(Generation and) Recognition of Digital Planes using Multidimensional Continued Fractions

Thomas Fernique

LIRMM – Univ. Montpellier 2

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Question: planarity of a given discrete object?

Many approaches already exist.

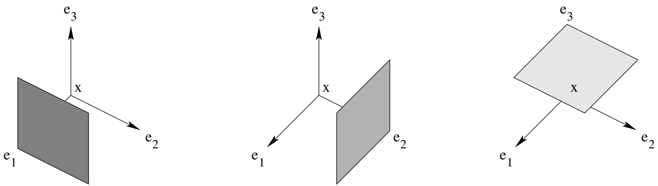
Here: generalization of Wu-Troesh algorithm for discrete lines.

Principe: local planarity checking and "backwards zooms".

1. Stepped planes
2. Local properties
3. Recoding (Backwards zooms)
4. A hybrid algorithm

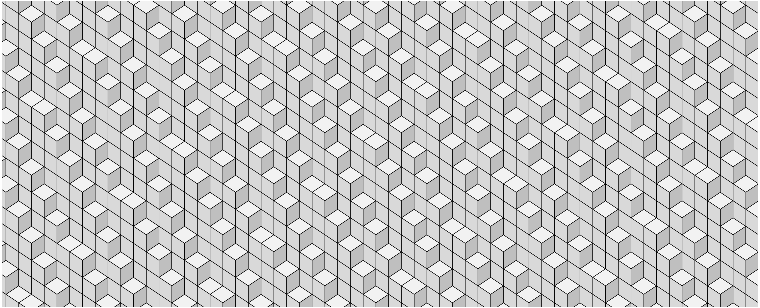
Let $(\vec{e}_1, \ldots, \vec{e}_d)$ be the canonical basis of $\mathbb{R}^d$.

Face of *type* $i \in \{1, \ldots, d\}$ located at $\vec{x} \in \mathbb{Z}^d$:



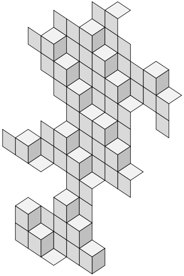
Discrete objects here considered: unions of such faces.

Stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$: boundary of the union of unit cubes of $\mathbb{Z}^d$ intersecting the real half-space $\{\vec{x} \mid \langle \vec{x} | \vec{\alpha} \rangle \leq \rho\}$.



Vertices of $\mathcal{P}_{\vec{\alpha},\rho}$: *discrete standard plane* of parameters $(\vec{\alpha}, \rho)$.

Patch of a stepped plane: union of faces included in this plane.



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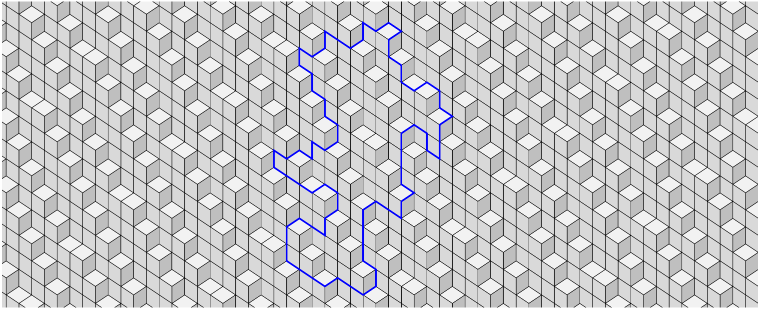

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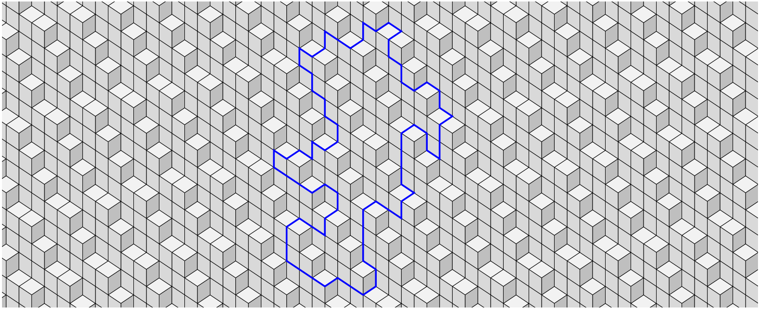

Let us stress that stepped planes can share patches.

More generally, given a union of faces $\mathcal{B}$, $P(\mathcal{B})$ denotes the set of parameters of stepped planes $\mathcal{B}$ is a patch of:

$$P(\mathcal{B}) = \{(\vec{\alpha}, \rho) \in \mathbb{R}_+^d \setminus \{\vec{0}\} \times \mathbb{R} \mid \mathcal{B} \subset \mathcal{P}_{\vec{\alpha},\rho}\}.$$

We speak about *admissibles parameters* of $\mathcal{B}$. It is a convex polytope of $\mathbb{R}^{d+1}$, non-empty iff $\mathcal{B}$ is a patch of stepped plane.

$\rightsquigarrow$ *Recognition* of $\mathcal{B}$: computation of $P(\mathcal{B})$.

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2. Local properties
3. Recoding (Backwards zooms)
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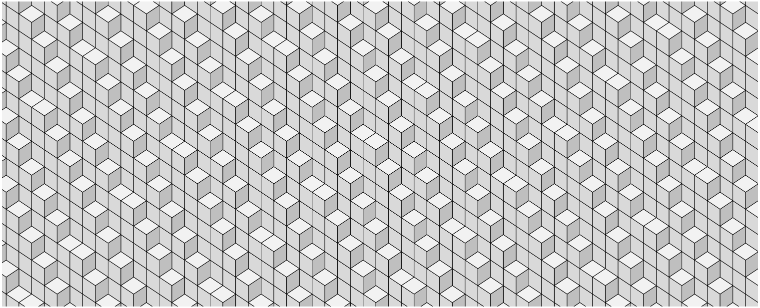
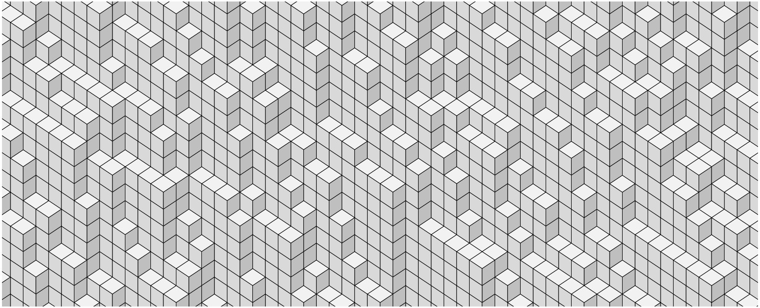
Stepped plane: constrained and regular object.



Looks planar.

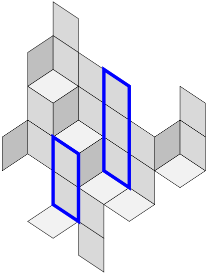
Stepped plane: constrained and regular object.



Does not look planar.

Definition (run)

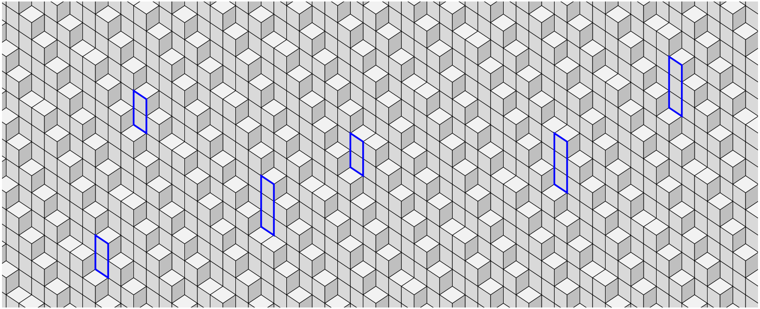
An (i, j)-run of a union of faces $\mathcal{B}$ is a maximal sequence of faces of type i, aligned in the direction $\vec{e}_j$ and included in $\mathcal{B}$.



$(1, 3)$-runs of length 2 and 3.

Proposition (Berthé-F. 2007)

(i,j)-runs of stepped plane $\mathcal{P}_{\vec{\alpha},\rho}$ have length $\lfloor \alpha_i/\alpha_j \rfloor$ or $\lceil \alpha_i/\alpha_j \rceil$.



If it is a stepped plane with normal $\vec{\alpha}$, then $2 < \alpha_1/\alpha_3 < 3$.

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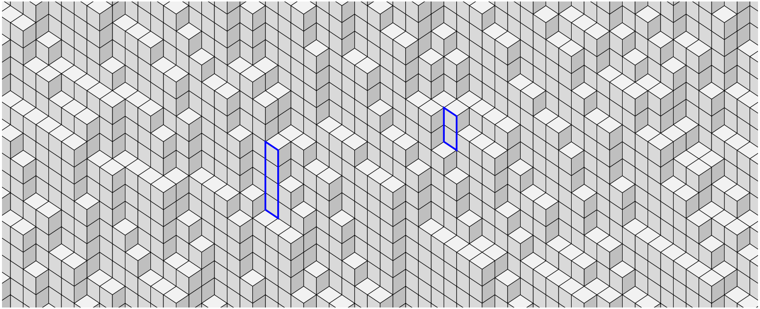
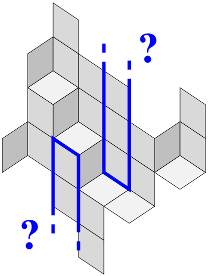


It is not a stepped plane (incompatible lengths of runs).

Planes and patches do not always have identical runs:



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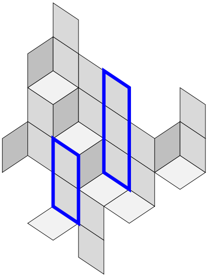

Weak conditions $\rightsquigarrow$ info. on the (eventual) normal vector.

We speak about a *recognizable* union of faces.

Recoding: map $\tilde{T}$ on union of faces which satisfies:

- $\mathcal{B}$ patch of $\mathcal{P}$ iff $\tilde{T}(\mathcal{B})$ patch of $\tilde{T}(\mathcal{P})$;
- $P(\mathcal{B})$ can be (easily) deduced from $P(\tilde{T}(\mathcal{B}))$;
- $\tilde{T}(\mathcal{B})$ contains (much) lesser faces than $\mathcal{B}$.

Aim: reducing the recognition of $\mathcal{B}$ to the one of $\tilde{T}(\mathcal{B})$.

More precisely, recoding by *dual maps* (see abstract):

$$\tilde{T}(\mathcal{B}) = E_1^*(\sigma_{\mathcal{B}})(\mathcal{B}).$$

Choice of $\sigma_{\mathcal{B}}$ relies on the runs of $\mathcal{B}$ (if recognizable).

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Intuition of the "whole" principe:

Compute the common part of multi-dimensional continued fraction expansion (Brun's algorithm) of the normal vectors of stepped planes $\mathcal{B}$ is a patch of.

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Solution: use the following equivalence notion:

$$\mathcal{B} \sim \mathcal{B}' \;\Leftrightarrow\; P(\mathcal{B}) = P(\mathcal{B}').$$

The equivalence classes are generally big.
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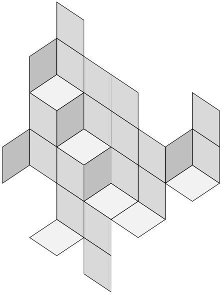
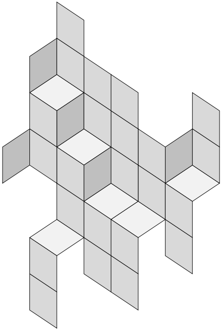
Note: each equivalence class has a structure of semi-lattice

Problems with boundaries



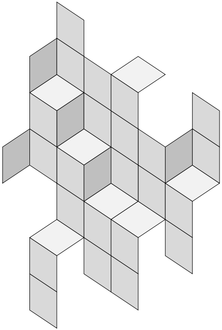
A stepped plane containing this has $(1, 3)$-runs of length 2 and 3.

Problems with boundaries



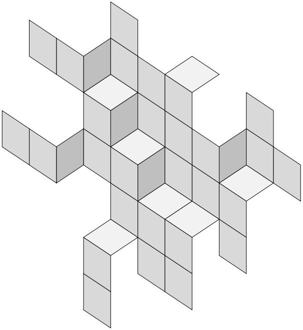
We thus can extend $(1, 3)$-runs of length less than 2,

Problems with boundaries



and “close” those of length 3.

Problems with boundaries



We can proceed similarly for $(1, 2)$-runs,

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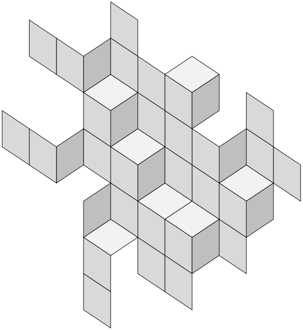

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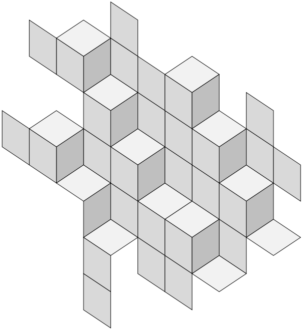
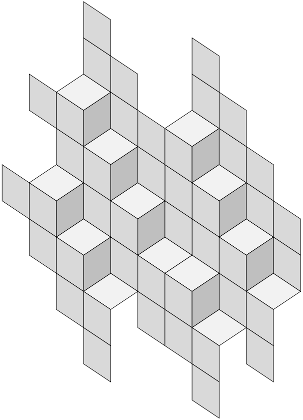


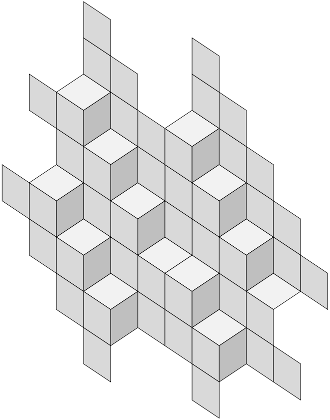
and for $(3,2)$-runs.

Problems with boundaries



We can then repeat these operations.

Problems with boundaries



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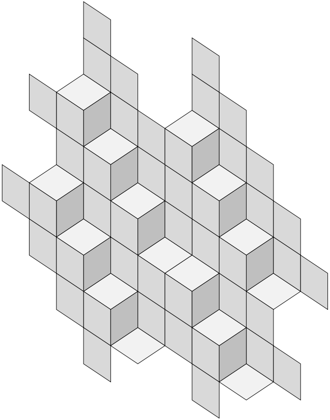

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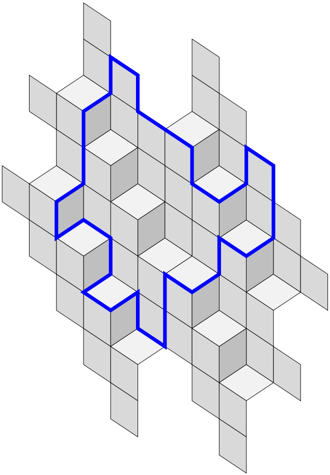
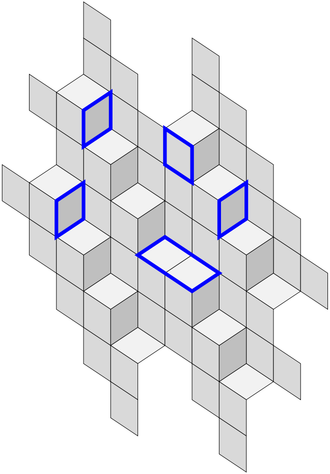


If the initial number is finite, then the process converges.

Problems with boundaries



We could also remove "deducible" faces. Thus, large choice.

Algorithm:

while $\mathcal{B}$ recognizable **do**
    $\tilde{\mathcal{B}} \leftarrow$ suitable equivalent of $\mathcal{B}$;
    $\mathcal{B} \leftarrow$ recoding of $\tilde{\mathcal{B}}$;
endwhile;
compute $P(\mathcal{B})$ thanks to another algorithm;
deduce the parameters of the initial $\mathcal{B}$;

Each loop:
reading runs + choosing an equivalent + recoding: $\mathcal{O}(|\mathcal{B}|)$.

Number of loops:
If $\mathcal{B}$ is a patch of stepped plane of normal $\vec{\alpha} = (p_1/q, \ldots, p_d/q)$:

$$\log_{\frac{d+2}{d+1}}(p_1 + \ldots + p_d + q).$$

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Otherwise, let D be the side of a bounding box for $\mathcal{B}$. Then, either $\mathcal{B}$ is a patch of a stepped plane of normal $\vec{\alpha} = (p_1/q, \ldots, p_d/q)$ with $p_i, q \leq D$, or it is not planar. Hence the general bound:

$$\log_{\frac{d+2}{d+1}}((d+1)D).$$

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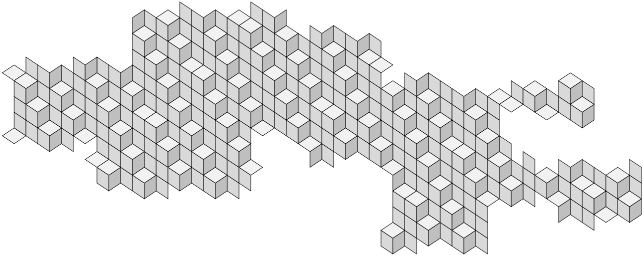
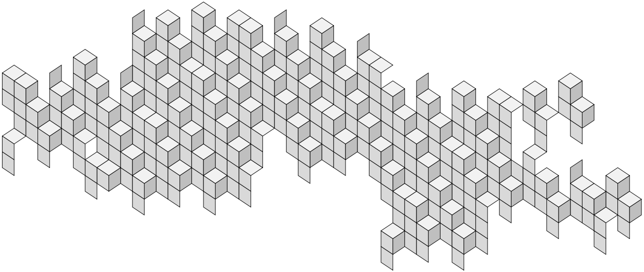
Total complexity: quasi-linear if D is polynomial in $|\mathcal{B}|$.

Example



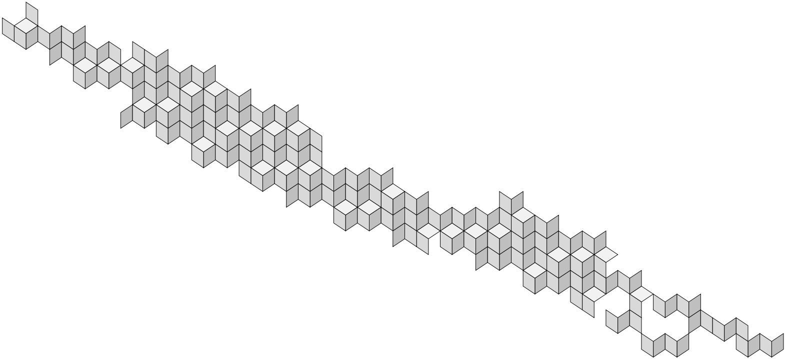
Union of 407 faces: is it a patch of stepped plane?

Example



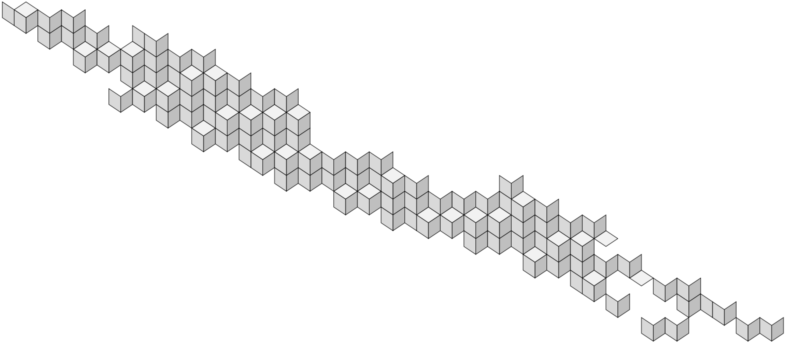
Recognizable union $\rightsquigarrow$ computation of a suitable equivalent.

Example



Recoding $\rightsquigarrow$ union of 216 faces. Is it a patch of stepped plane?

Example



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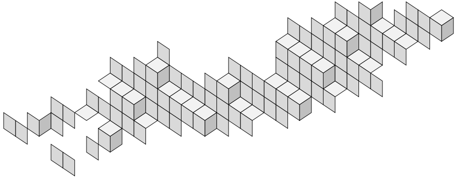
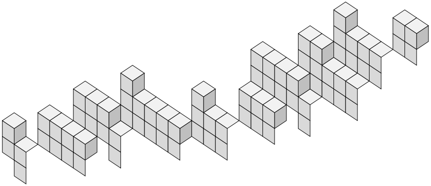


Recoding $\rightsquigarrow$ union of 129 faces. Is it a patch of stepped plane?

Example



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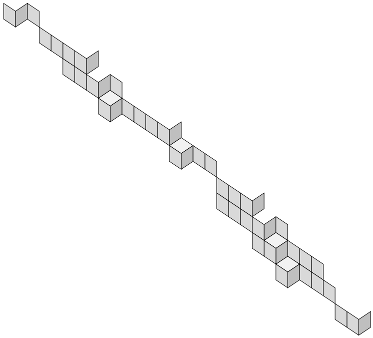
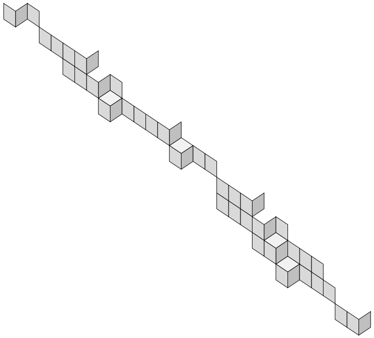


Recoding $\rightsquigarrow$ union of 52 faces. Is it a patch of stepped plane?

Example



Unrecognizable union $\rightsquigarrow$ use another algorithm.

Advantages of recognition by recoding:

- new generalization of a result for discrete lines;
- good theoretical complexity;
- allows partial recognition (convergence of continued fractions);
- "backwards" recoding $\rightsquigarrow$ generation (see abstract).

Open problems:

- practical complexity?
- robustness?