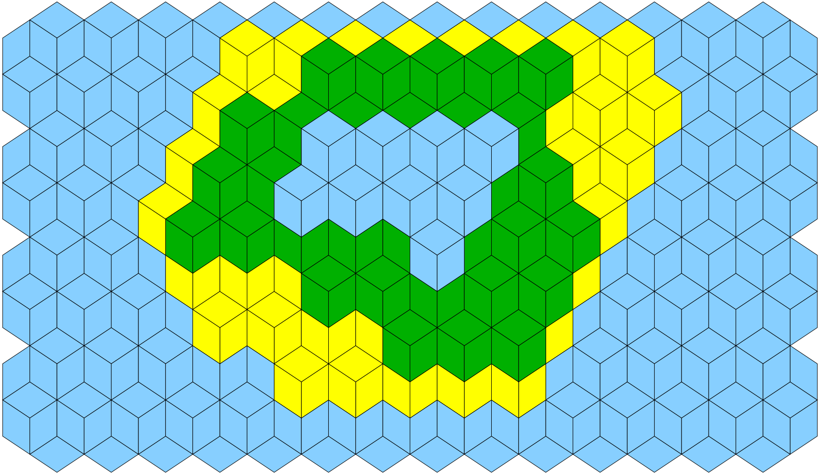

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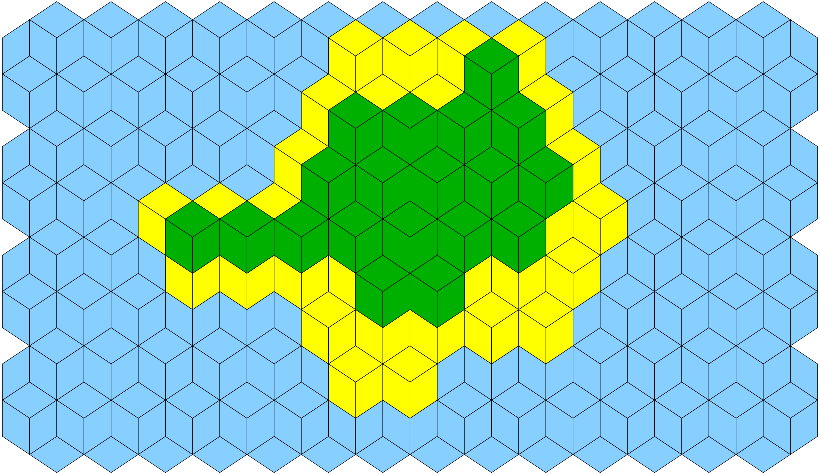

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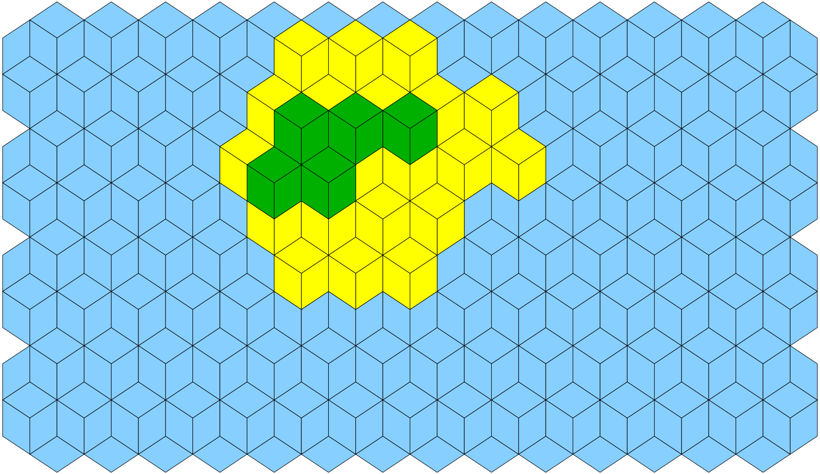
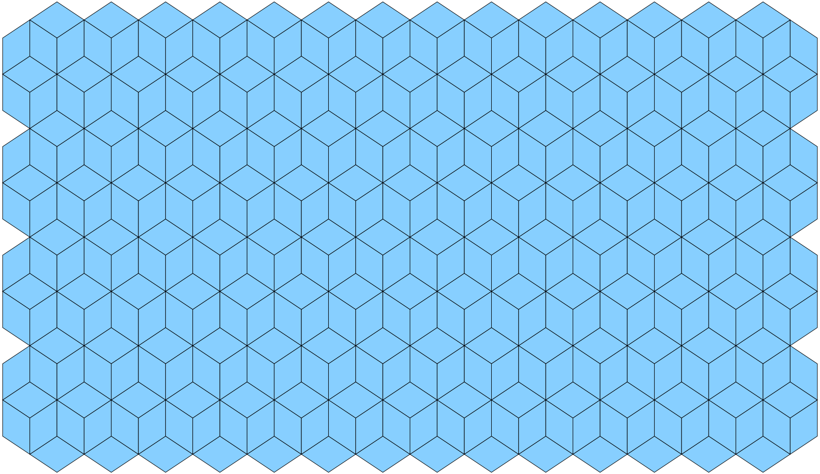


Numerical simulations

Numerical simulations $\rightsquigarrow$ conjecture: $\widehat{T} = \Theta(n^2)$.



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Tool

To bound $\widehat{T}$:

Proposition

Let $(\omega_t)_{t\geq 0}$ be a Markov chain over Ω.
If there are $\varepsilon > 0$ and a "potential function" $\phi : \Omega \to \mathbb{R}_+$ s. t.

$$\phi(\omega_t) > 0 \quad \Rightarrow \quad \mathbb{E}[\phi(\omega_{t+1}) - \phi(\omega_t)|\omega_t] \leq -\varepsilon\phi(\omega_t),$$

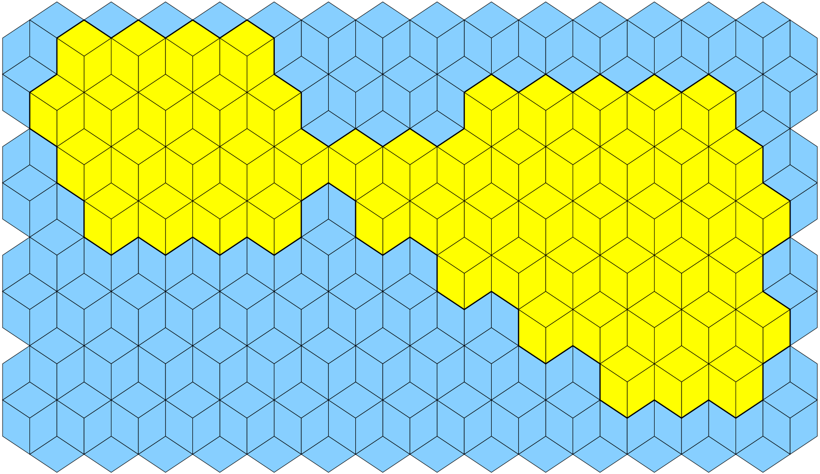
then

$$\mathbb{E}(\min\{t \mid \phi(\omega_t) = 0\}) \leq \frac{\log \phi(\omega_0)}{\varepsilon}.$$

How to find ϕ which satisfies such a "differential equation"?

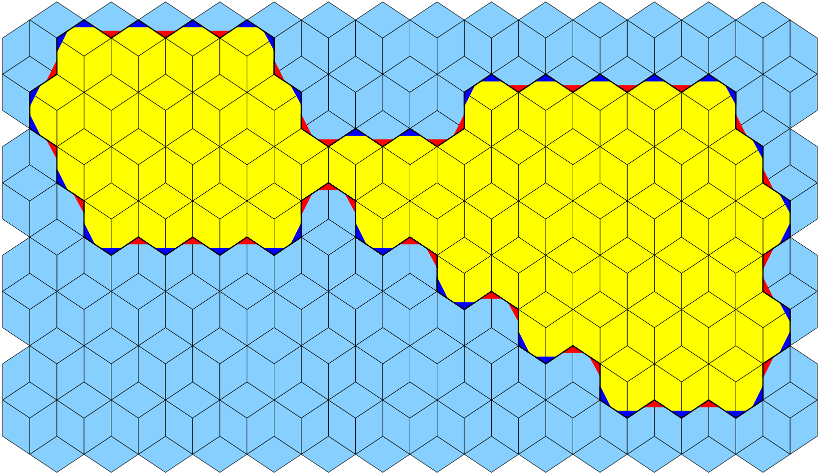
One island

Consider the border of a hole-free islande with F possible flips.



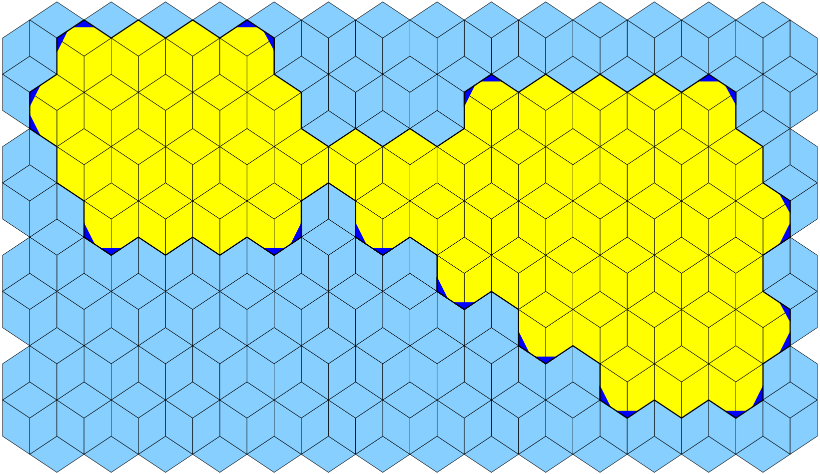
One island

There is always 6 more salient than reflex angles (induction).



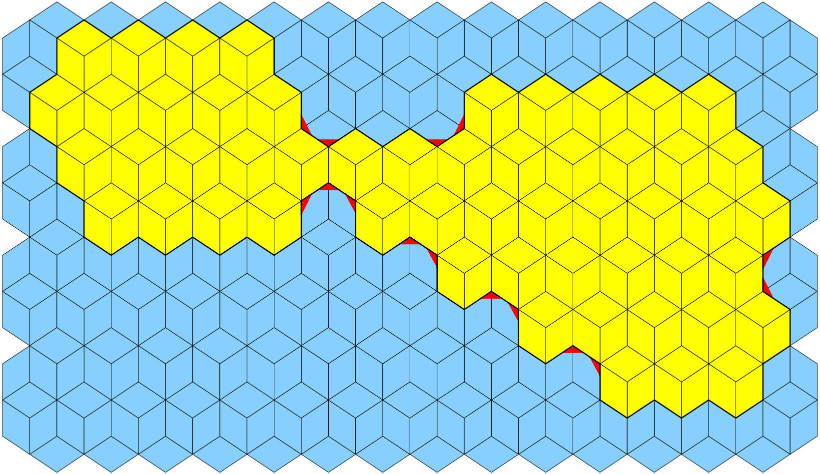
One island

$i \geq 2$ salient angles in a row $\rightsquigarrow$ flip s.t. $\Delta(4V + E) = -2i$.



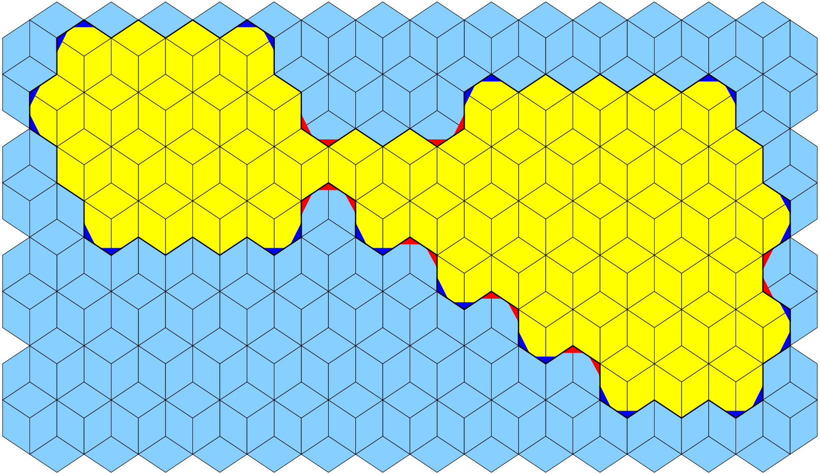
One island

$i \geq 2$ reflex angles in a row $\rightsquigarrow$ flip s.t. $\Delta(4V + E) = +2i$.



One island

Let $\phi = 4V + E$. We thus have $\mathbb{E}(\Delta\phi) \leq -6 \times 2/F$.



More islands

Applied to each of the k island of a tiling:

$$\mathbb{E}(\Delta\phi) \leq -\frac{12k}{F}.$$

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Triple scam!

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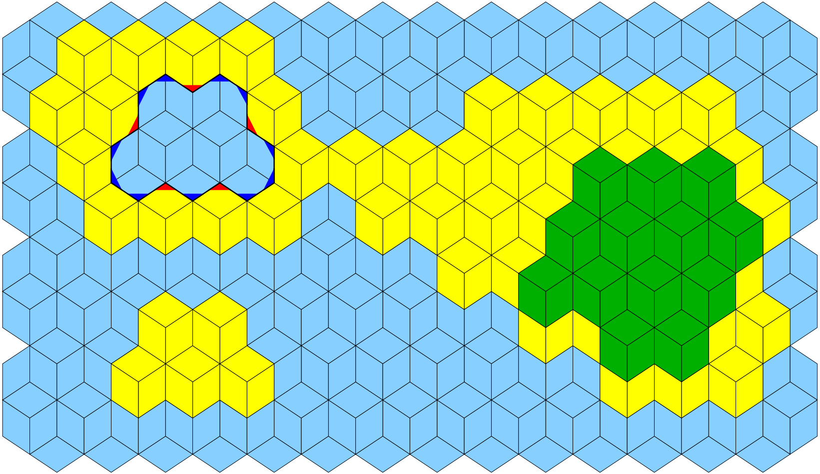
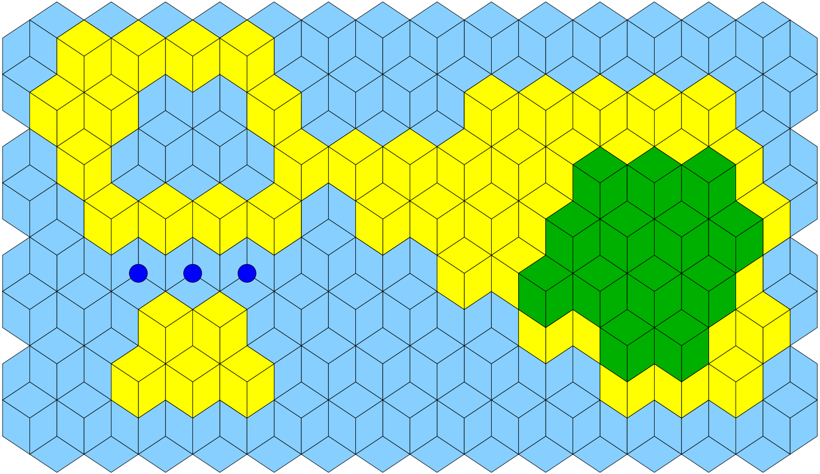
1 - Holes have an adverse effect on the volume, hence on ϕ.



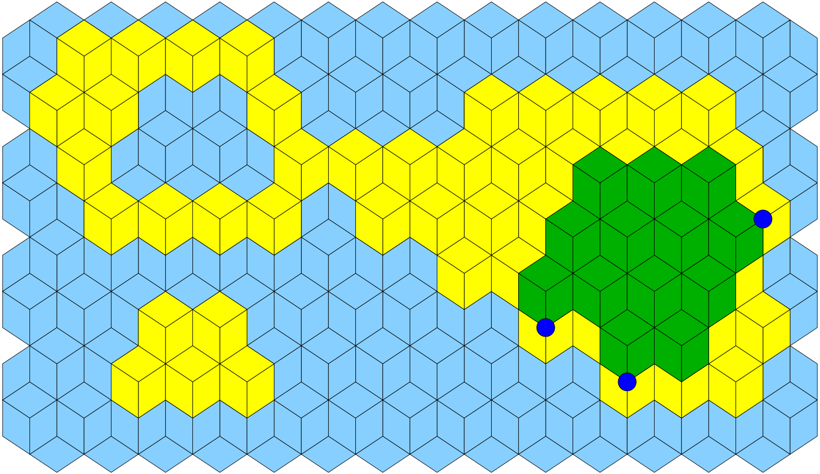
Triple scam

2 - Islands merging may increase ϕ.



Triple scam

3 - Obstructions between stacked island can prevent ϕ to decrease.



1 Tilings and flips

2 Cooling process

3 An upper bound

4 A second upper bound

Second tool

To bound $\widehat{T}$:

Proposition

Let $(\omega_t)_{t\geq 0}$ be a Markov chain over Ω.
If there are $\varepsilon > 0$ and a "potential function" $\phi : \Omega \to \mathbb{R}_+$ s. t.

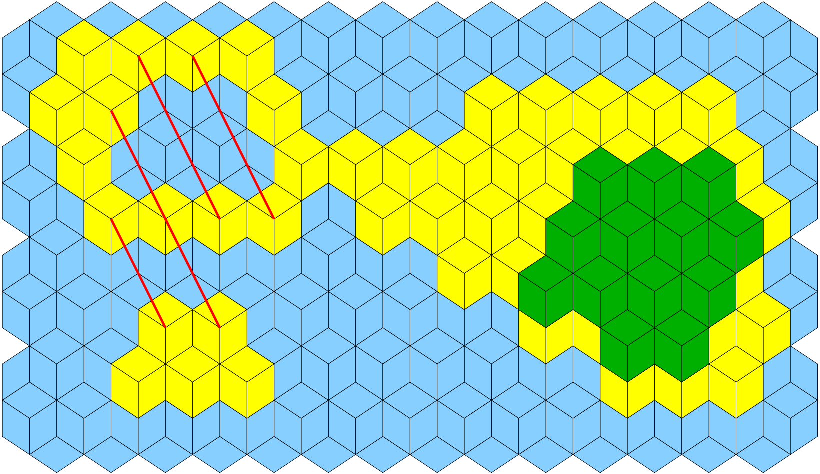
$$\phi(\omega_t) > 0 \quad \Rightarrow \quad \mathbb{E}[\phi(\omega_{t+1}) - \phi(\omega_t)|\omega_t] \leq -\varepsilon,$$

then

$$\mathbb{E}(\min\{t \mid \phi(\omega_t) = 0\}) \leq \frac{\phi(\omega_0)}{\varepsilon}.$$

Trick: triconvex hull

Triconvex hull $\overline{\omega}$ of ω: convexity in three directions.



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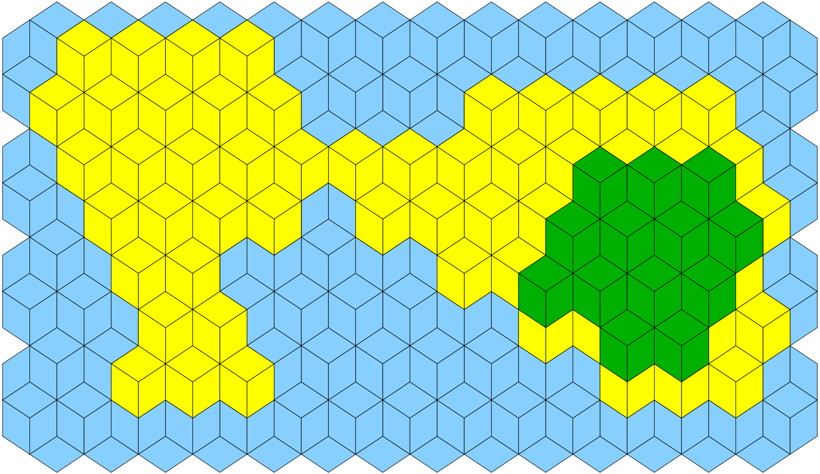

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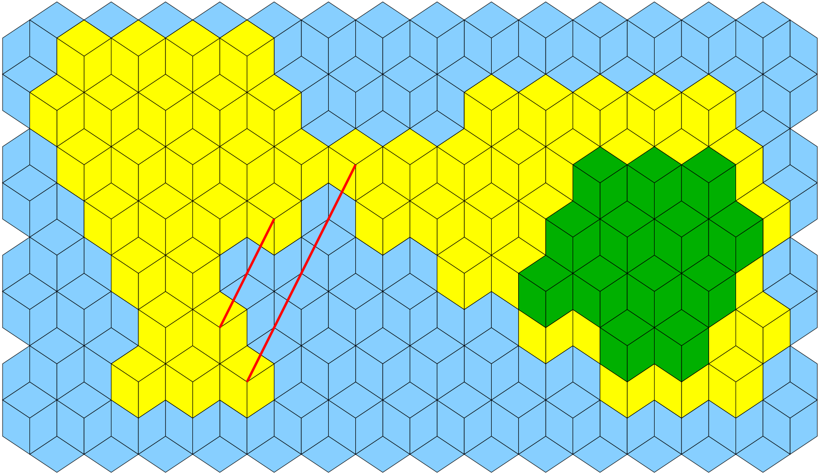

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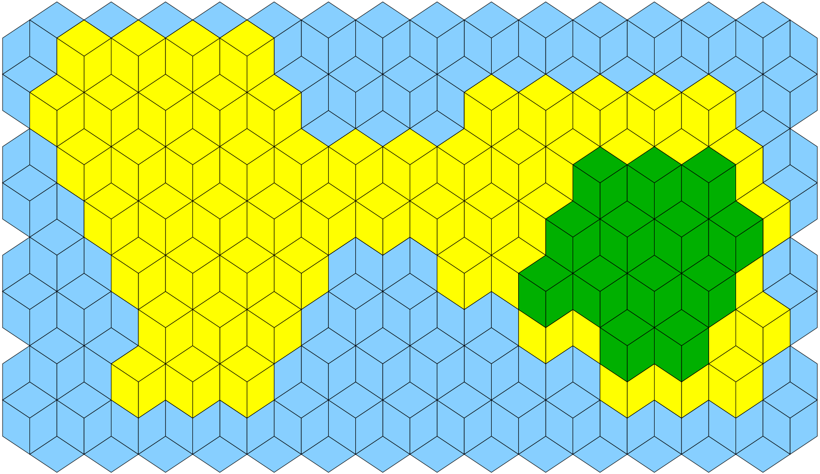

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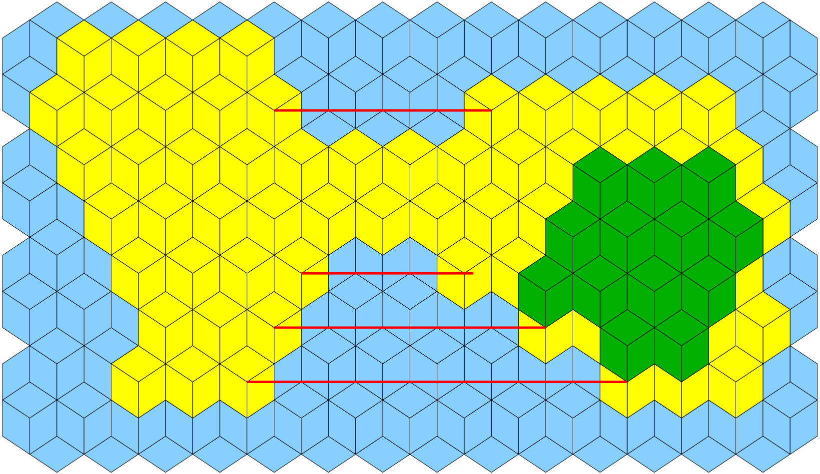

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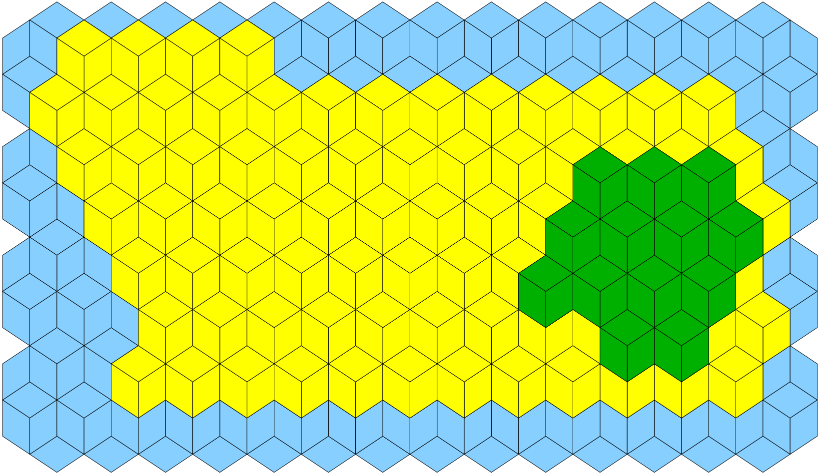

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If ω has height 1, the hull brings us back to the one island case

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For height $k \leq \sqrt{n}$, the highest islands disappear in $O(n^2)$. Thus

$$\widehat{T} = O(n^2\sqrt{n}).$$

Where comes the $\sqrt{n}$ factor from?

No more scam here, but we somewhere lost a $\sqrt{n}$ factor.

Top-down argument forgets lower flips that can decrease ϕ.
However, similuations suggest $\widehat{T} = \Theta(n^2)$ even for height 1.

Triconvex hull forgets inner flips that can decrease ϕ.
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Can we get a tight bound?

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Can we get a tight bound?

And what about the *average average convergence time*?

Th. Fernique, D. Regnault, *Stochastic flips on dimer tilings*, Disc. Math. Theor. Comput. Sci. (2010).