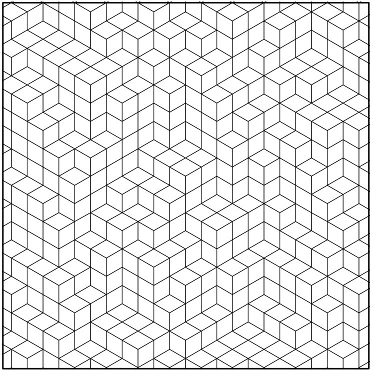
Weak Local Rules for the N-Fold Tilings

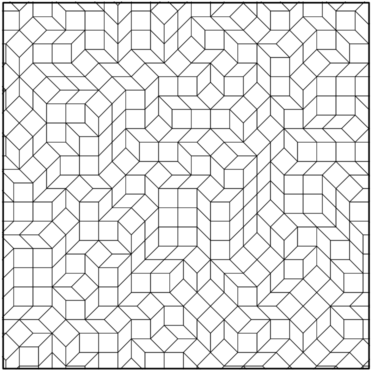
Nicolas Bédaride & Thomas Fernique

Outline

Outline

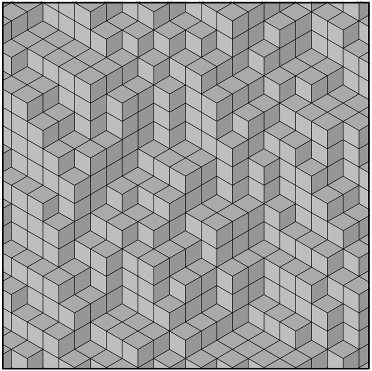
Rhombus tiling

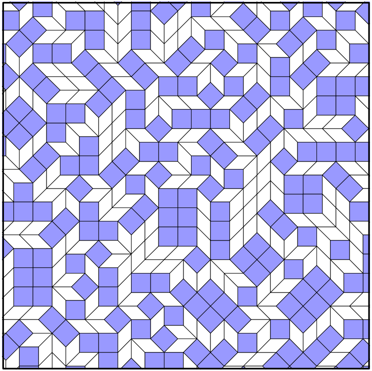




n pairwise non-colinear vectors of $\mathbb{R}^2 \rightsquigarrow$ tiling of $\mathbb{R}^2$ by $\binom{n}{2}$ rhombi.

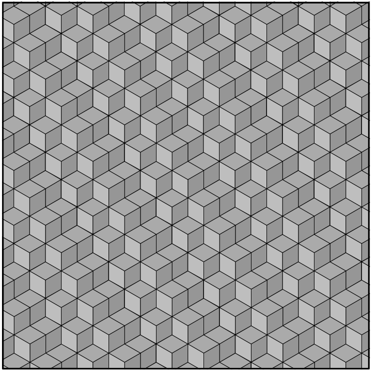
Lift

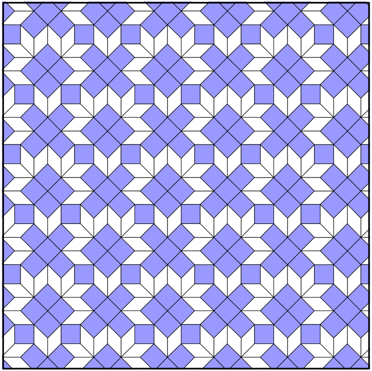




Lift: homeomorphism which maps tiles on 2-faces of unit n-cubes.

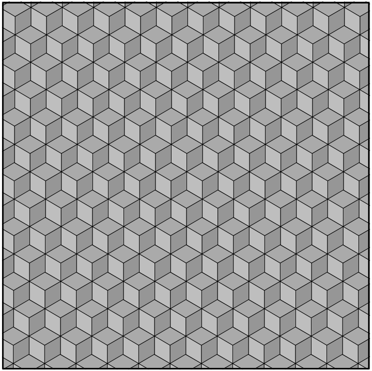
Planarity

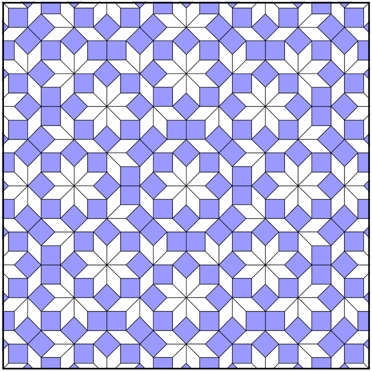




Planar: lift in $E + [0, t]^n$, where E is the *slope* and t the *thickness*.

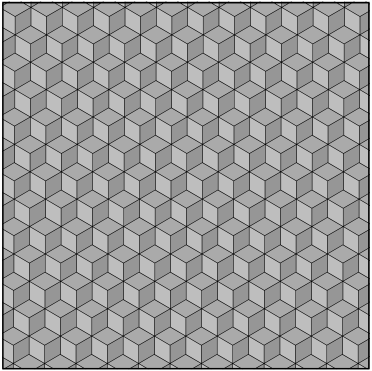
N-fold tiling

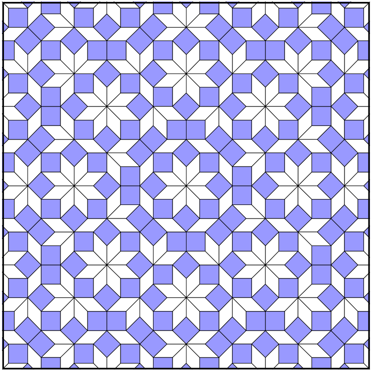




N-fold: same finite patterns as its image under a rotation by $\frac{2\pi}{N}$.

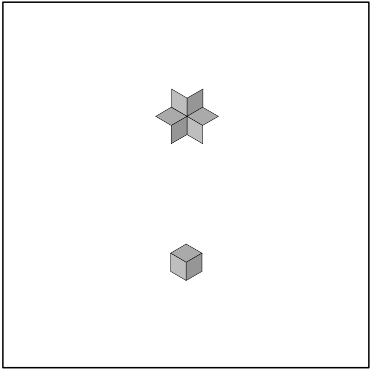
Weak local rules (Levitov, 1988)

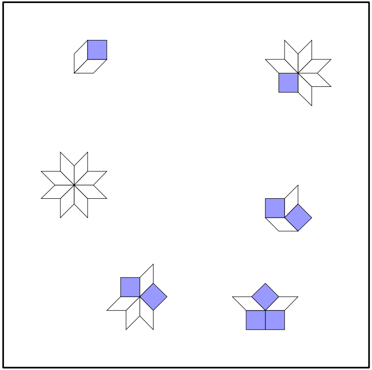




Weak local rules: when patches of radius R characterize the slope.

Weak local rules (Levitov, 1988)

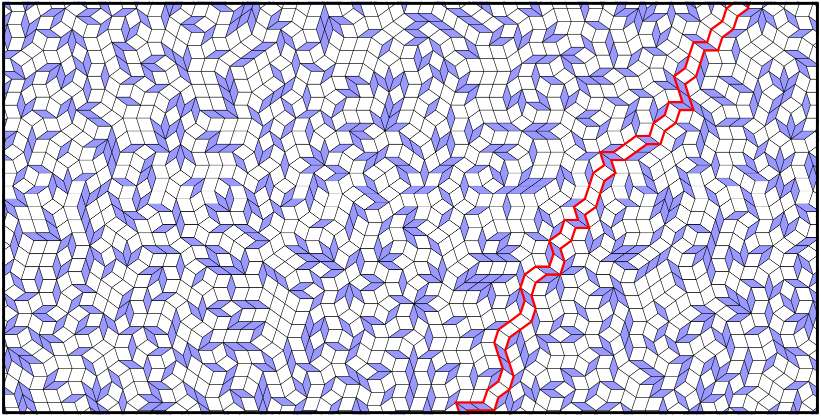




Weak local rules: when patches of radius R characterize the slope.

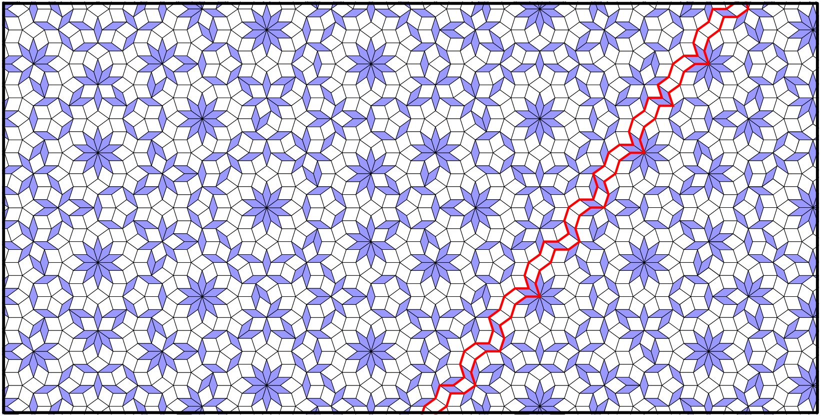
Outline

Stripes



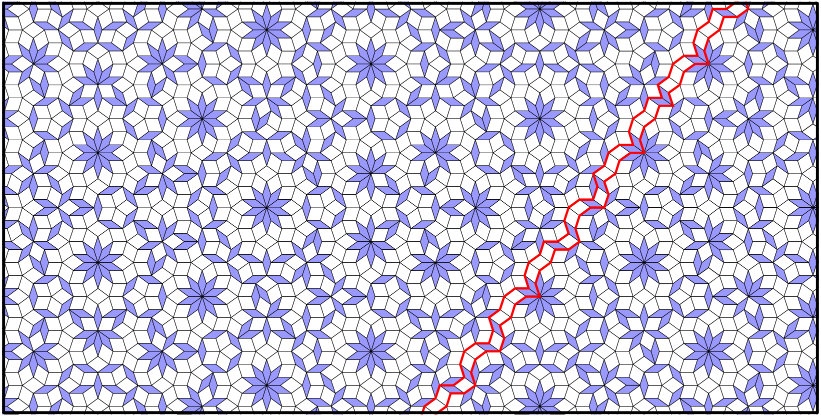
Stripe: each tile is adjacent to the next one along parallel edges.

Alternation Condition (Socolar, 1990)



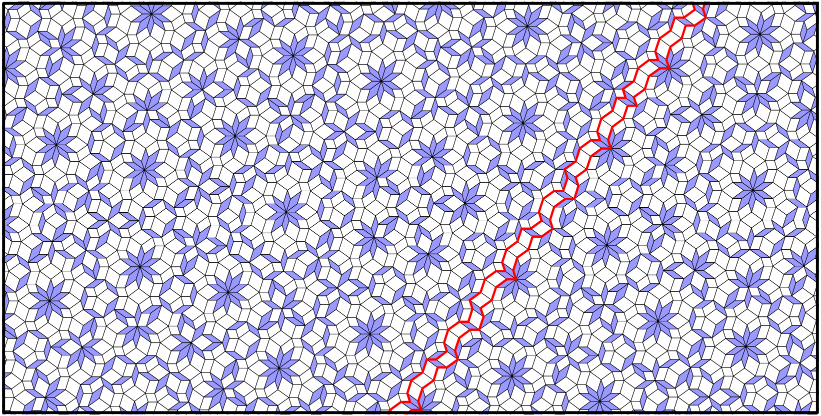
AC: in each stripe, each tile must alternate with its mirror image.

Subperiods



AC enforces the projections on three basis vectors to be periodic.

Subperiods



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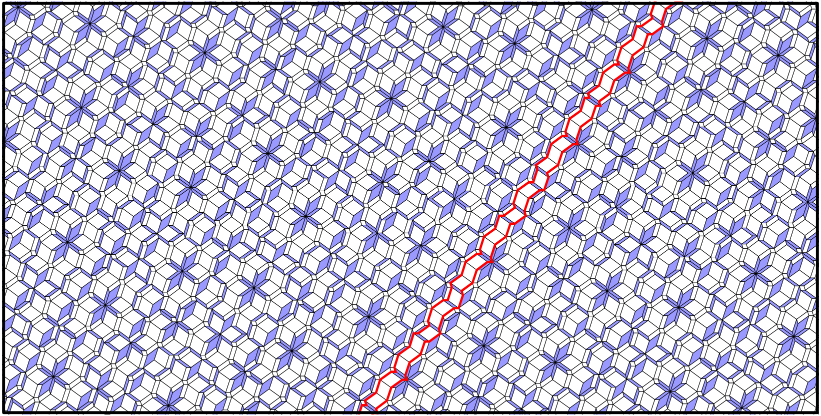

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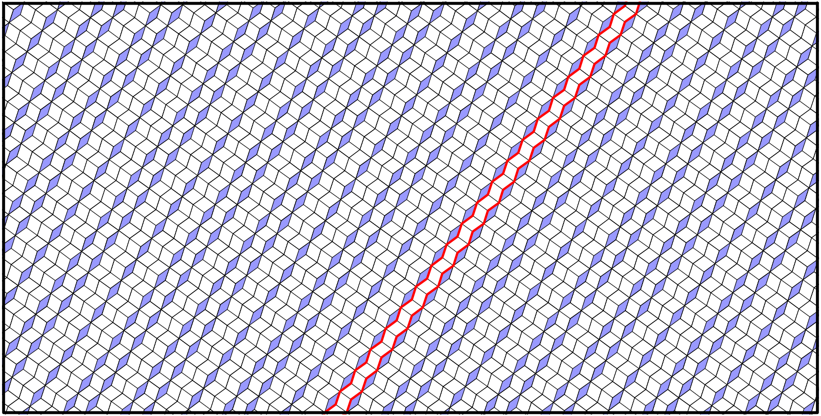


AC enforces the projections on three basis vectors to be periodic.

Slope

Consider a planar tiling whose slope is generated by $\vec{u}$ and $\vec{v}$. Let $G_{ij} = u_i v_j - u_j v_i$ be the Grassmann coordinates of the slope.

- One proves that the AC enforces the relations

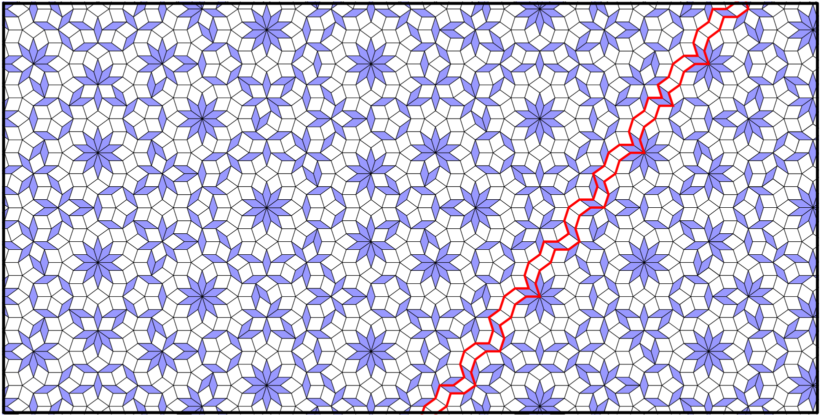
$$G_{ij} = G_{j,2j-i}.$$

- Grassmann coordinates moreover always satisfy the relations

$$G_{ij} G_{kl} = G_{ik} G_{jl} - G_{il} G_{jk}.$$

One proves that this characterize the n-fold slope when $n \neq 4p$.

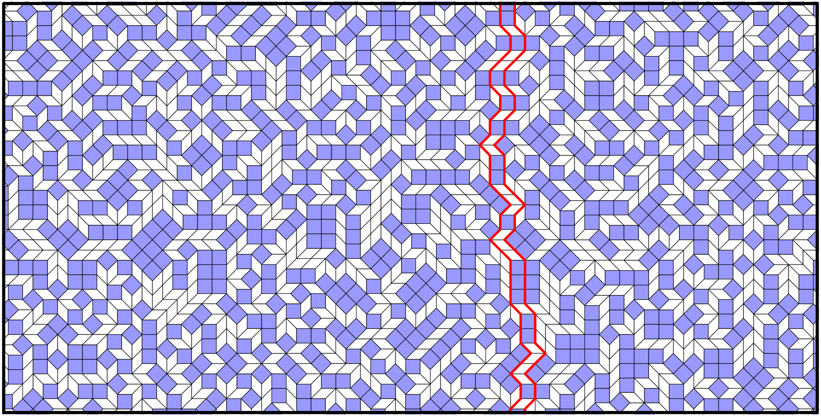
Planarity



AC enforces straight stripes. One shows that it enforces planarity.

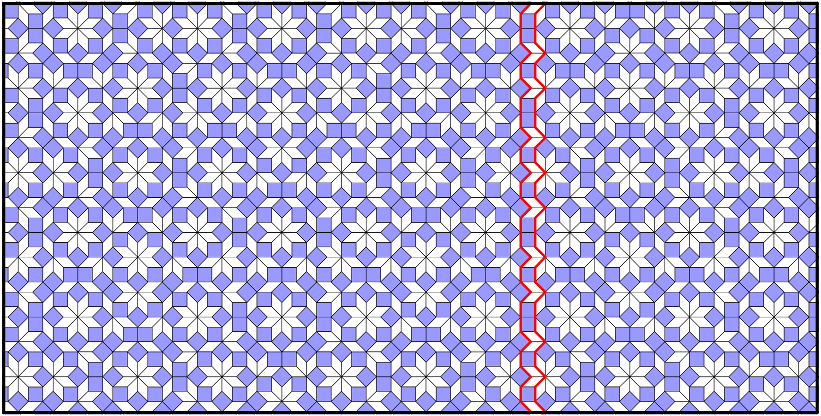
Outline

Evenly even n



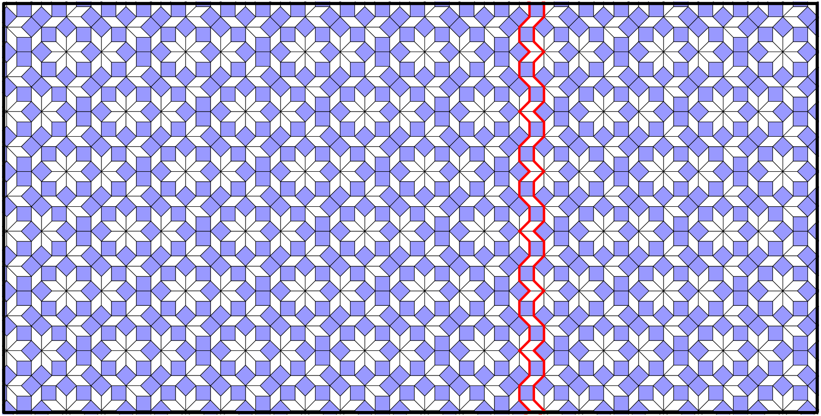
Tiles can be squares when $n = 4p$. What AC does now enforce?

Slopes



One proves that AC characterizes a one-parameter family of slopes.

Slopes



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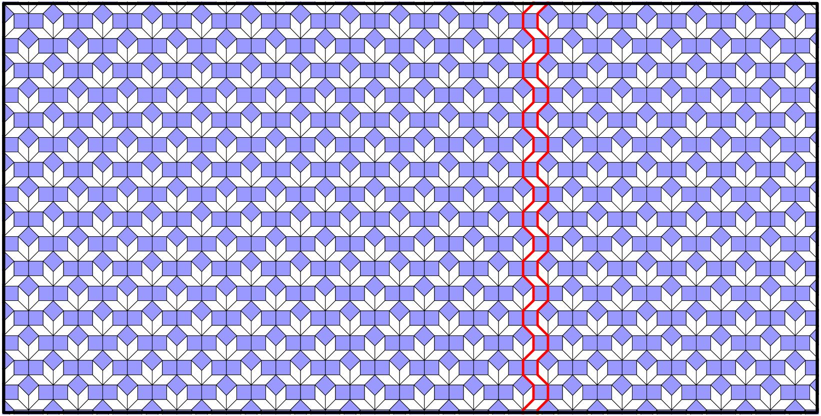

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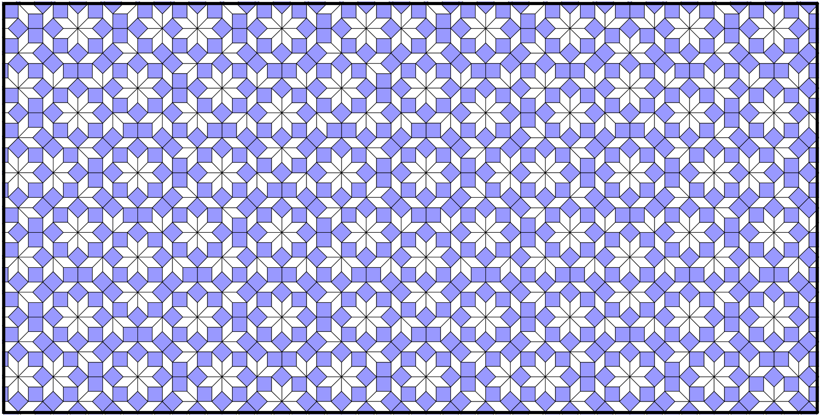
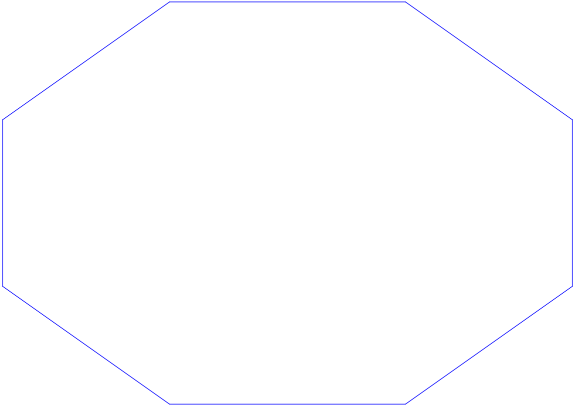
Slopes



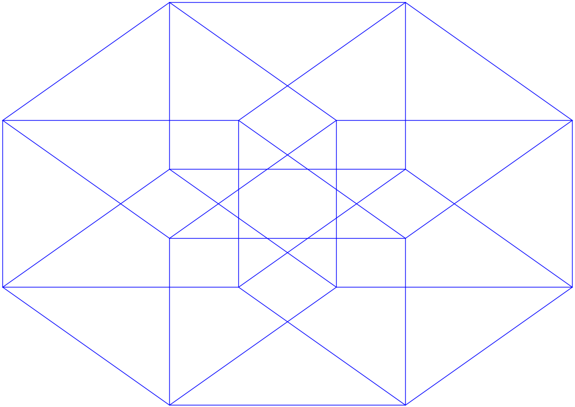
Can we find a finite patch which distinguishes the $4p$-fold tilings?

Window



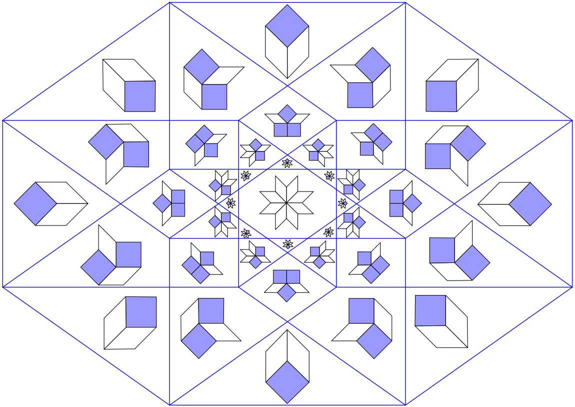
Window of a planar tiling of slope E: projection of $[0,1]^n$ onto $E^\perp$.

Window



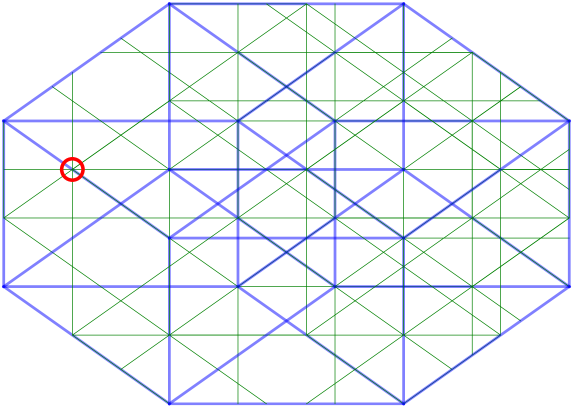
Projecting also the $(n-3)$-faces of $\{0,\ldots,k\}^n$ yields a partition.

Window



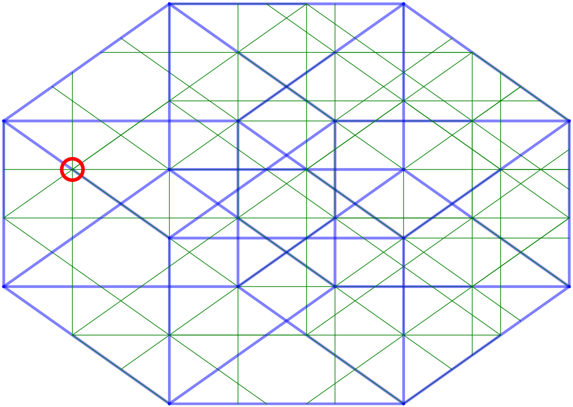
One proves that its parts are in bijection with the size k patches.

Coincidence



Coincidence: intersection of at least $n-1$ projected $(n-3)$-faces.

Sliding coincidences



One proves that AC preserves the coincidences of any $4p$-fold tiling.

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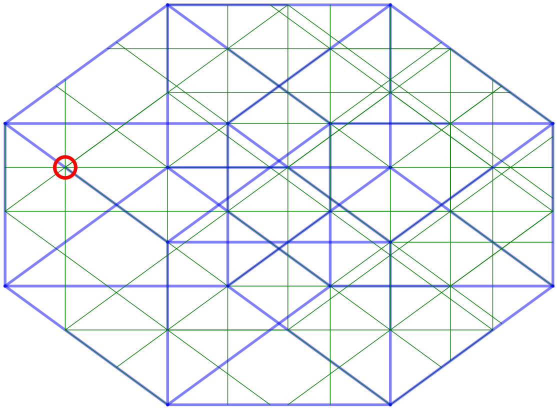

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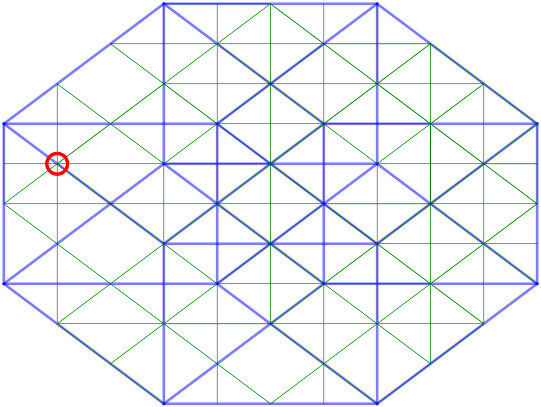

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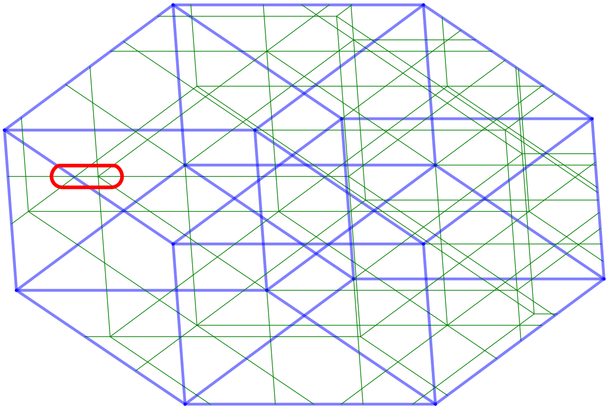

This is not necessarily the case outside the one-parameter family.

Conclusion

Theorem (Socolar 1990, Bédaride-Fernique 2014)

The N-fold tilings admit weak local rules iff N is not evenly even.

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Some questions:

- Does AC nevertheless enforce planarity?
- What about other planar tilings?
- What if tiles can be *decorated*?

Conclusion

Theorem (Socolar 1990, Bédaride-Fernique 2014)

The N-fold tilings admit weak local rules iff N is not evenly even.

Some questions:

- Does AC nevertheless enforce planarity?

  Holds for $N = 8p$ and $N = 12p$.

- What about other planar tilings?

  The slope must be algebraic.

- What if tiles can be *decorated*?

  Any computable slope can be enforced (Sablik-Fernique 2012).

Some references

- L. S. Levitov, *Local rules for quasicrystals*, Comm. Math. Phys. **119** (1988)
- S. E. Burkov, *Absence of weak local rules for the planar quasicrystalline tiling with the 8-fold rotational symmetry*, Comm. Math. Phys. **119** (1988)
- J. E. S. Socolar, *Weak matching rules for quasicrystals*, Comm. Math. Phys. **129** (1990)
- N. Bédaride, Th. Fernique, *When periodicities enforce aperiodicity*, to appear in Comm. Math. Phys.
- N. Bédaride, Th. Fernique, *No weak local rules for the 4p-fold tilings*, arXiv:1409.0215 (2014)