Markov Chains on Tilings: From Chaos to Order

Thomas Fernique

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Quasicrystals and tilings

Definition (IUCr, 1992)

Crystal = ordered material = essentially discrete diffraction.

1982: discovery of the first non-periodic crystal ($\rightsquigarrow$ Nobel in 2011).

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Definition

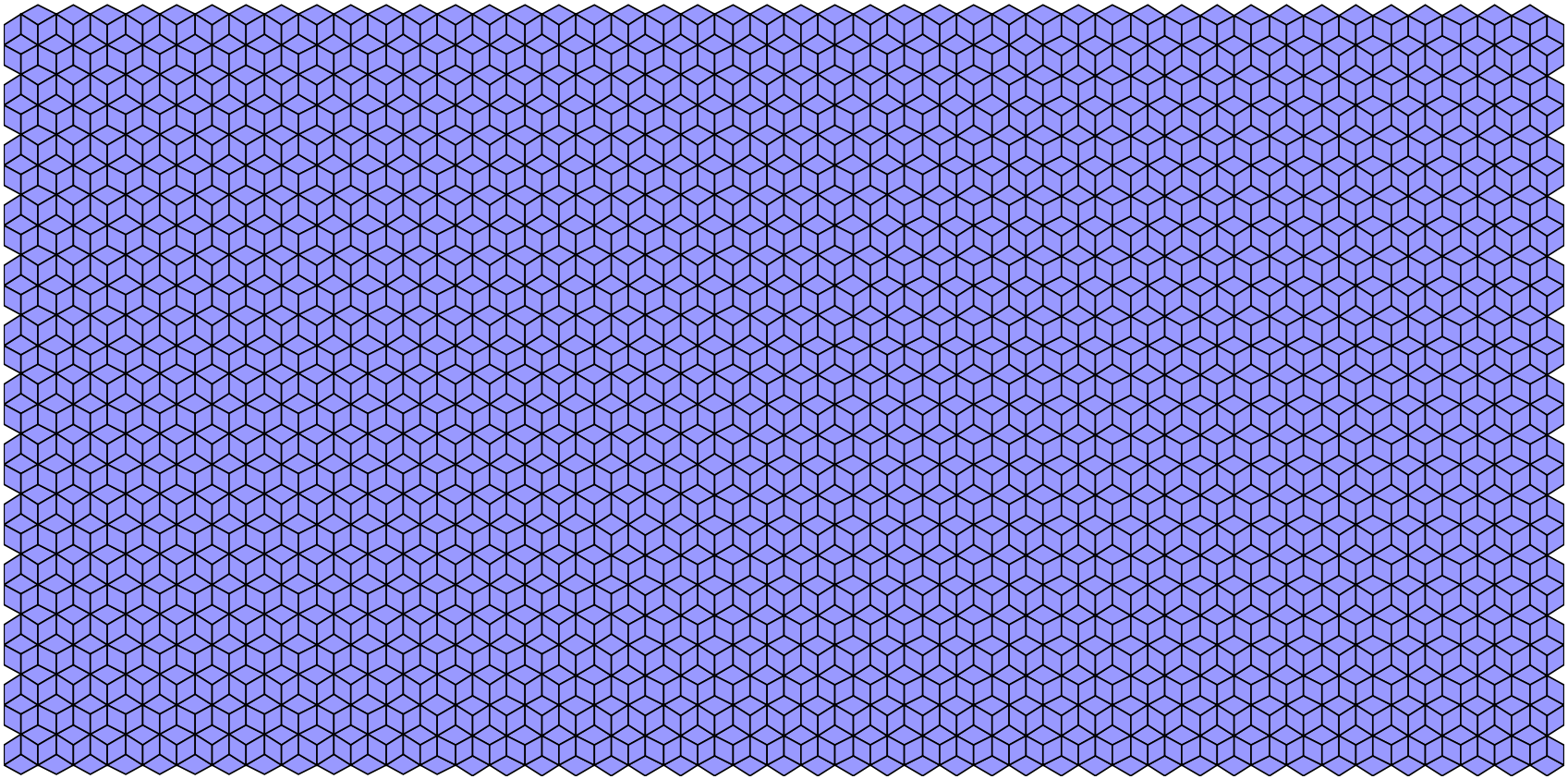
Tiling = covering of the plane by non-overlapping compact sets.

Example: digitizations of affine planes in higher dimensional space.

Quasicrystal stability (at low T): finite range energetic interaction.
Modelled on tilings by constraints on the way things locally fit.

Example 1: Dimer tilings

Rows alternate rhombi (between their orientation):



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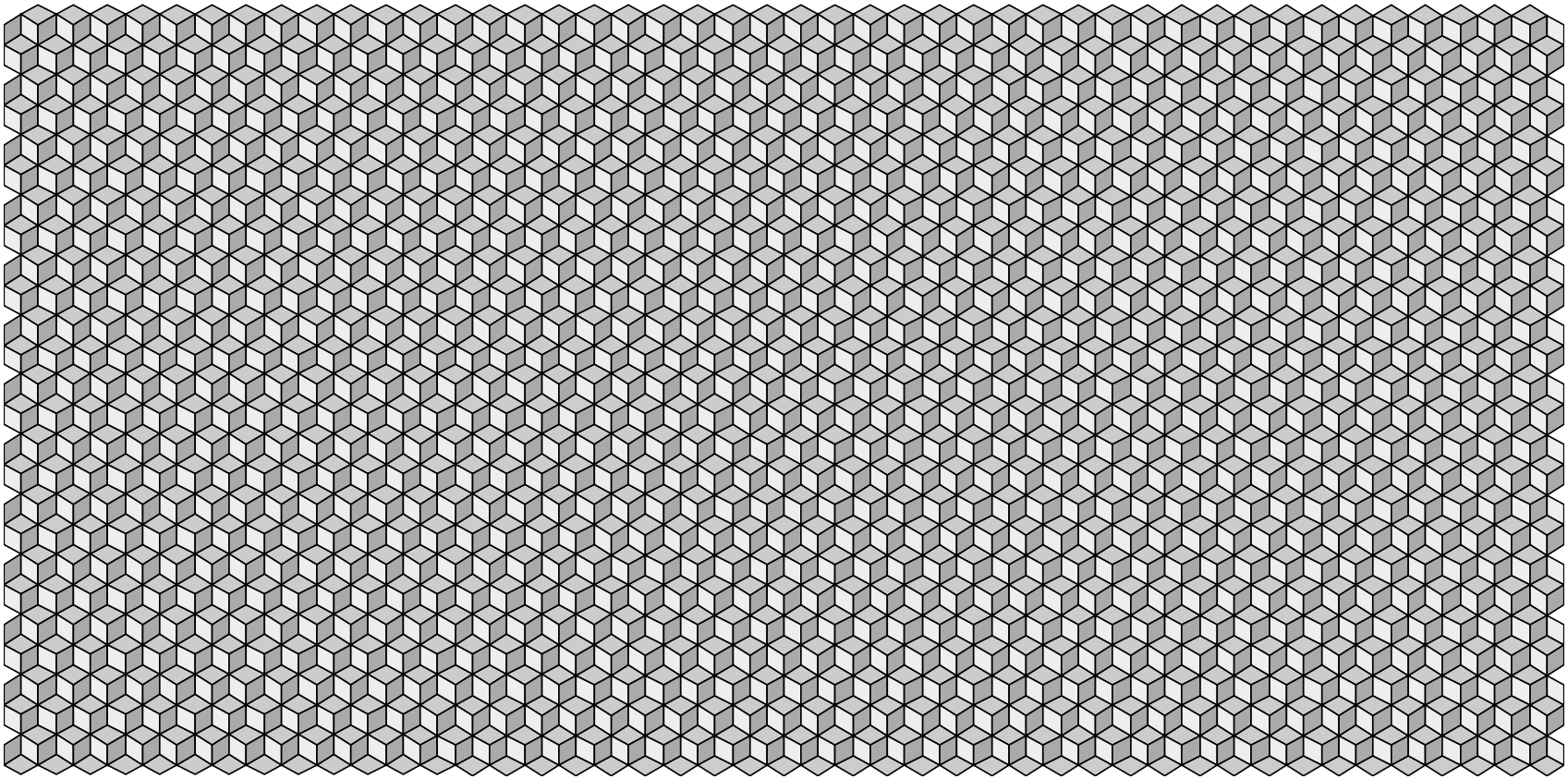

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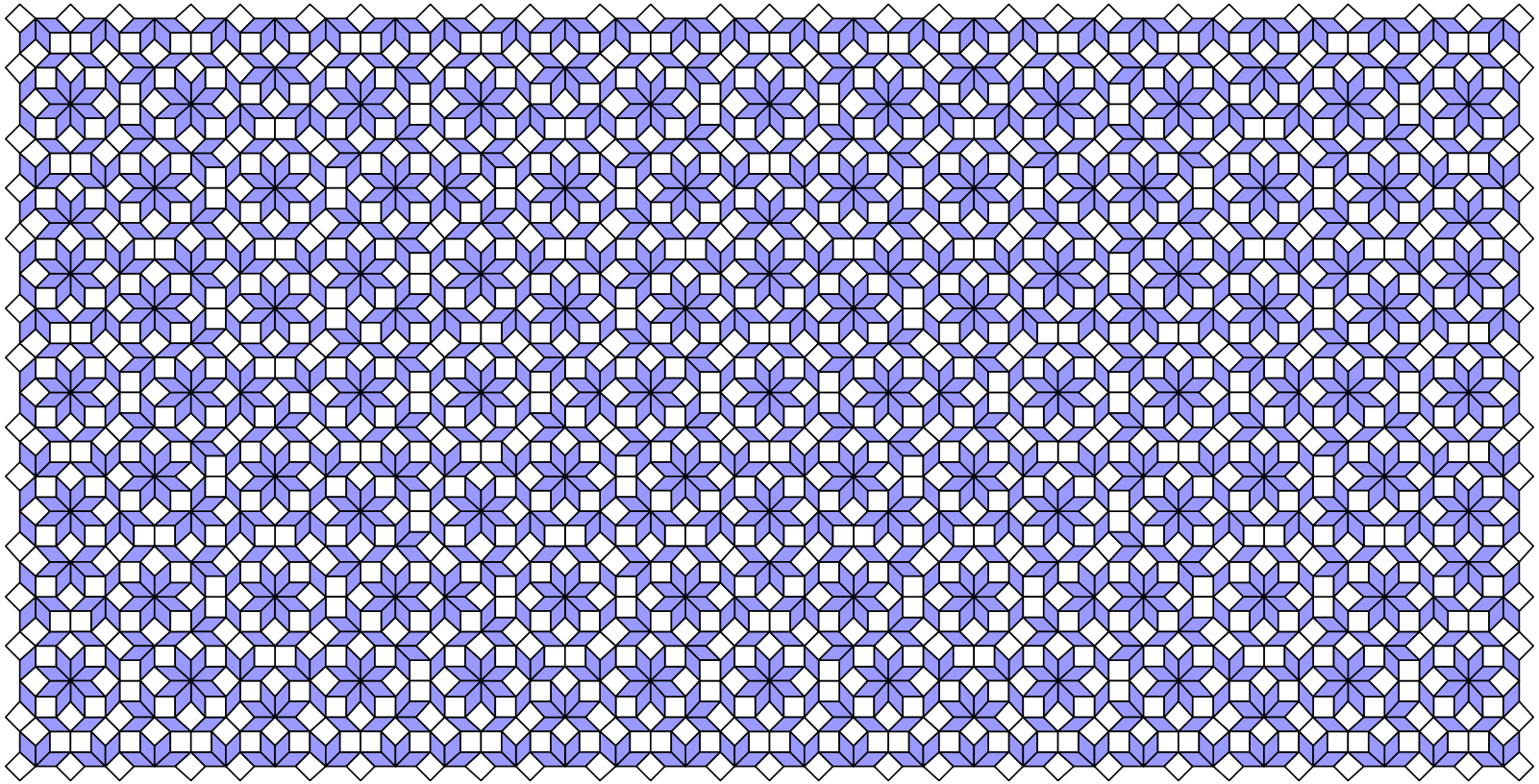

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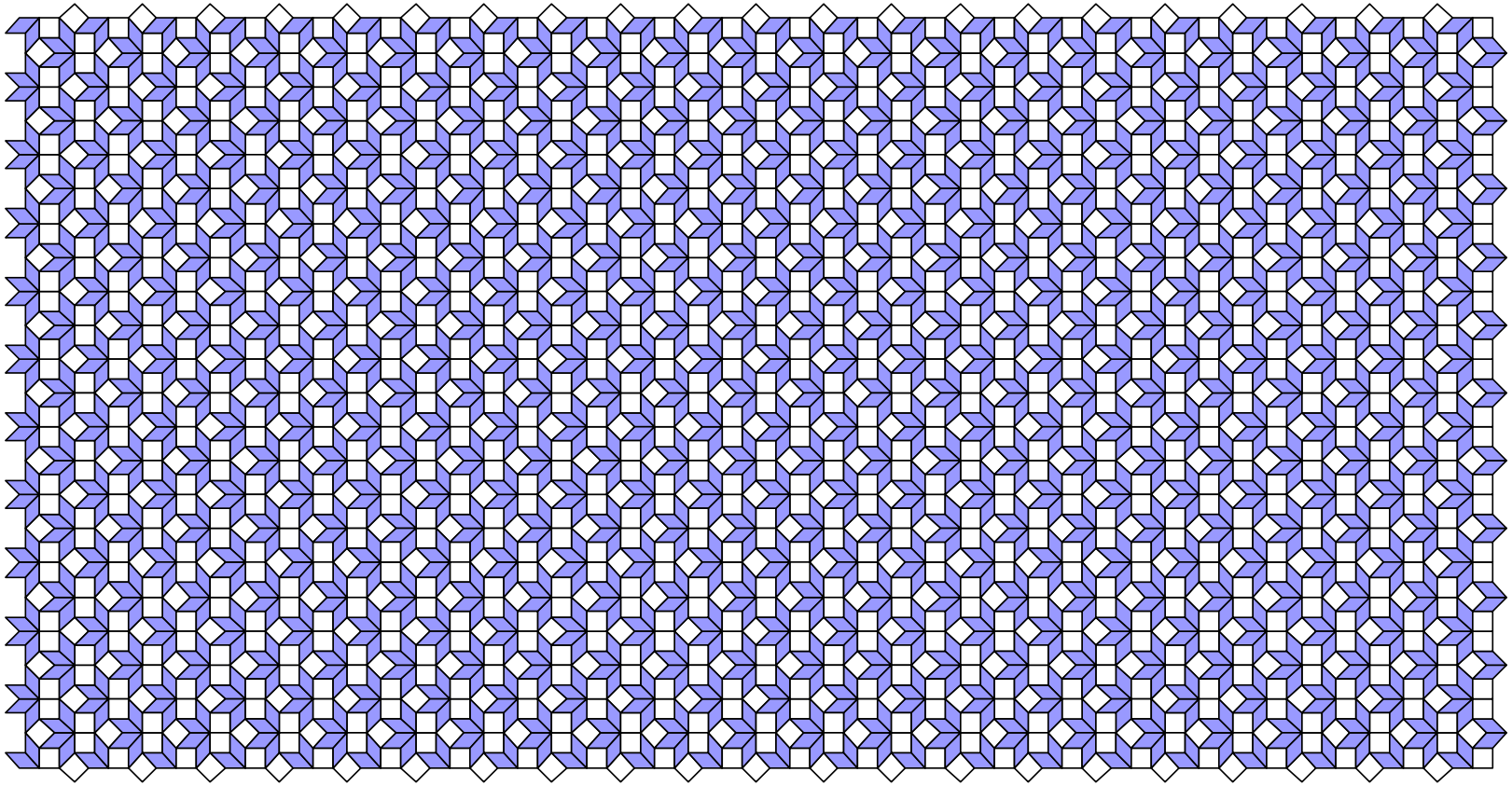

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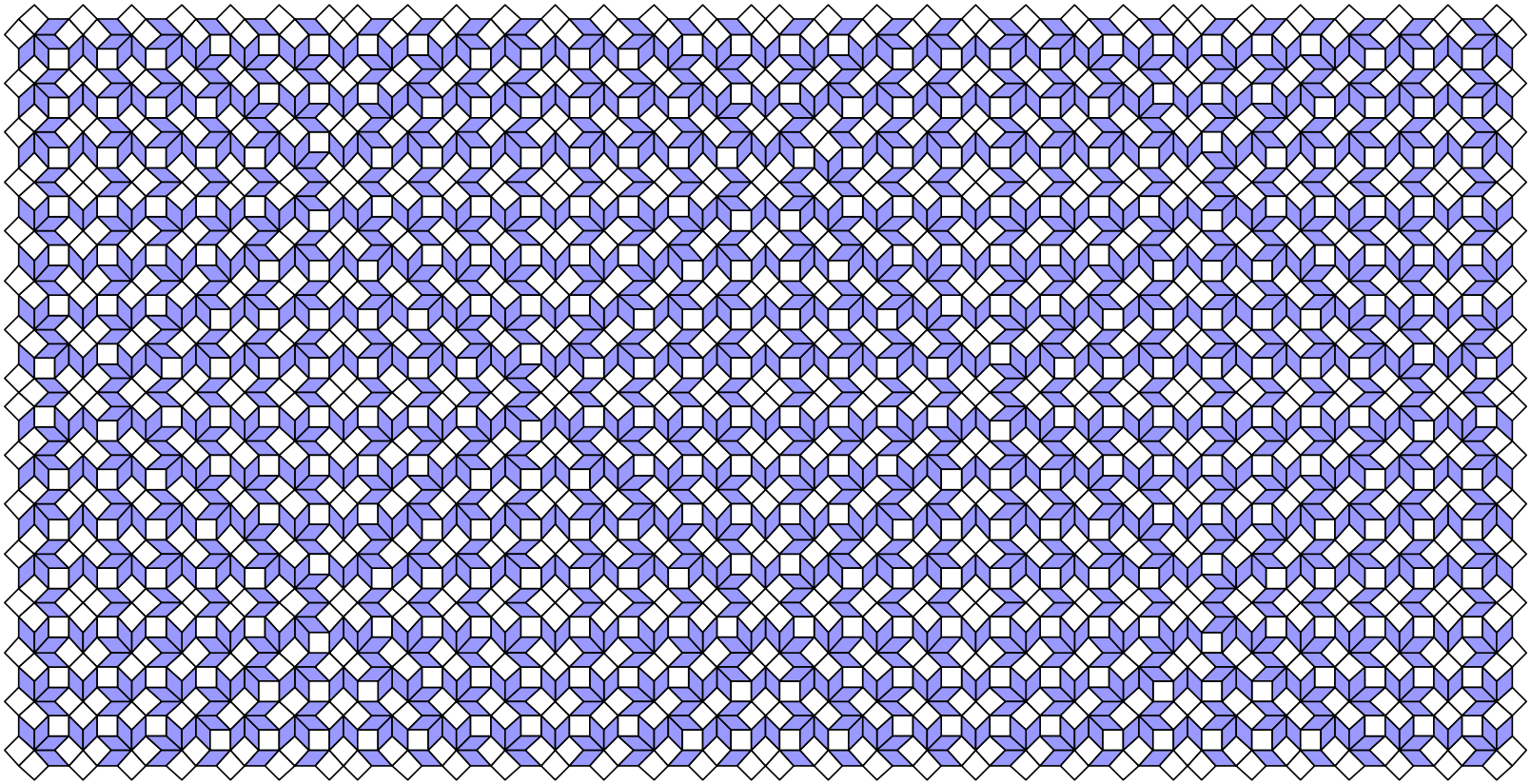

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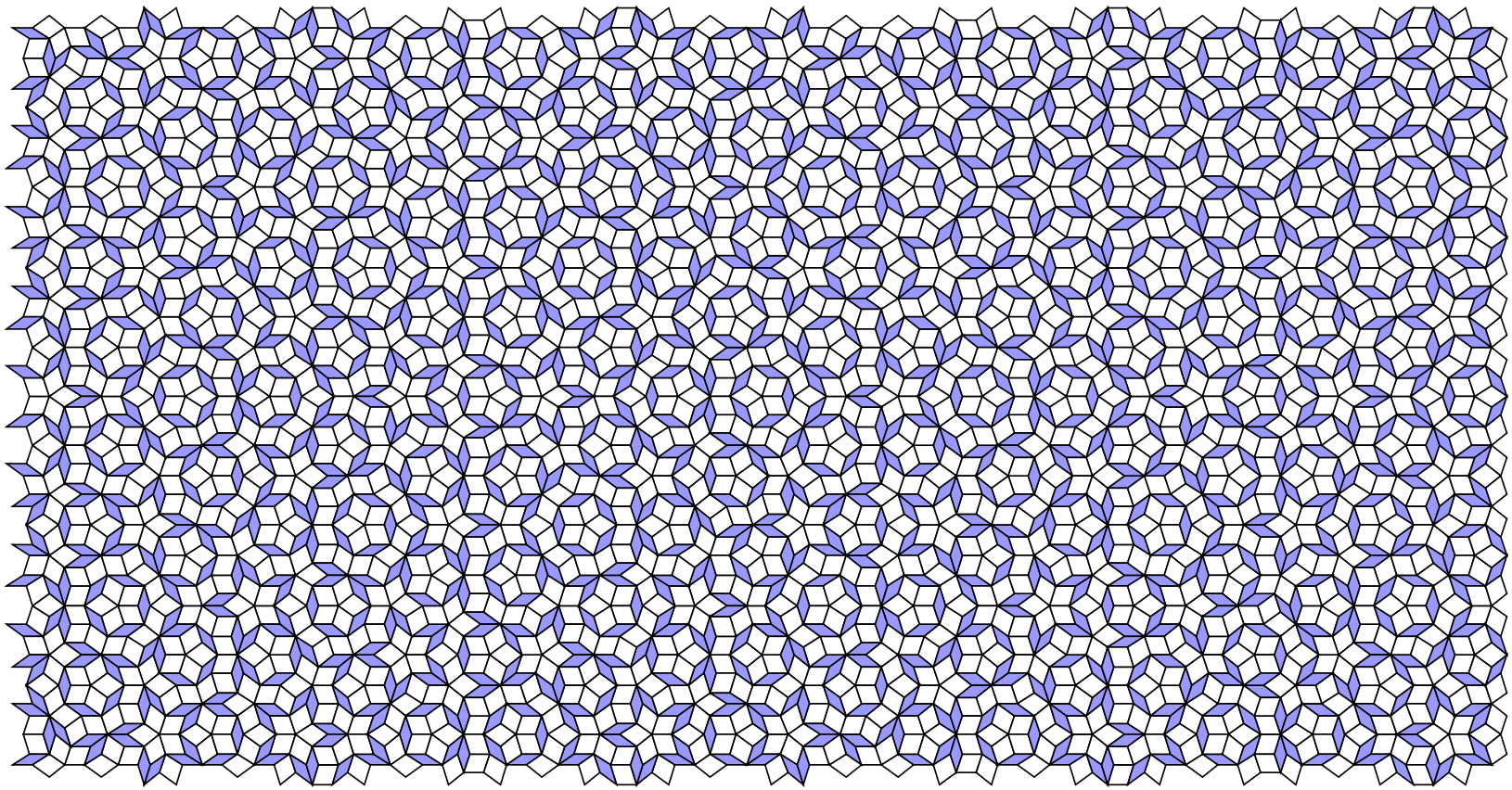

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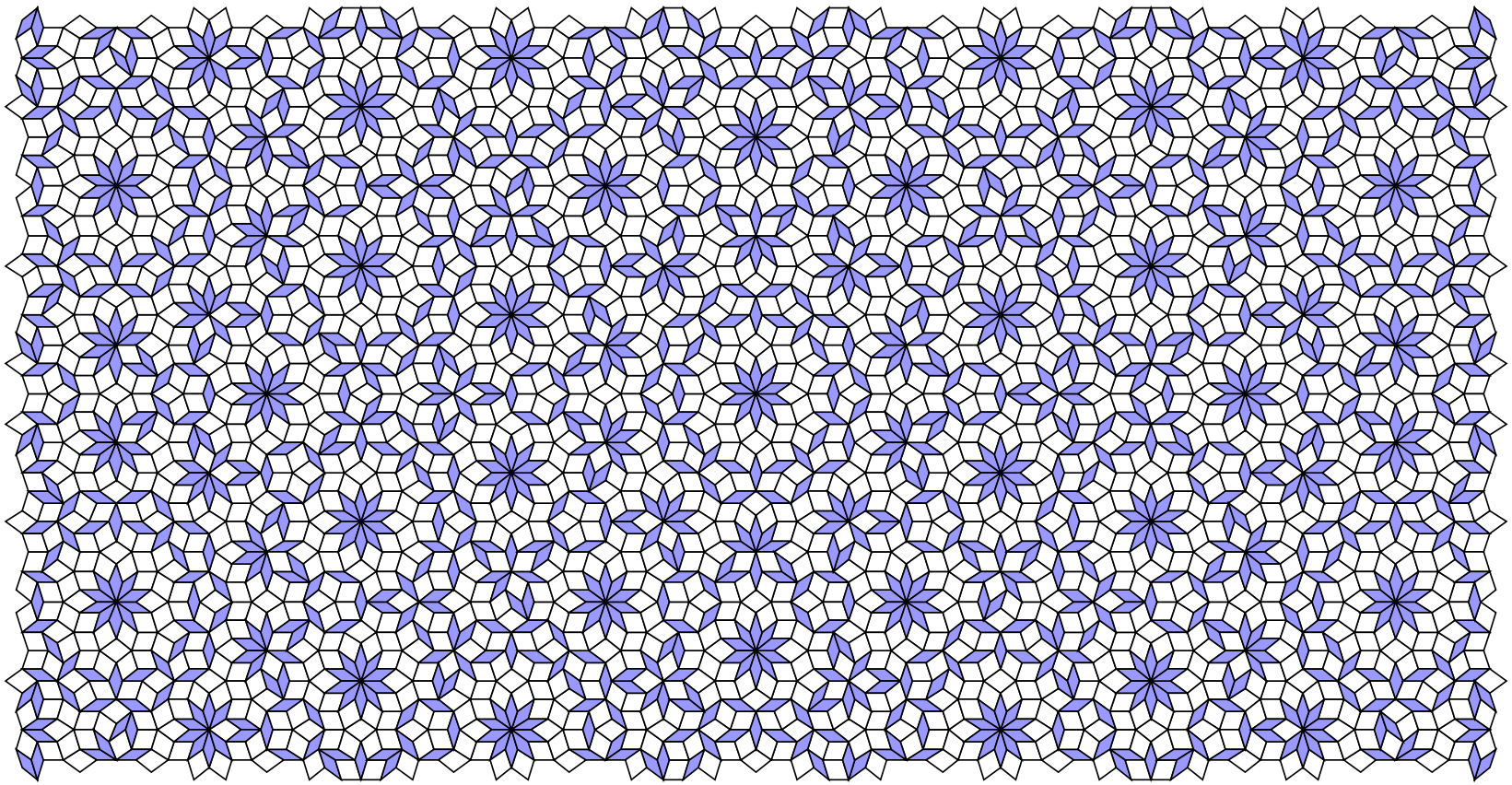

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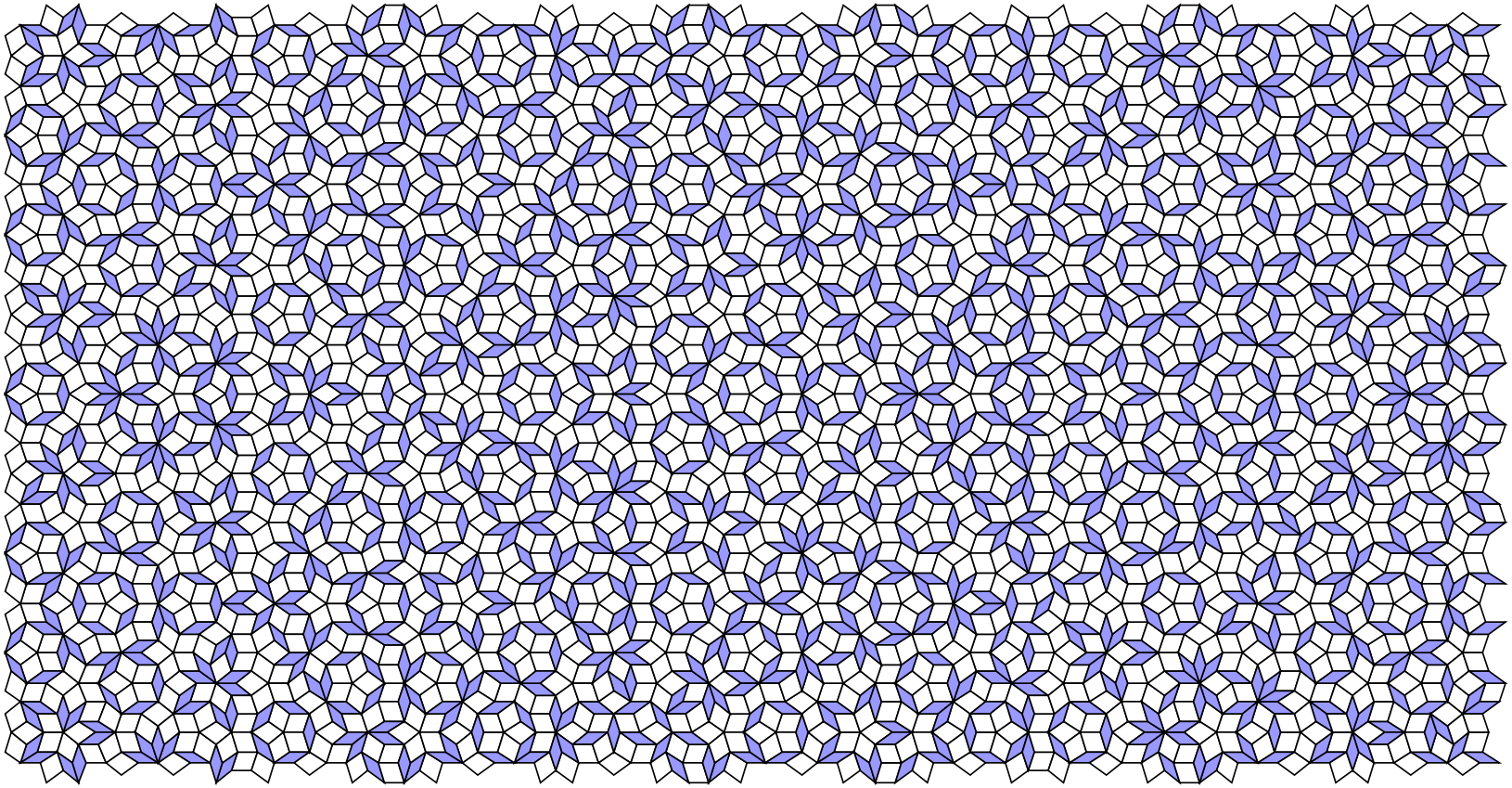

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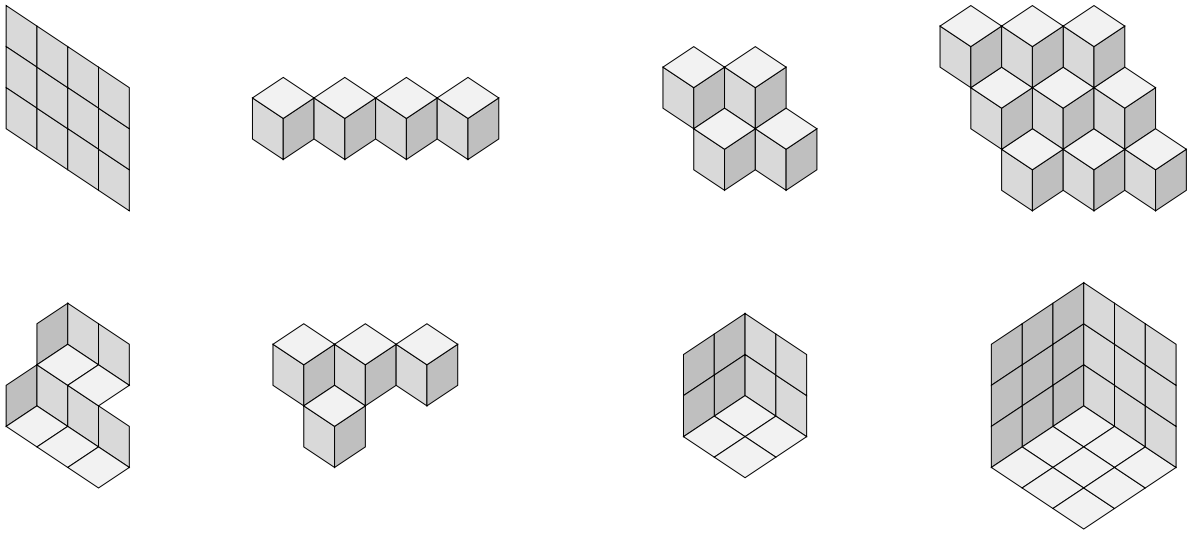
Definition (Configurational entropy of a tiling T)

$S(T) := \log(\text{nb tilings of the same domain as } T)/\text{nb tiles in } T.$



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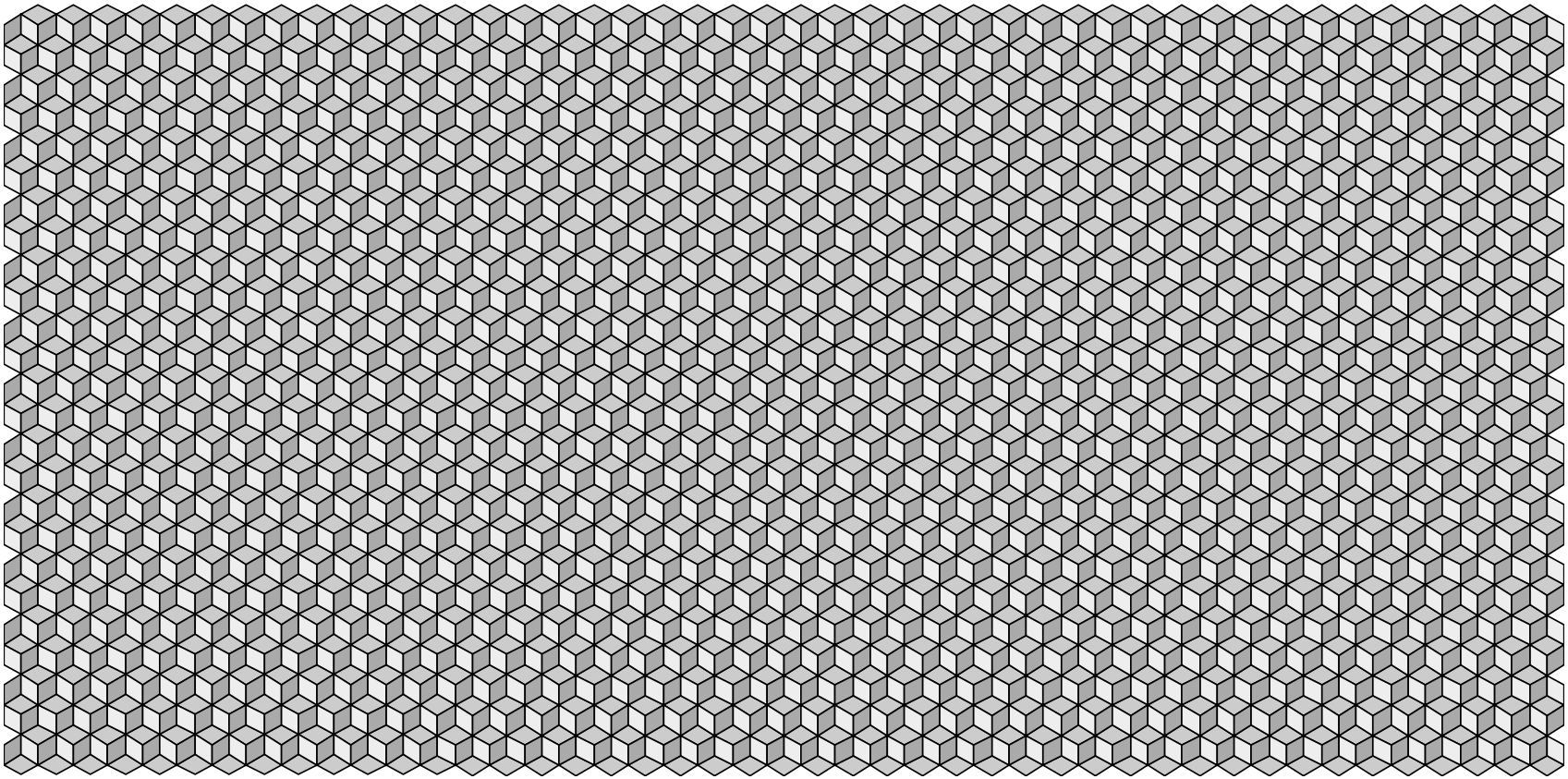
Maximal entropy tilings are planar. There are efficiently sampled.



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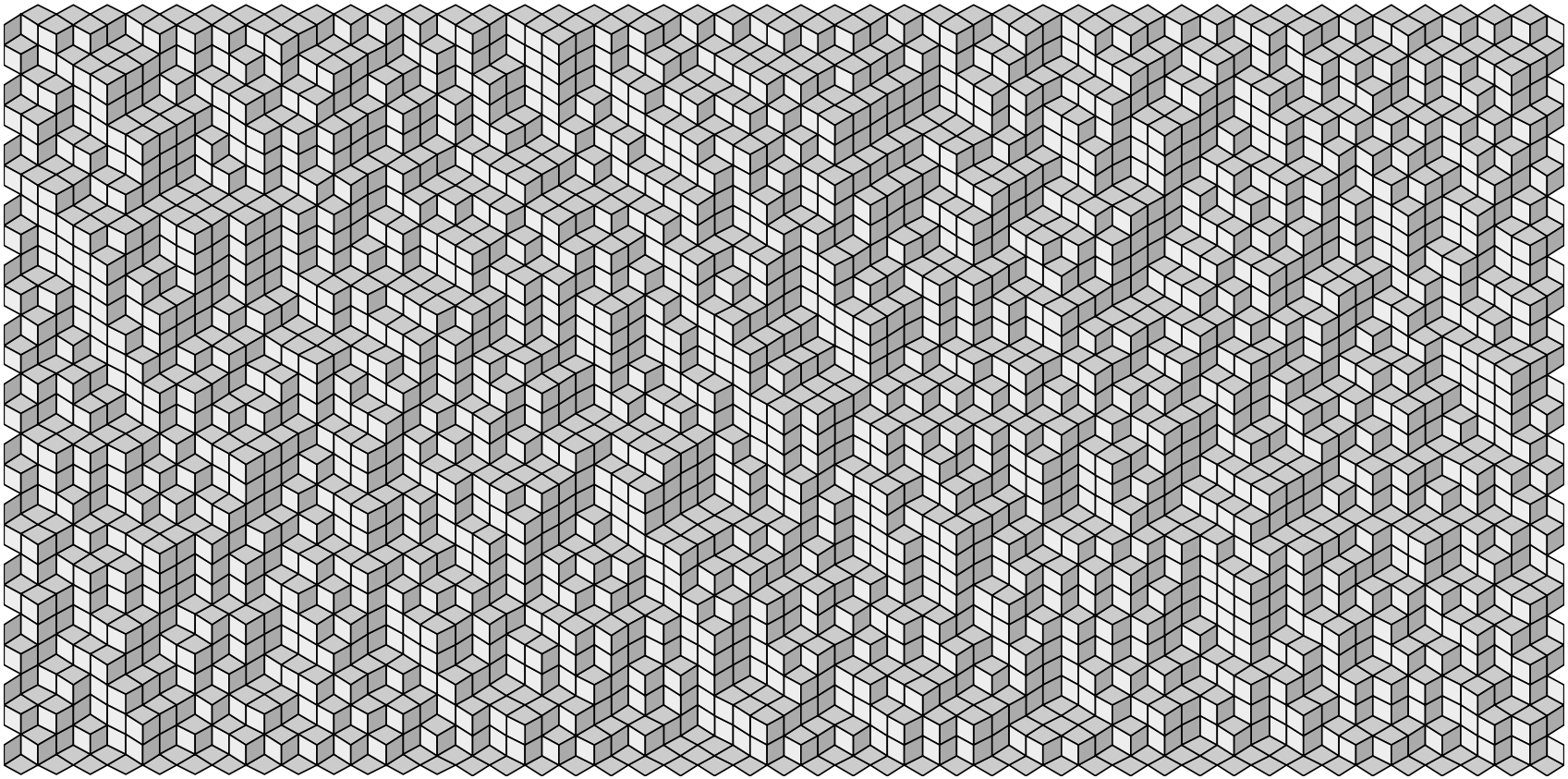

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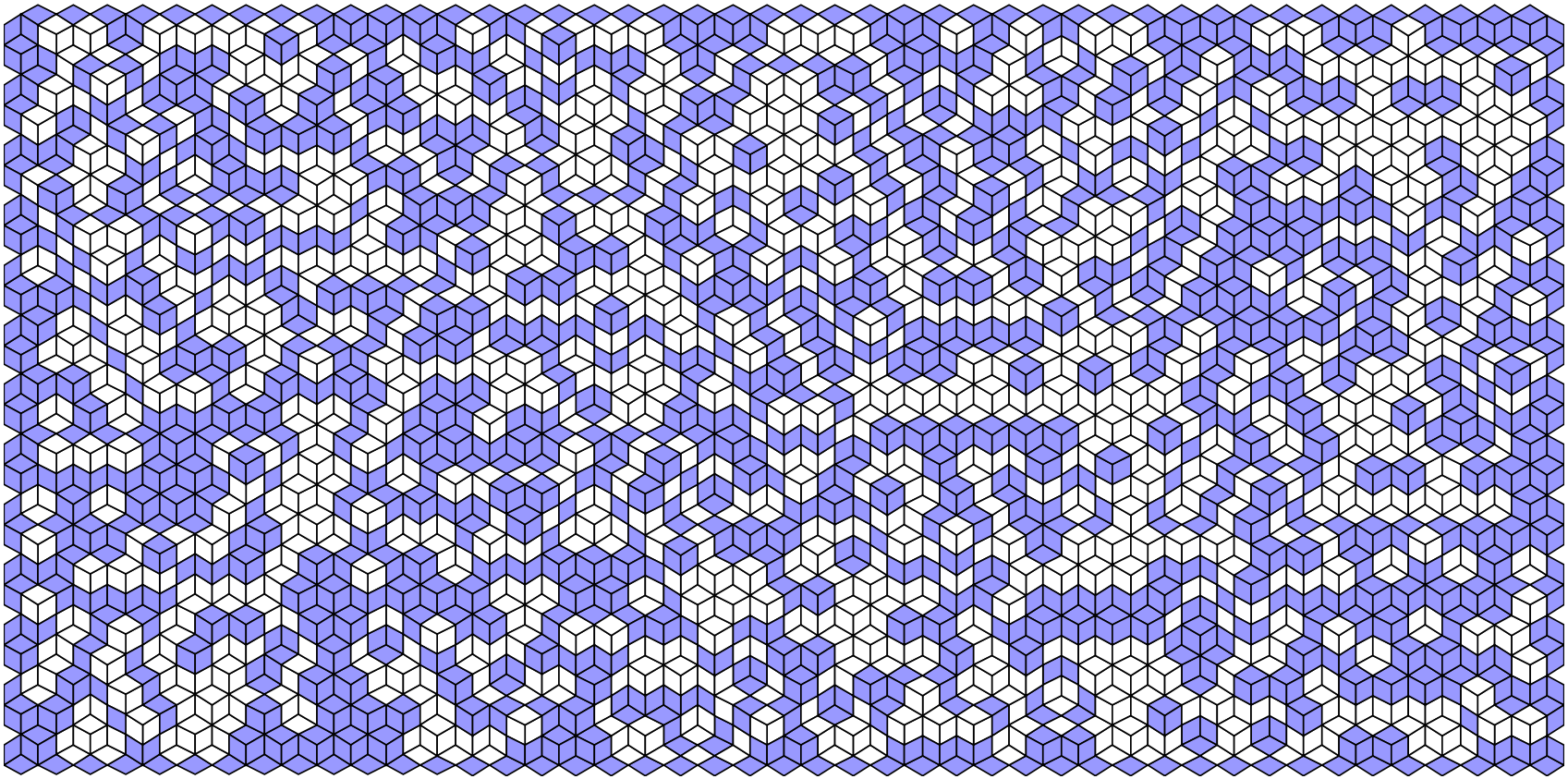

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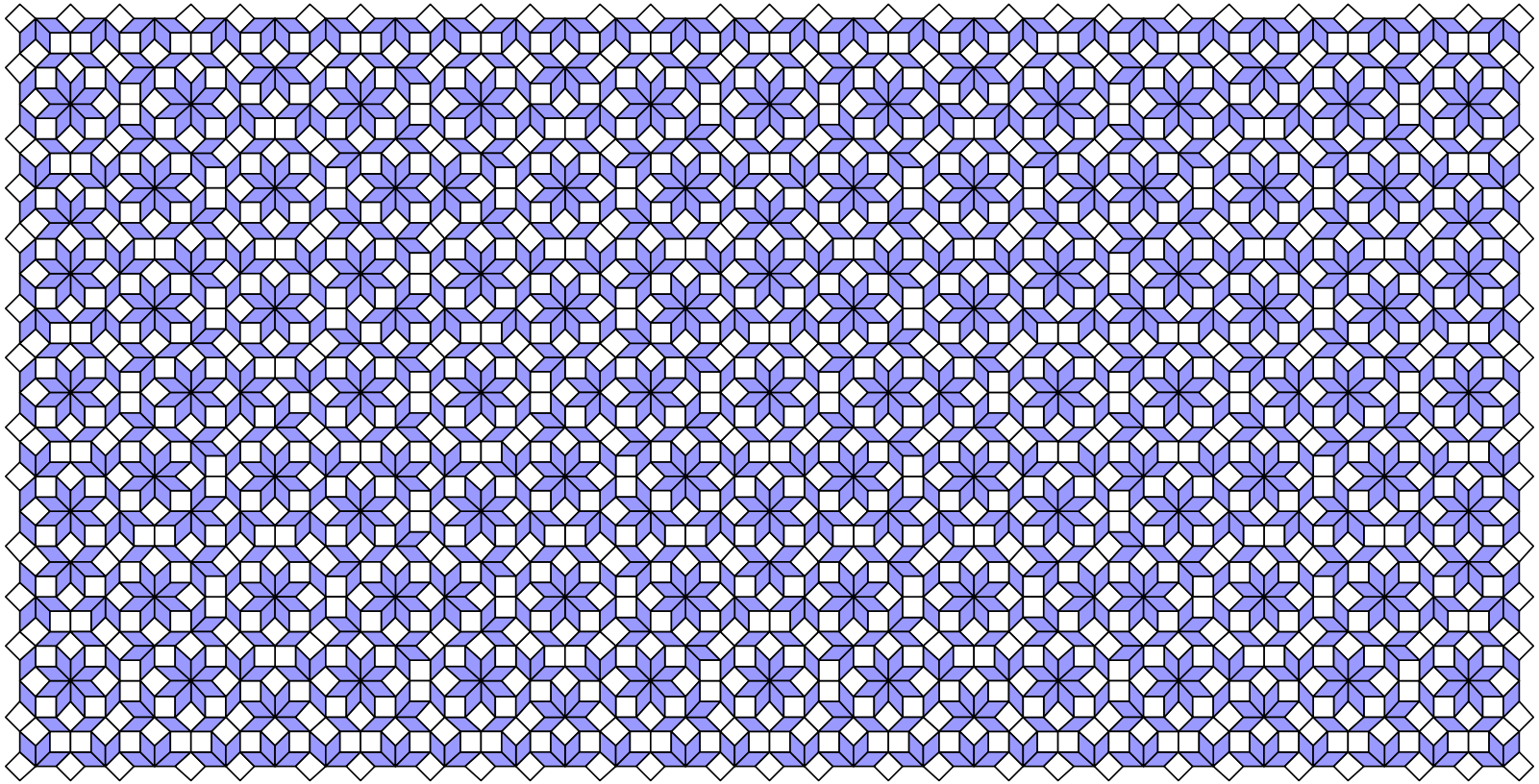

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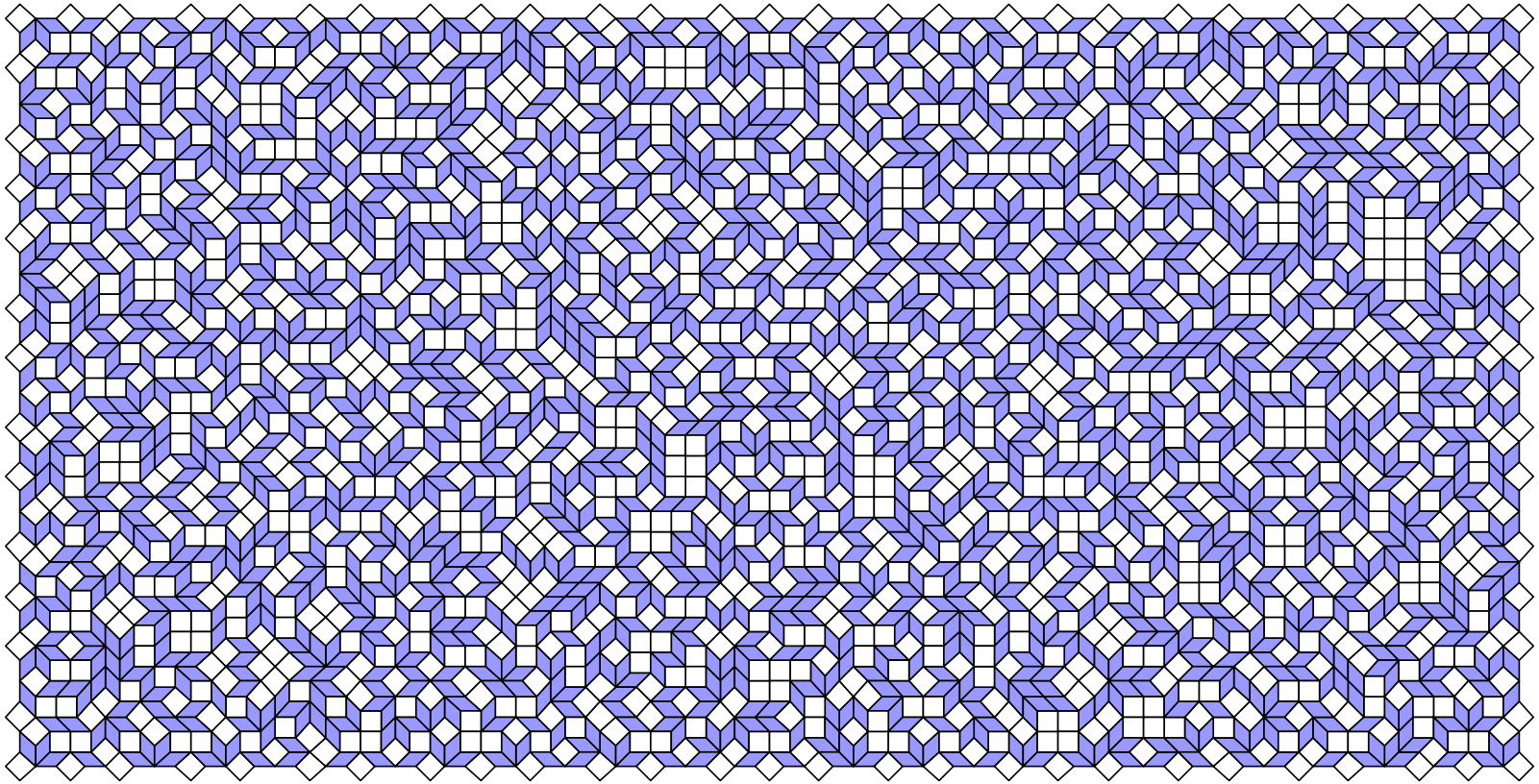

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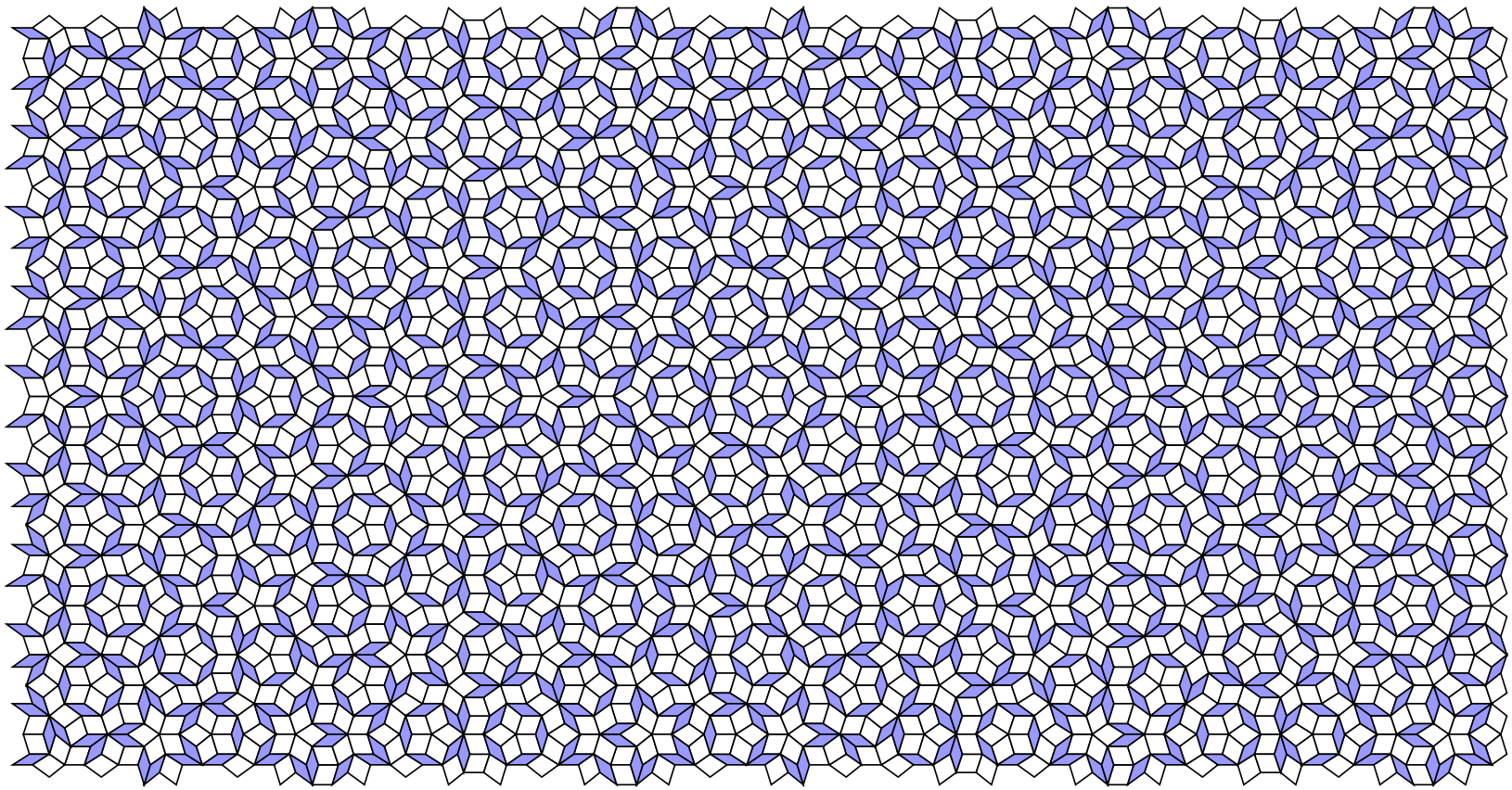

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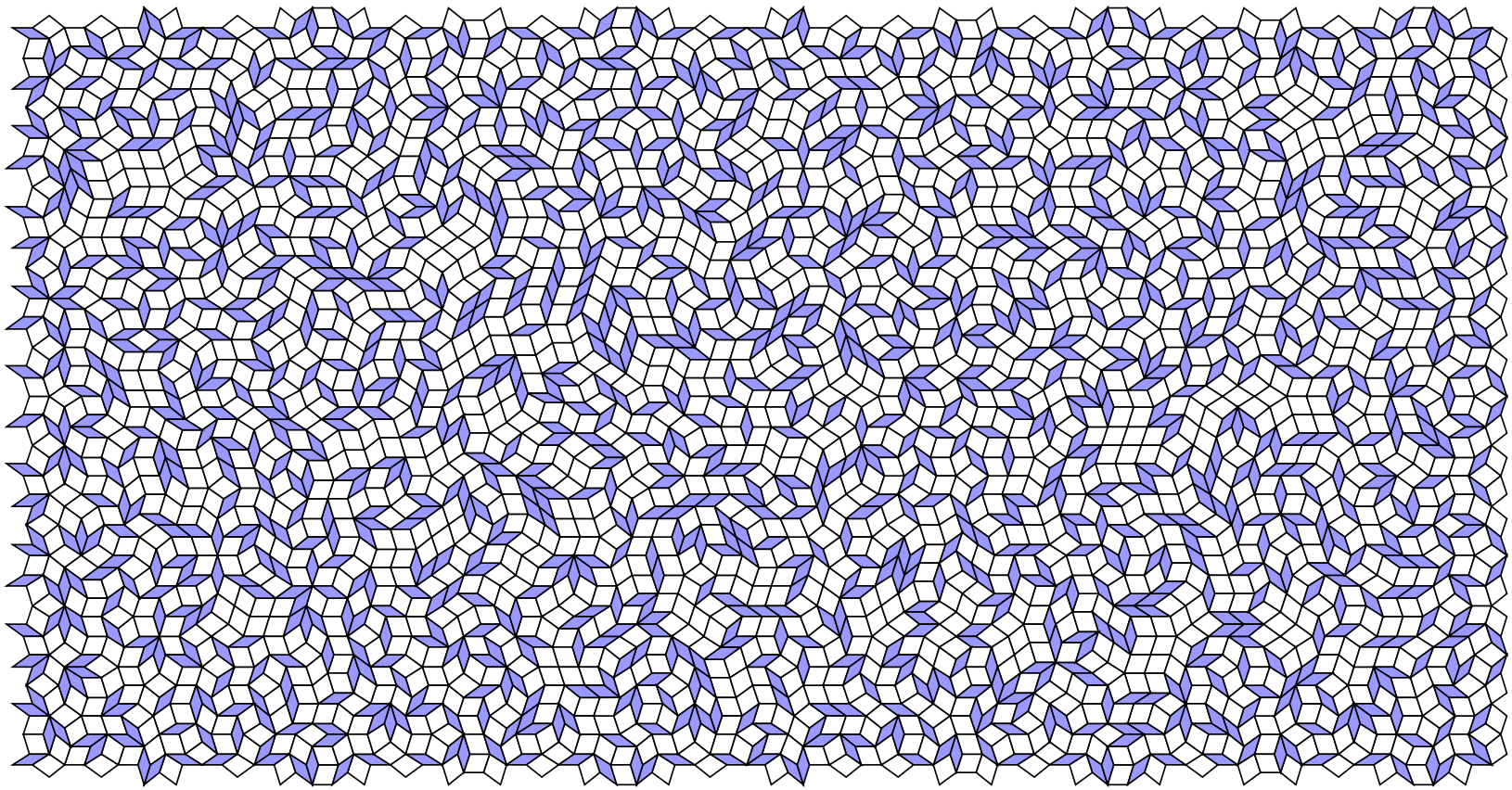

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Diffusion mechanism which makes the cooling correct the defects?

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This is the Metropolis-Hastings algorithm for the Gibbs distribution

$$P(\text{tiling}) = \frac{1}{Z(T)} \exp\left(-\frac{E(\text{tiling})}{T}\right).$$

Chaos for high T, order for low T, but what about convergence?

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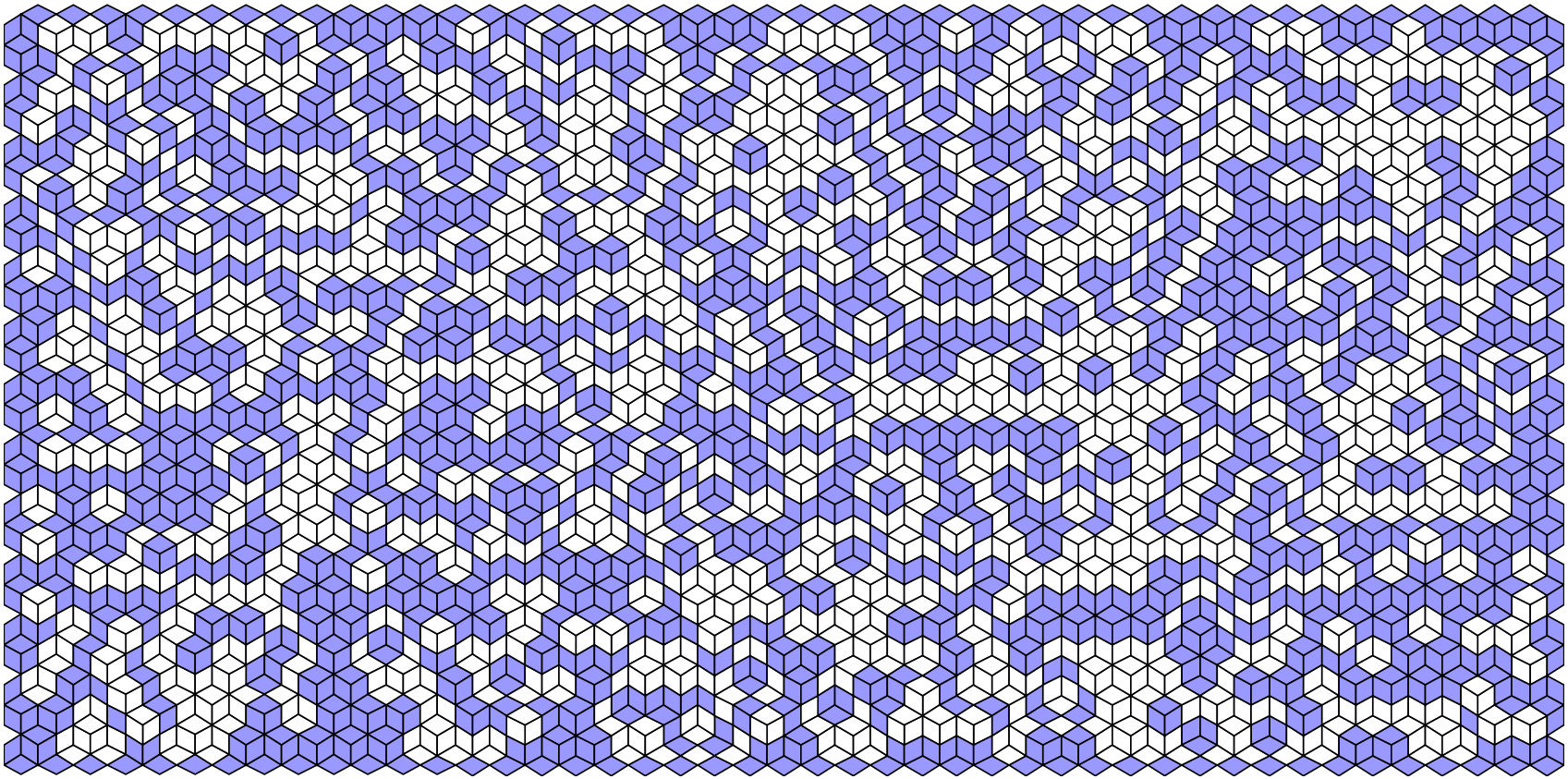
Ergodic at $T > 0$, $\Theta(n^2 \ln n)$ mixing at $T = \infty$, $O(n^{2.5})$ at $T = 0$.



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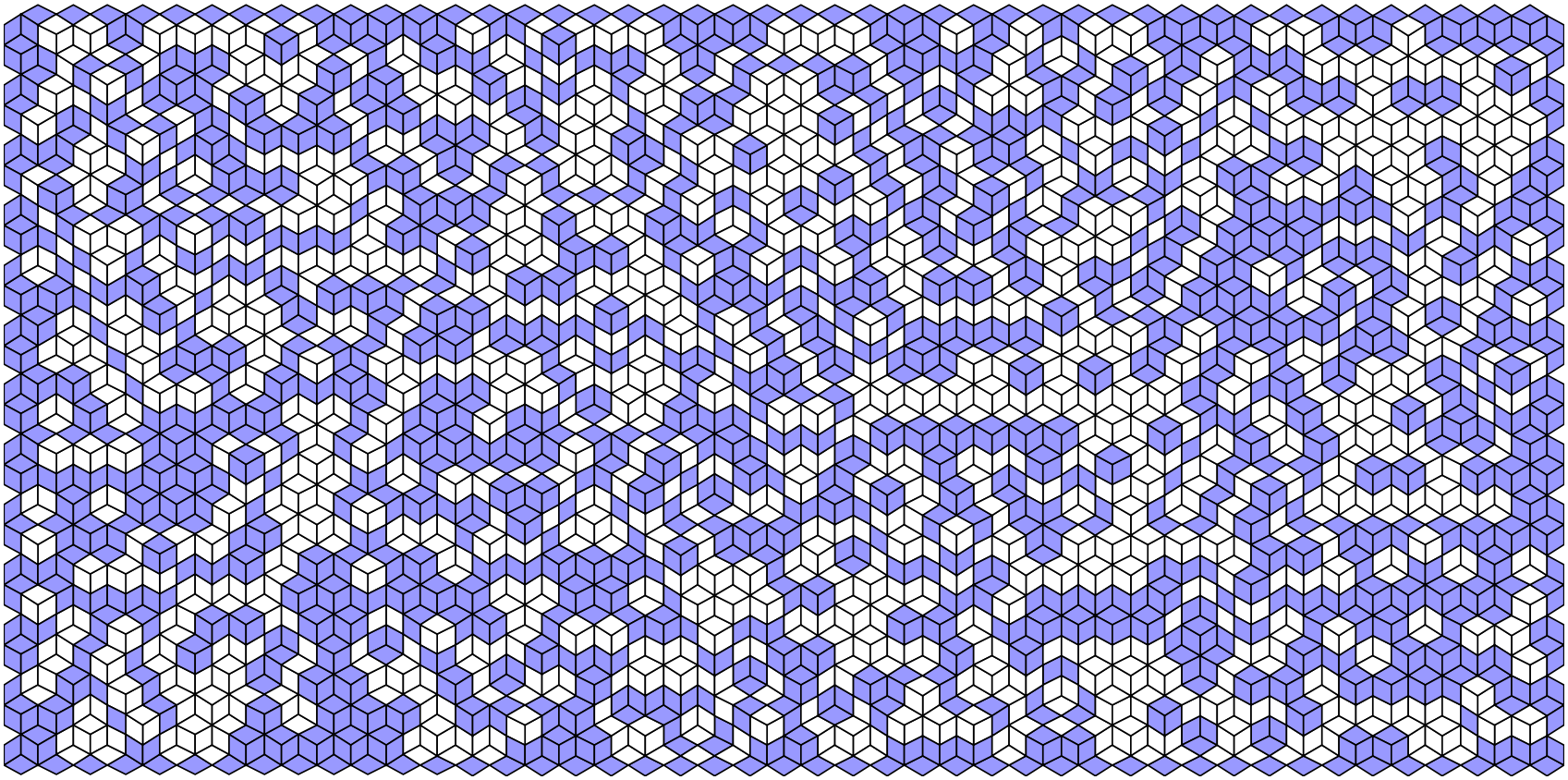

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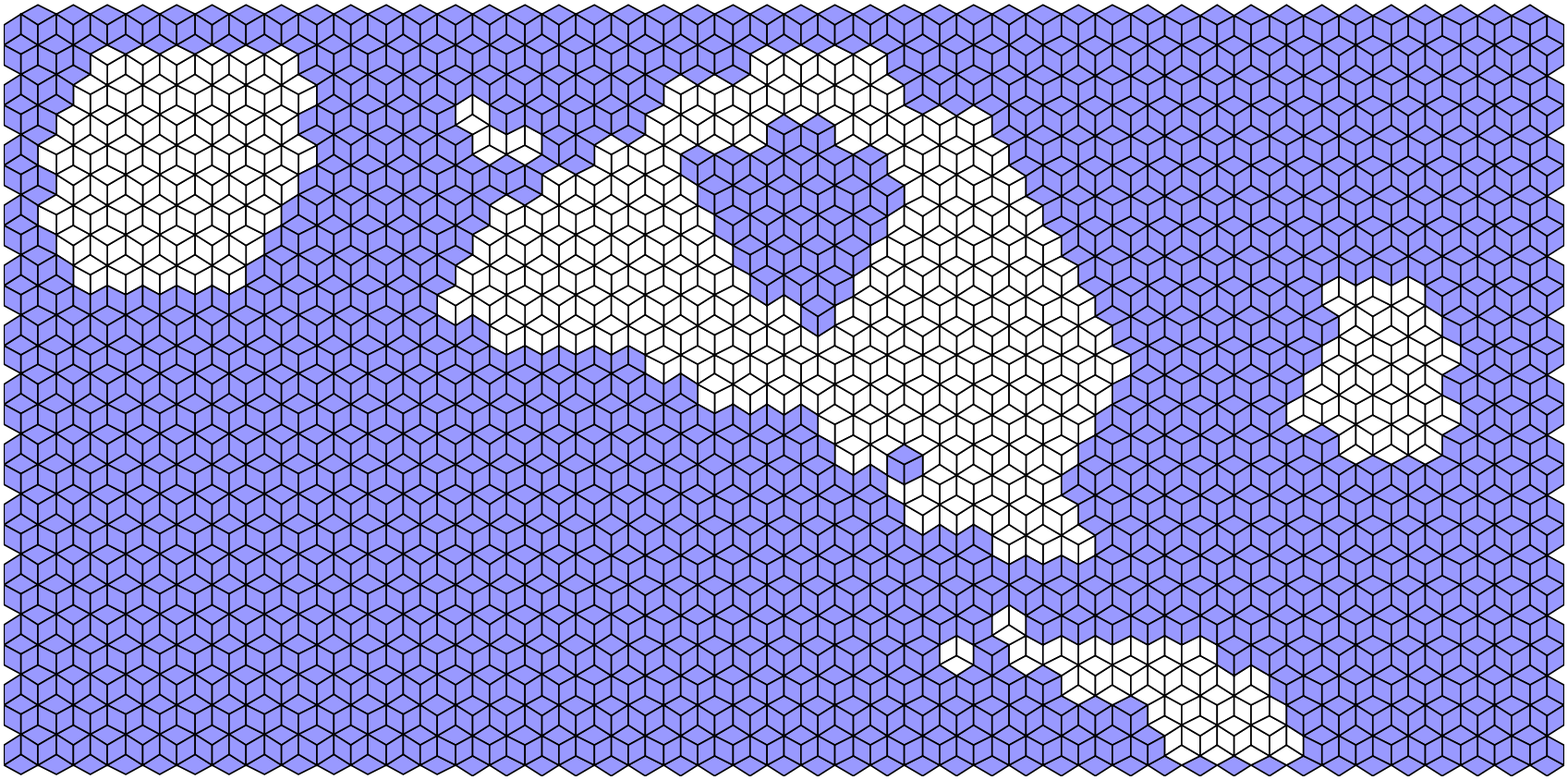

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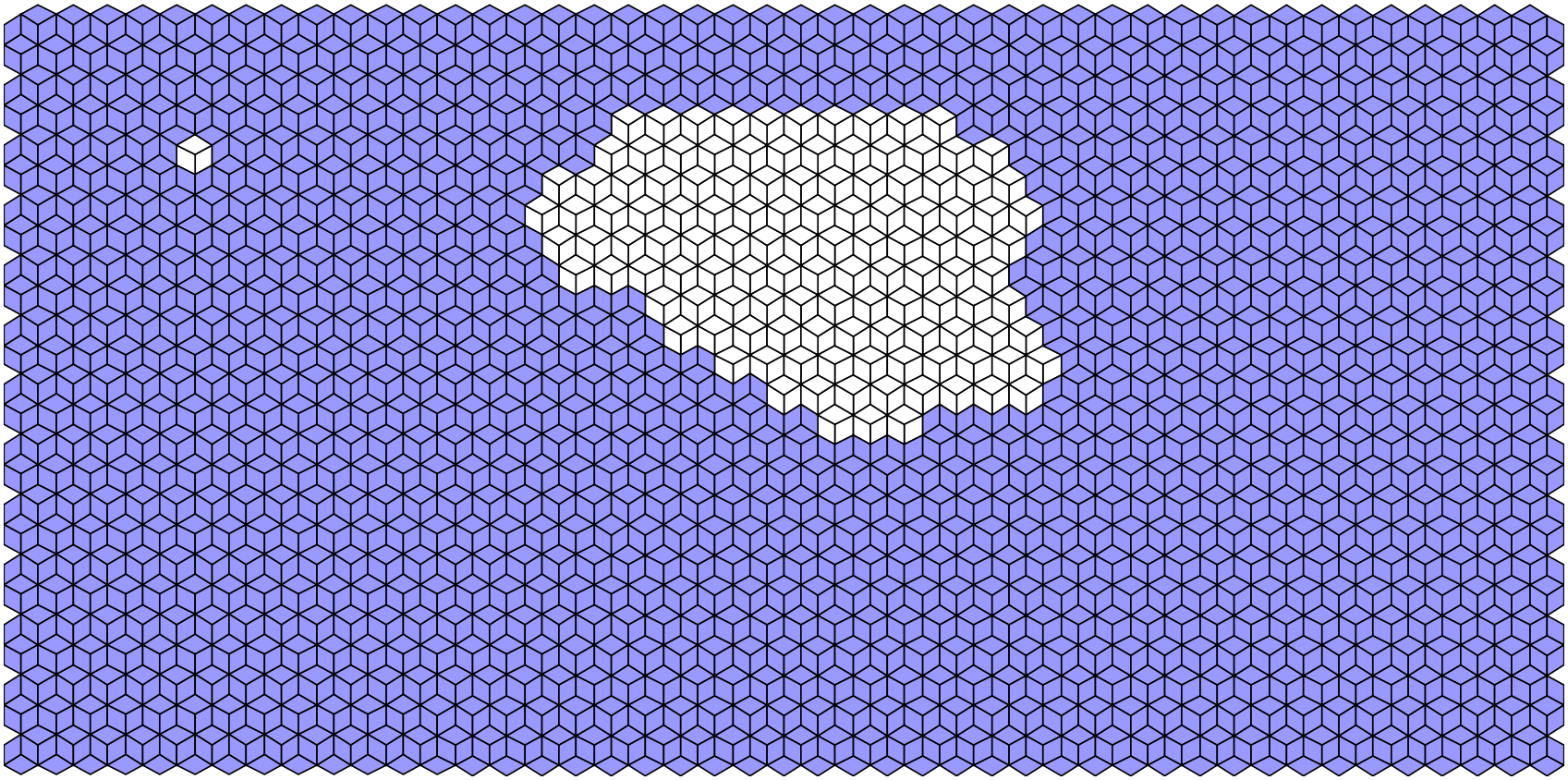

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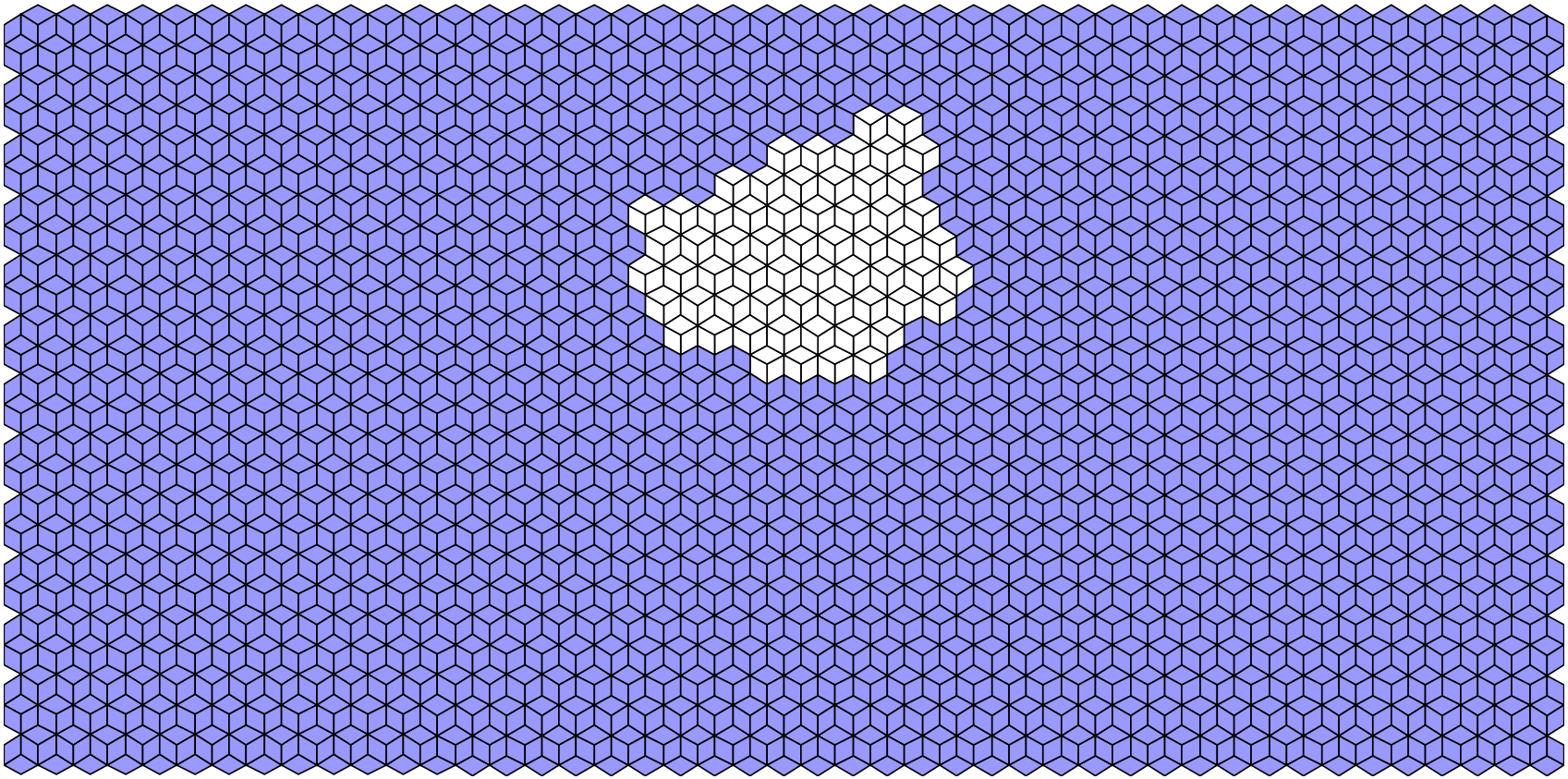

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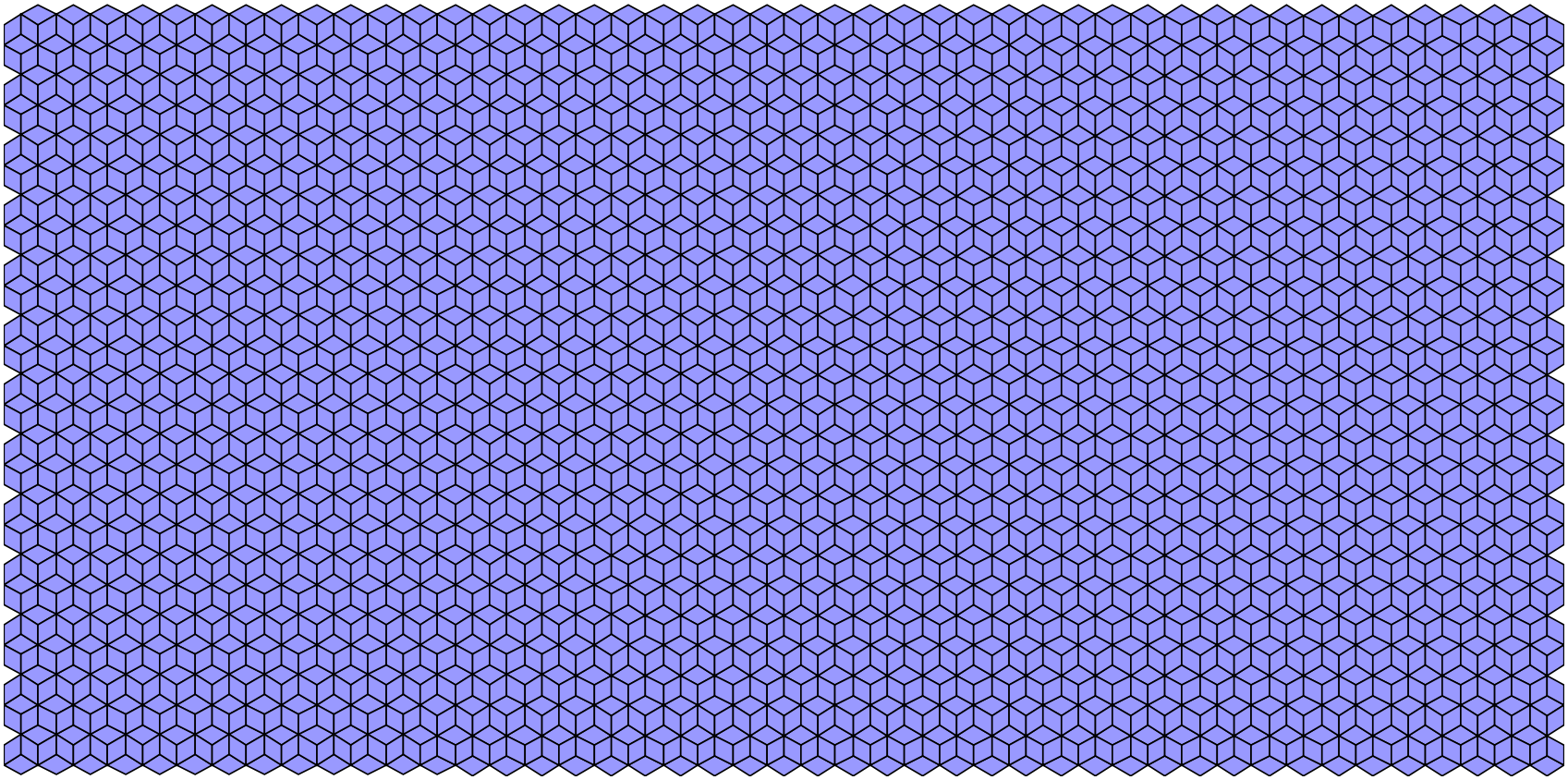

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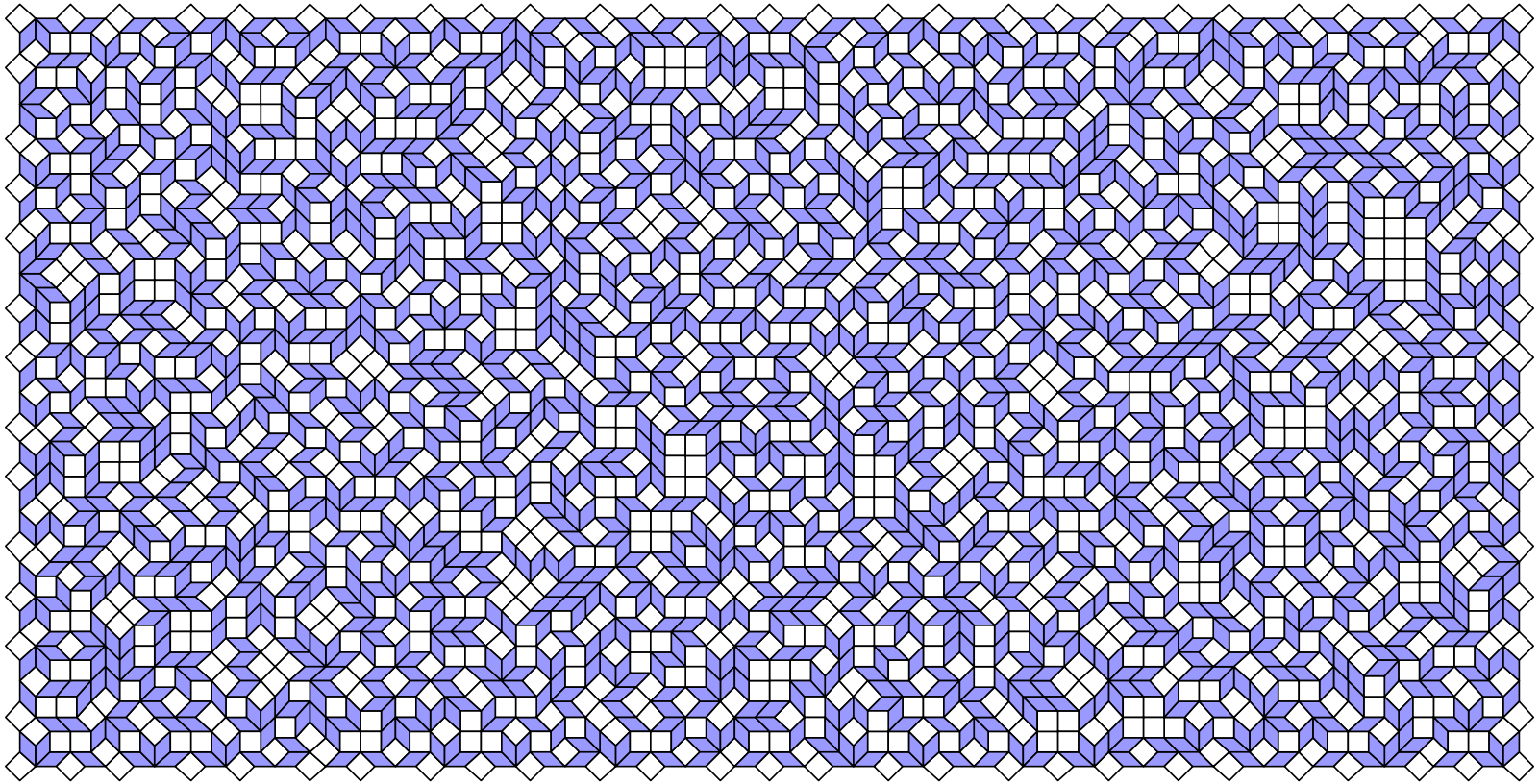

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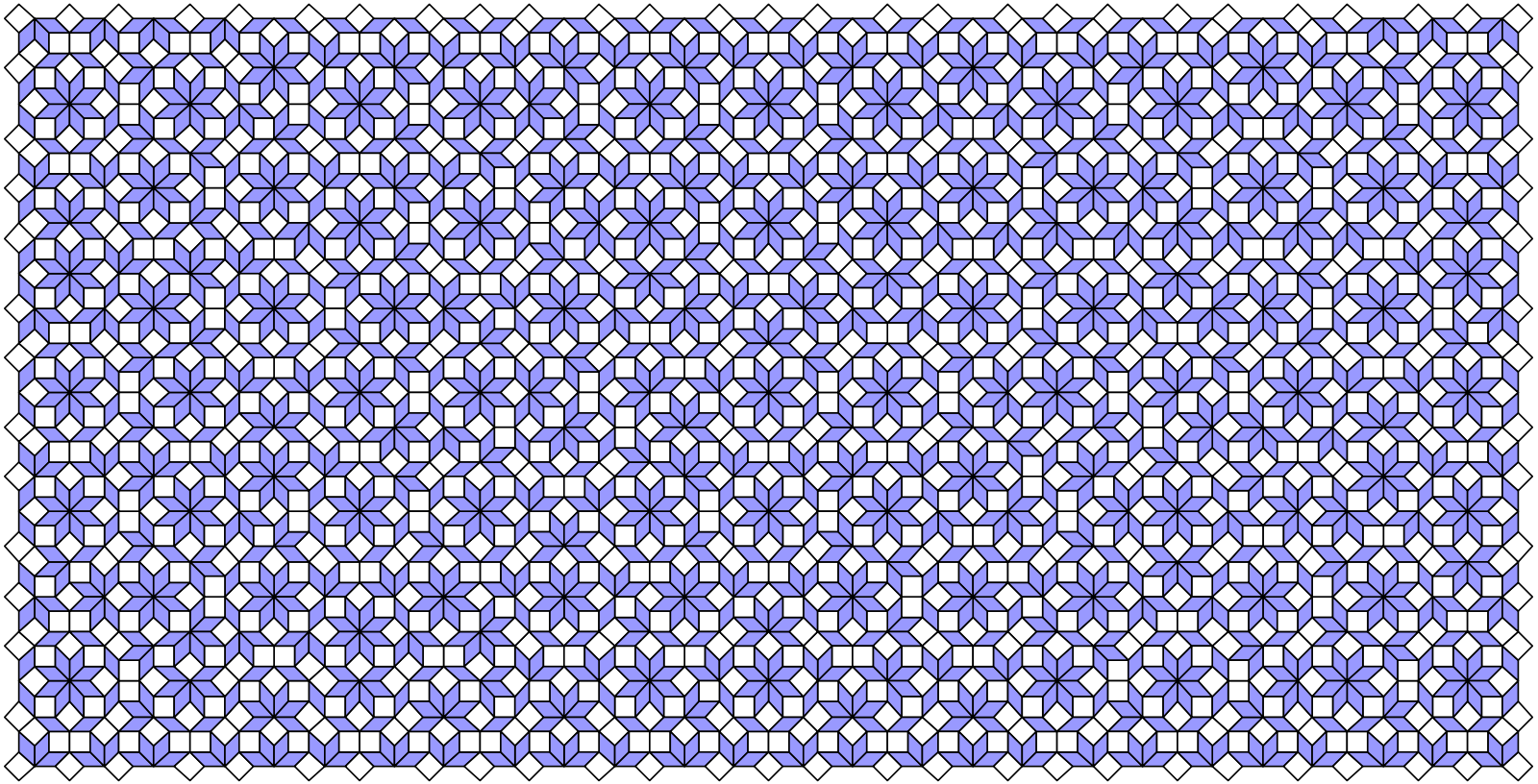

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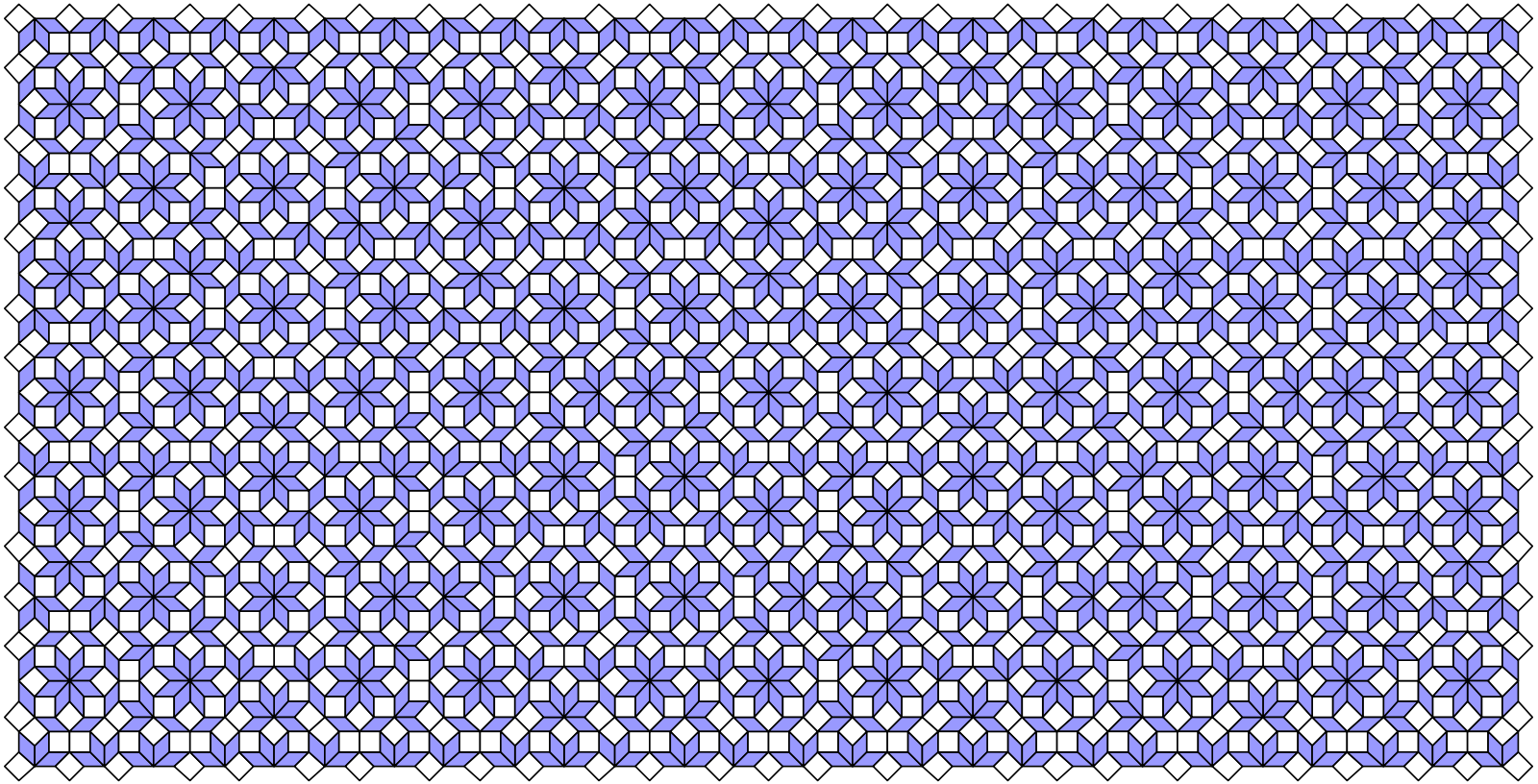

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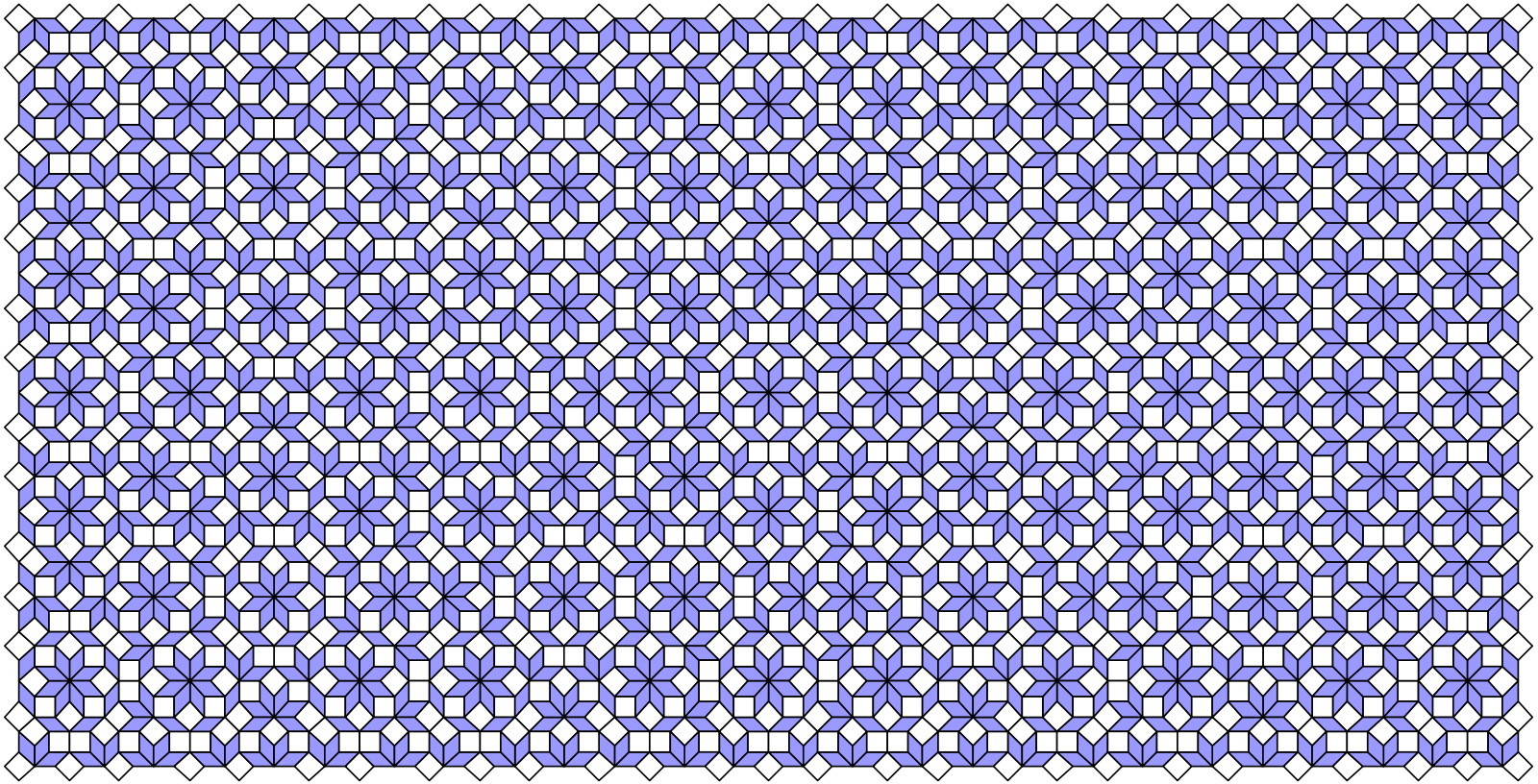

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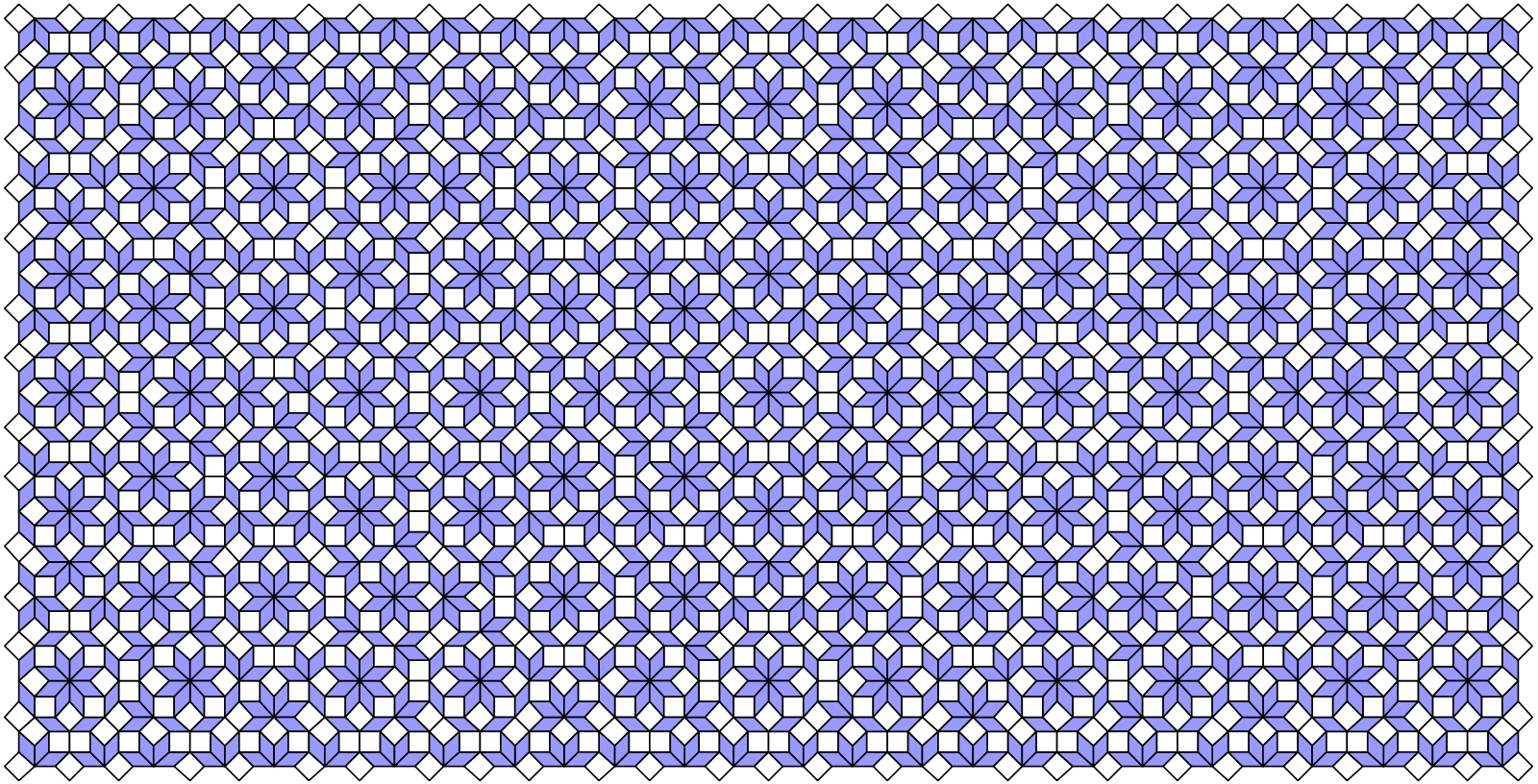

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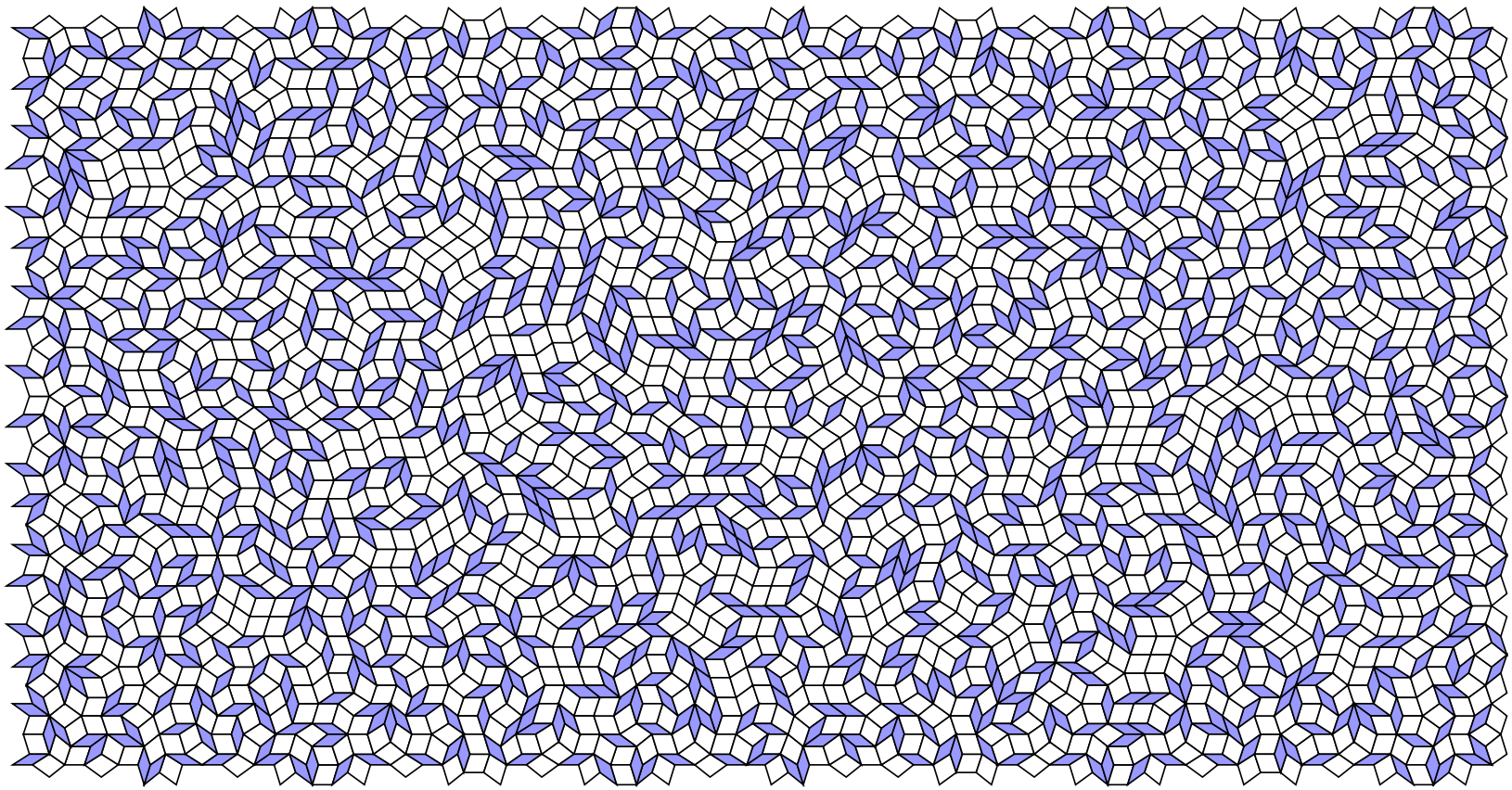

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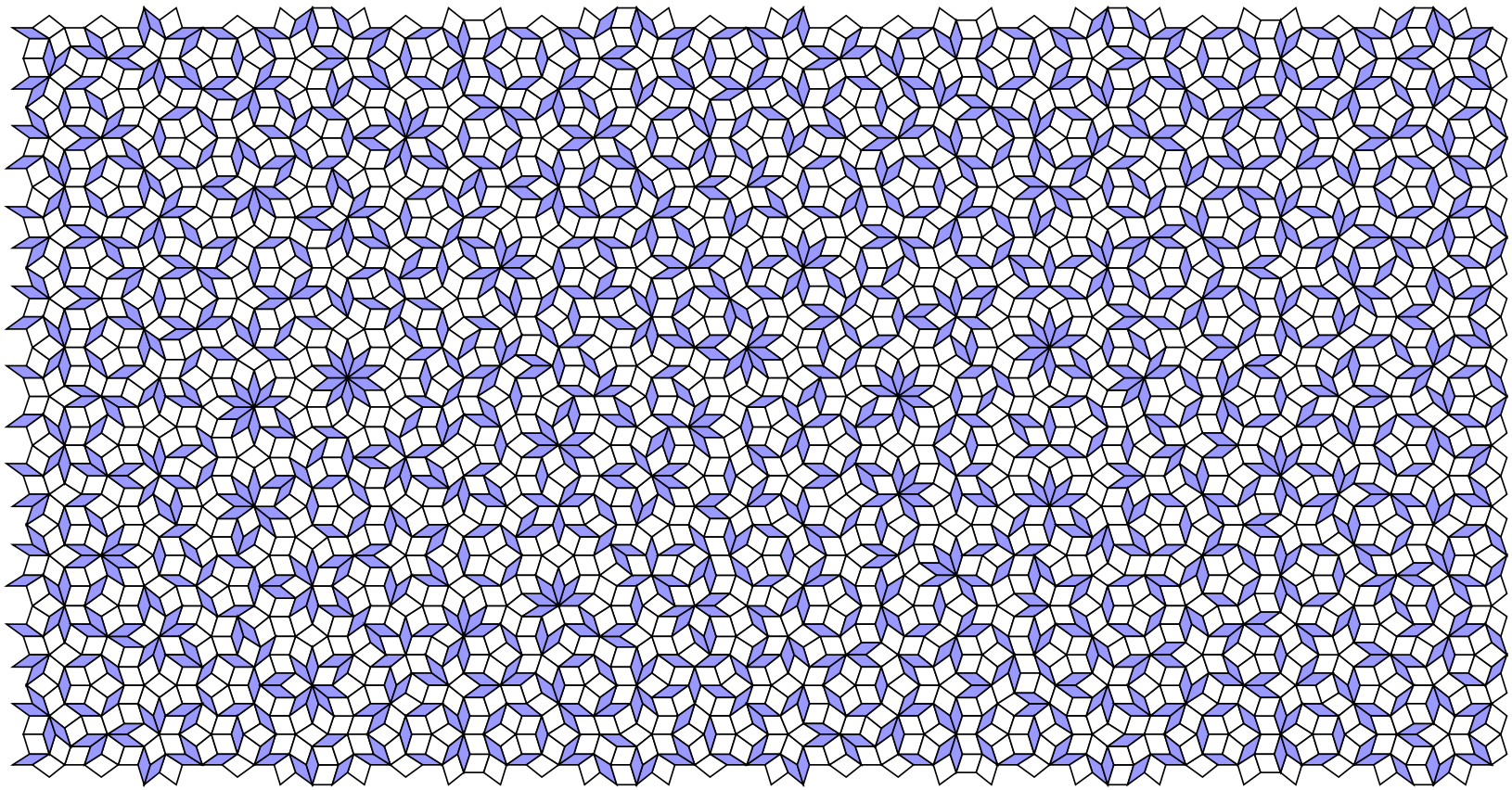

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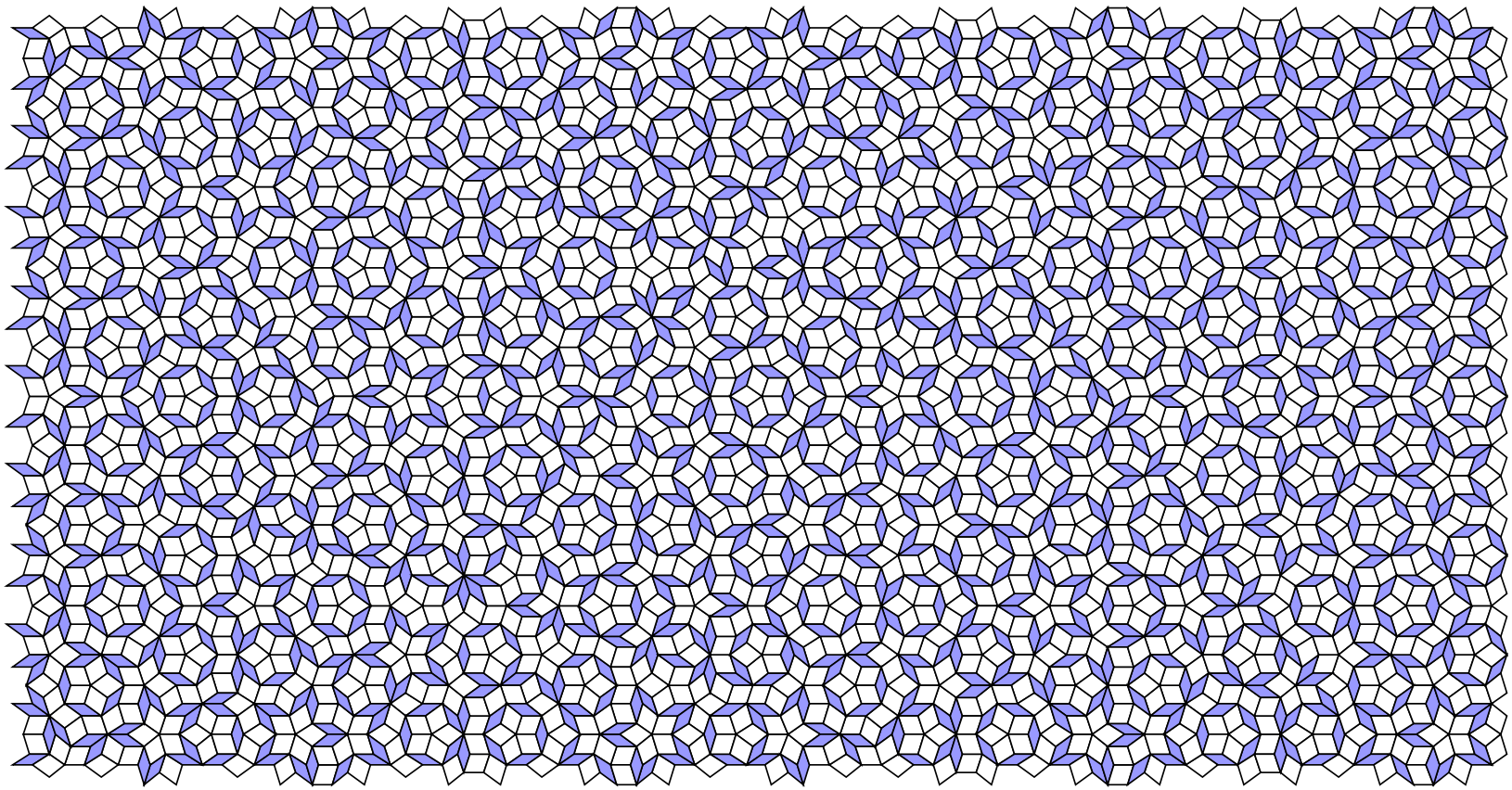

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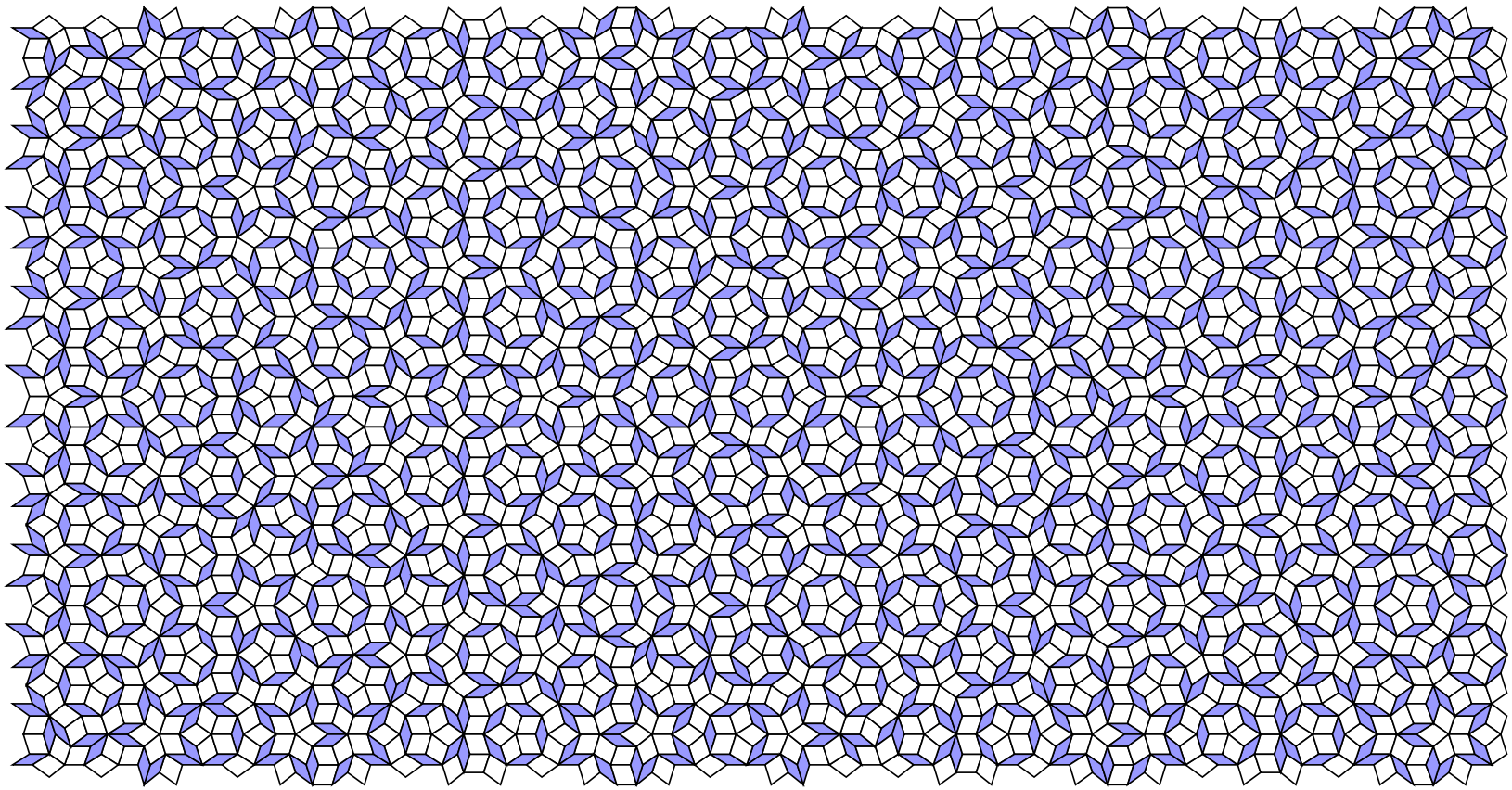

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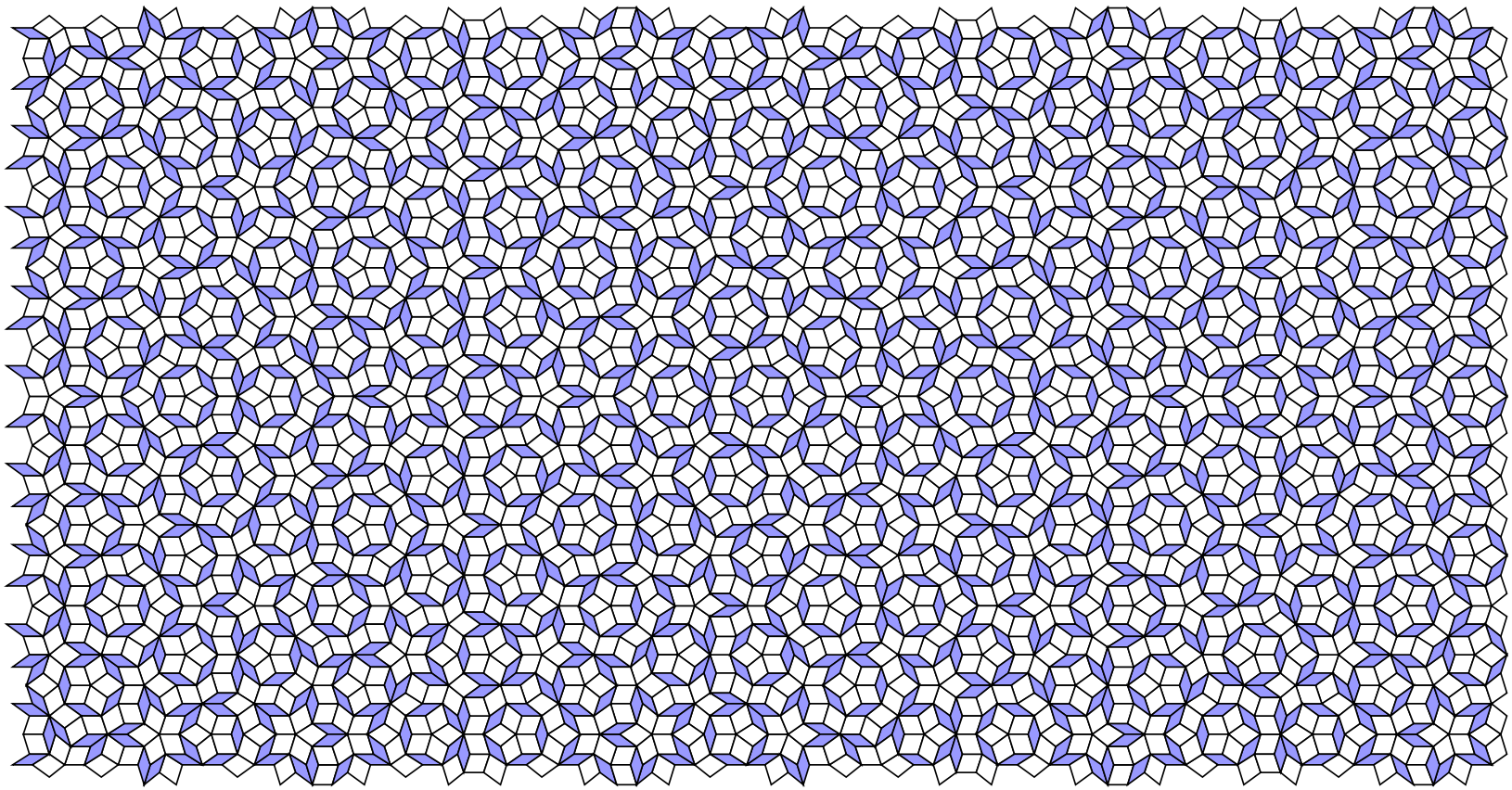

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