Stochastic flips on two-letter words

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Joint work with O. Bodini & D. Regnault

Analco, January 16, 2010.

1. Our problem
2. Motivations
3. Main result

Configuration: word w over $\{1, 2\}$ with as many 1 as 2.

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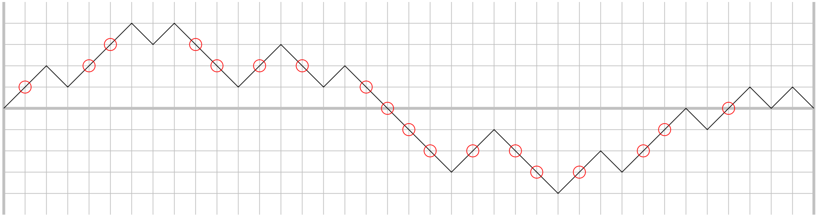

Error: two identical consecutive letters. Counted by $E(w)$

Flips

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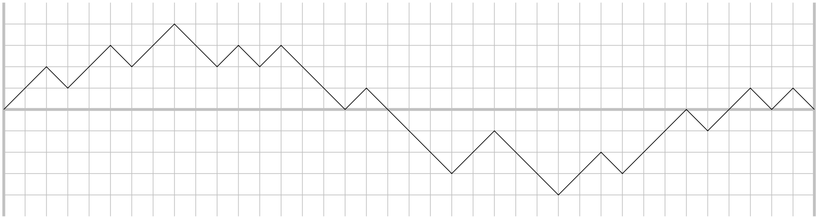

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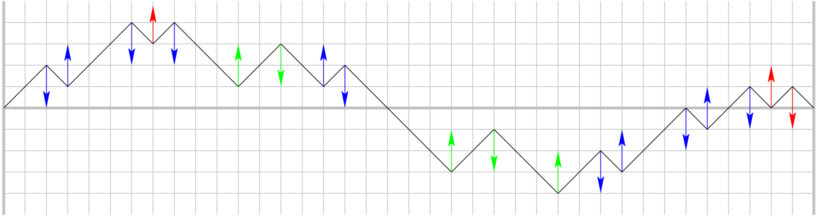
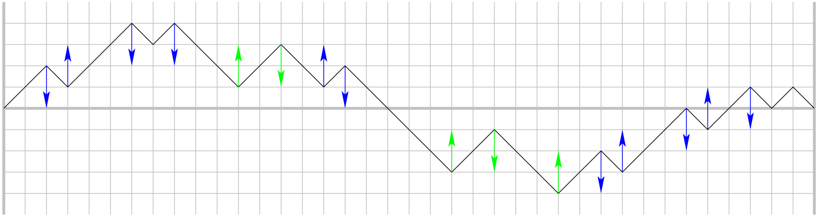


Flips can delete, shift or create errors.

Process: $w_t \to w_{t+1}$ by a random flip which does not create errors.



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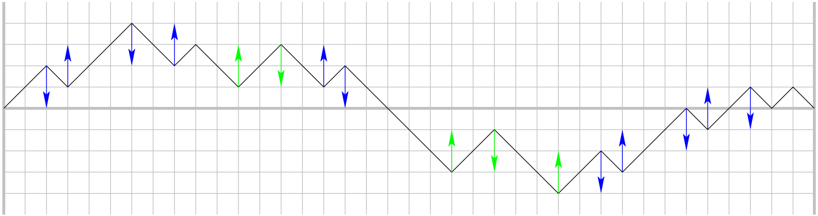

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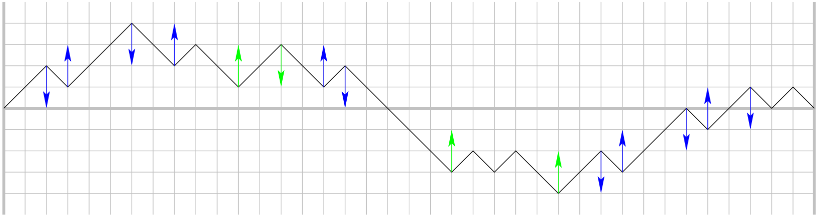

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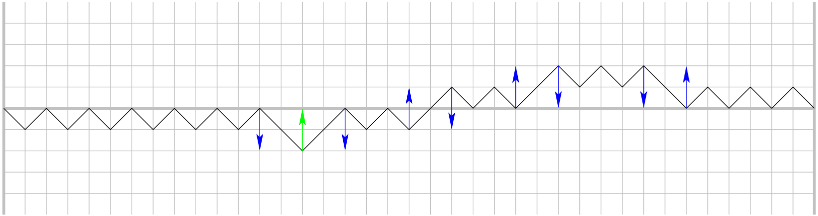

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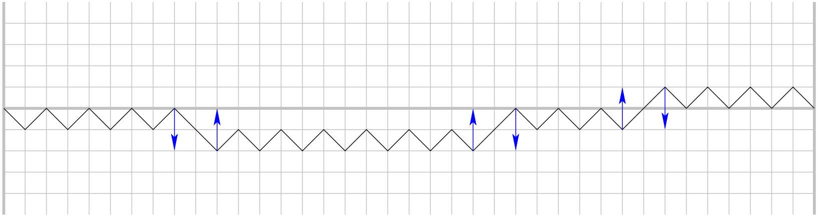

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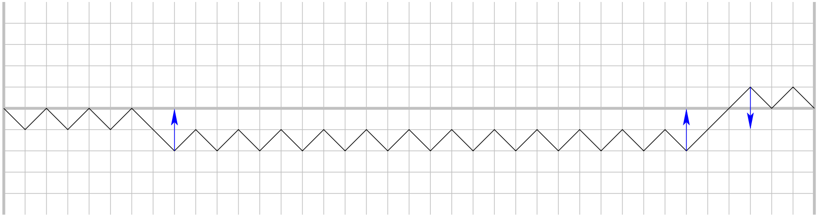

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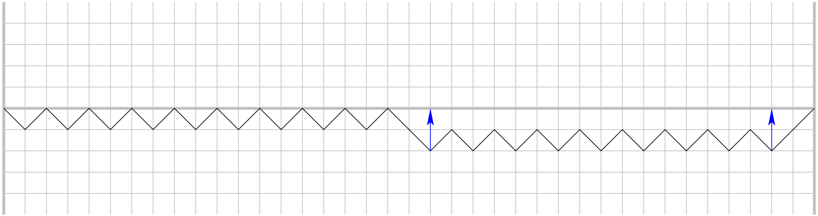

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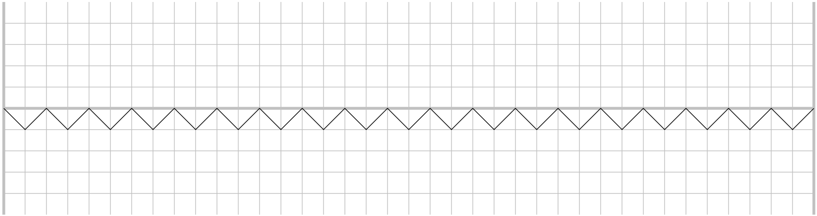

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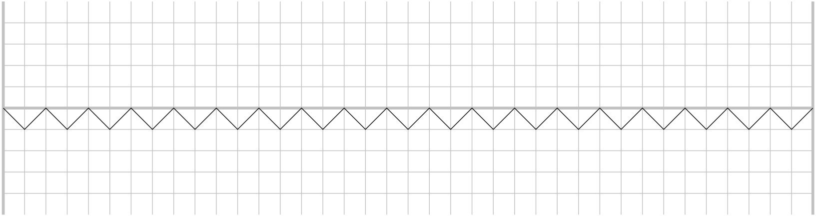
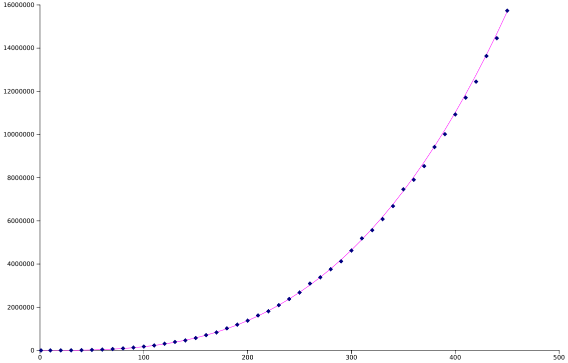


Convergence time: $T(w_0) := \min\{t \geq 0 \mid E(w_t) = 0\}$.

Expected convergence times of $1^n 2^n$ and $2^n 1^n$ seem to be $\Theta(n^3)$.



Here, we show that the worst expected convergence time is $O(n^3)$.

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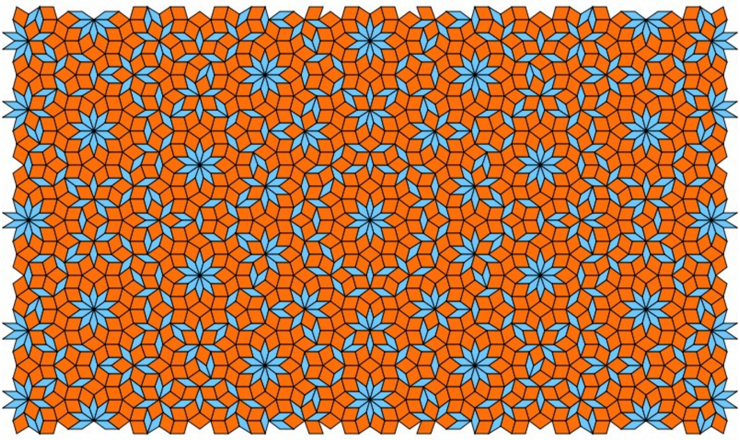
Non-periodic tilings model the structure of quasicrystals, with forbidden patterns modelling finite range interaction:



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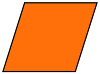
Add one tile at time:



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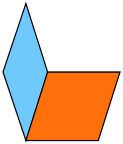

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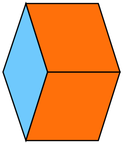

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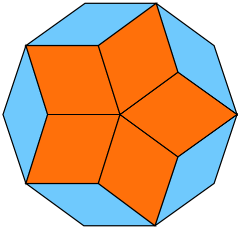

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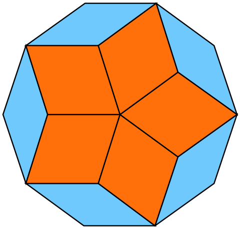

Hard to avoid forbidden patterns

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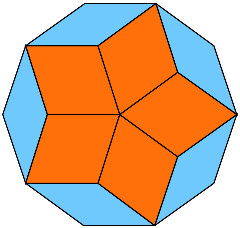
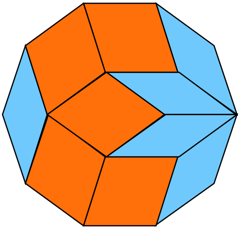


Perform local corrections:



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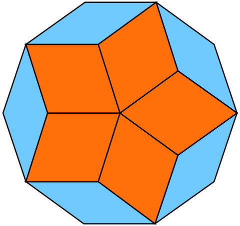

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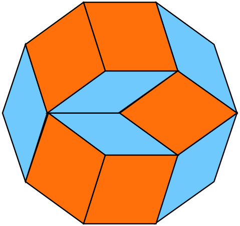

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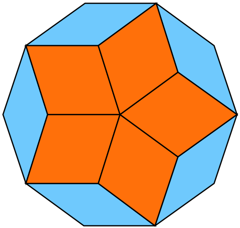

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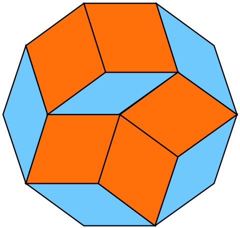

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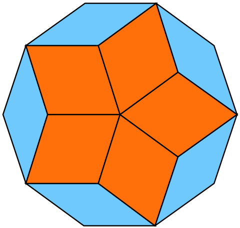

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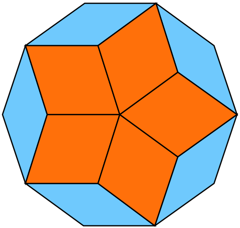

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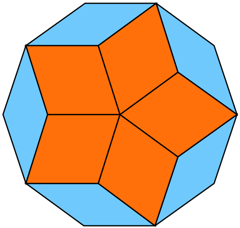

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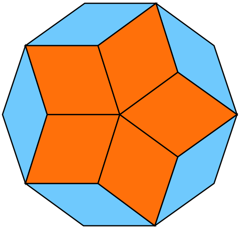
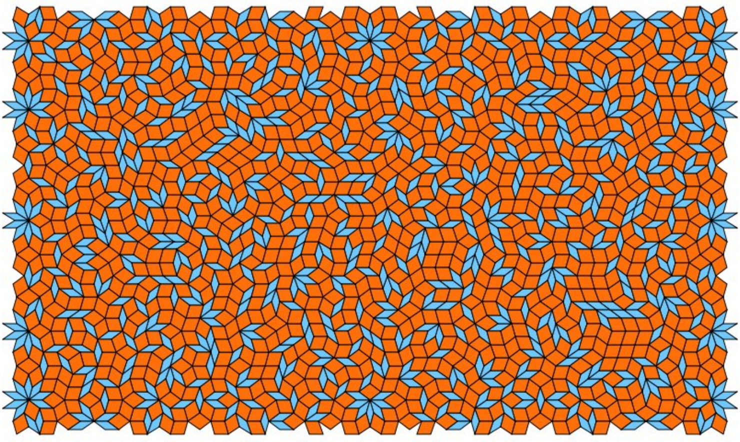


Does it converges? Rate?

Experimentally converges rather fast, but remains to be proven:



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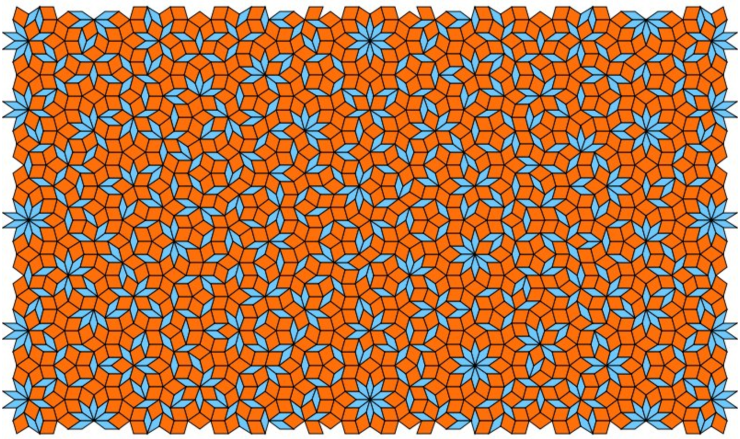

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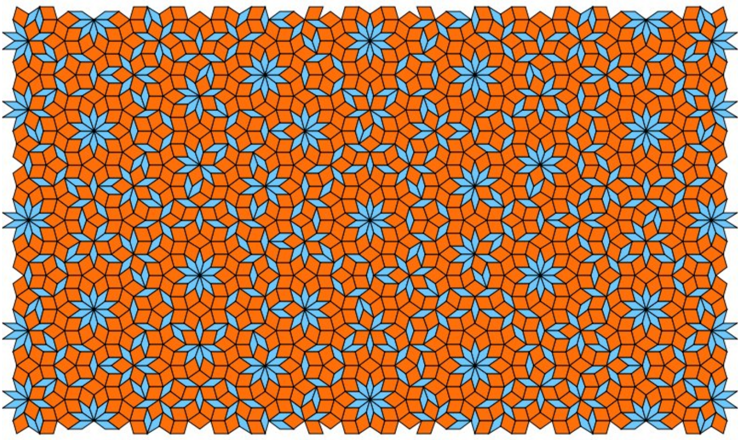

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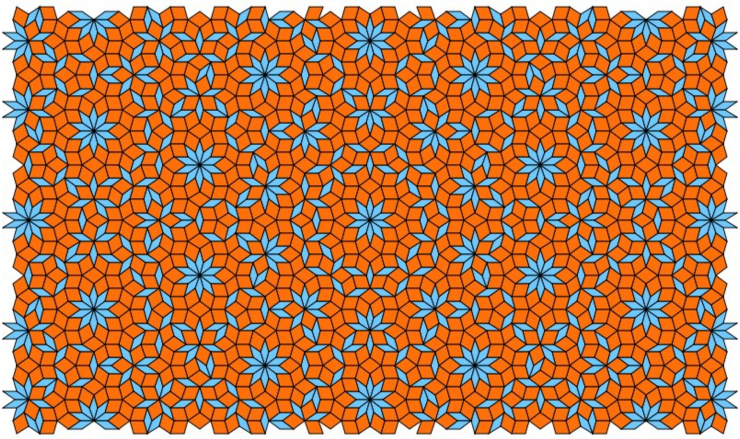

Our problem turns out to be the most simple case:

- tiling of the line with two tiles (two-letter word);
- forbidden patterns: two identical consecutive letters (errors)
- local corrections: swap letters (flip)

Main tool:

Decrease on expectation

Let $(w_t)_{t\in\mathbb{N}}$ be a stochastic process over a space $\mathcal{W}$.
Let $\psi : \mathcal{W} \to \mathbb{R}_+$ and $\varepsilon > 0$ such that, whenever $\psi(w_t) > 0$,

$$\mathbb{E}\left(\Delta\psi(w_t)\middle| w_t, \ldots, w_0\right) \leq -\varepsilon.$$

Then

$$\mathbb{E}(\min\{t \geq 0 \mid \psi(w_t) = 0\}) \leq \frac{\psi(w_0)}{\varepsilon}.$$

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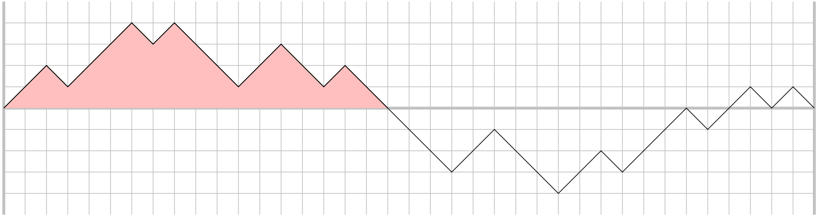
Main task: find a suitable ψ. Unfortunately, E does not suit!

We introduce Dyck factor:



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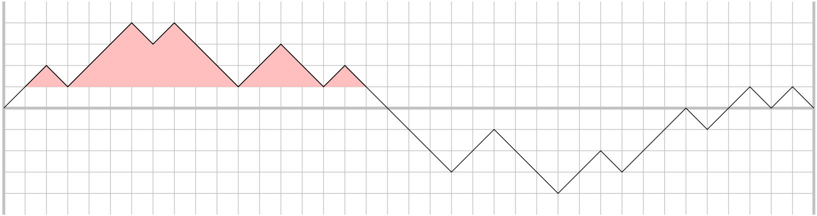

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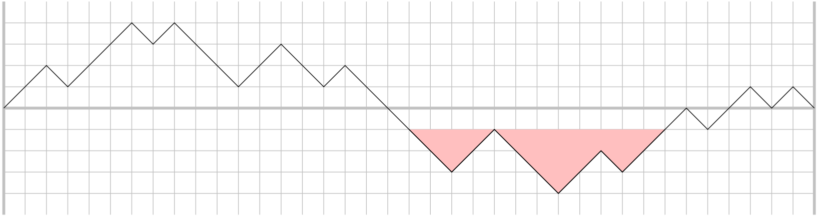

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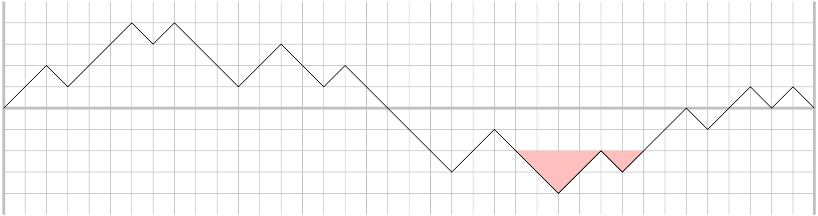

Definition

Let $0 < \alpha < 1$. Let $DF(w)$ be the Dyck factors of w. One sets:

$$\psi_\alpha(w) := \sum_{v \in DF(w)} (1 + |v|_1)^\alpha.$$

One proves ($n = |w|$):

1. $(1 + \frac{n}{2})^\alpha \leq \psi_\alpha(w) \leq n^{1+\alpha}$;

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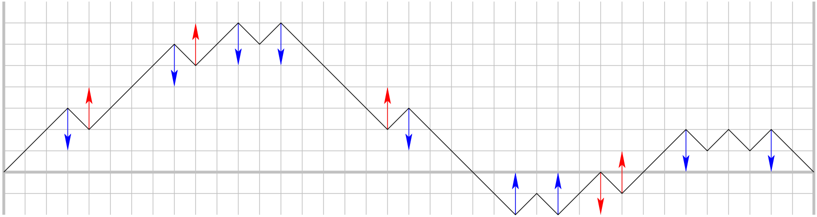
This yields:

Theorem (Bodini-F-Regnault)

The expected convergence time is at most cubic:

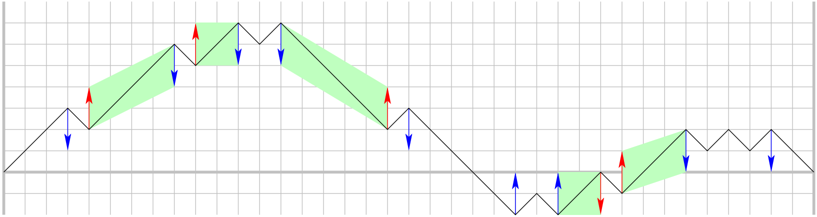
$$\mathbb{E}(T(w)) \leq \frac{2n^3}{\alpha(1-\alpha)}.$$

Main idea ensuring the decrease on expectation (sketch):



A flip can increase (red) or decrease (blue) ψ_α.

Main idea ensuring the decrease on expectation (sketch):



With each red flip is associated a “higher” blue flip.

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Whenever the red flip increases ψ_α by $(p+1)^\alpha - p^\alpha \dots$

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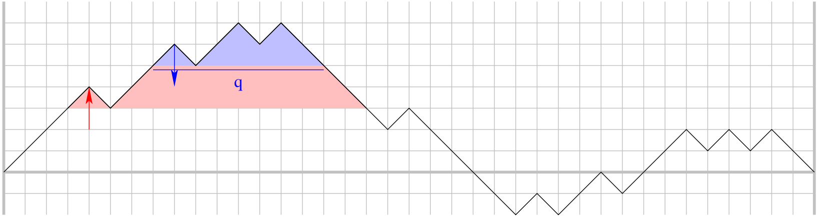

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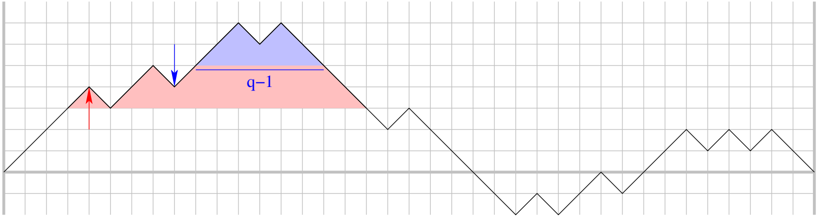

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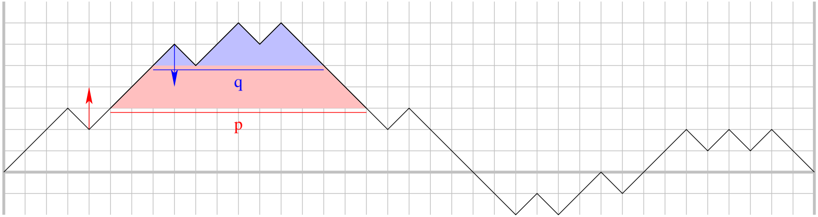

...and the concavity of $x \to x^{\alpha}$ yields a negative total variation.

Thank you for your attention

In the abstract: average case analysis with a well-chosen α