On Rhombus Tilings

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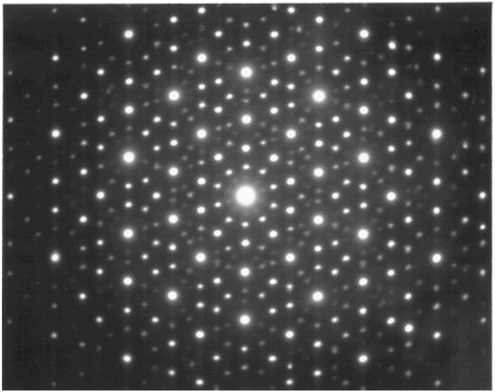
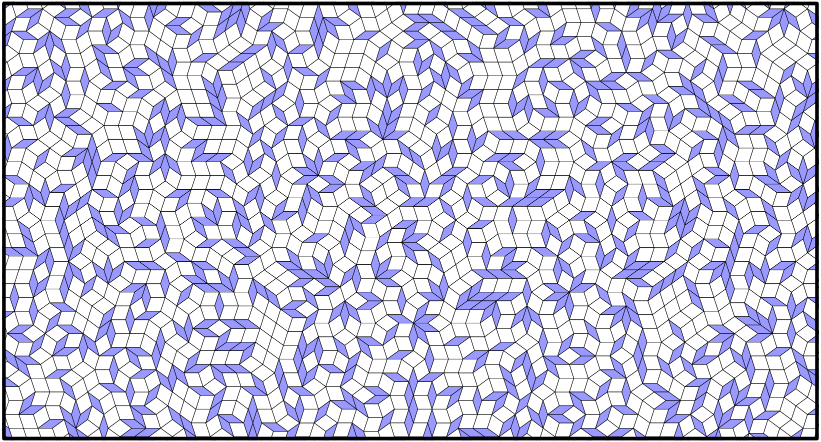
Quasicrystals (Shechtman discovery, 1982)



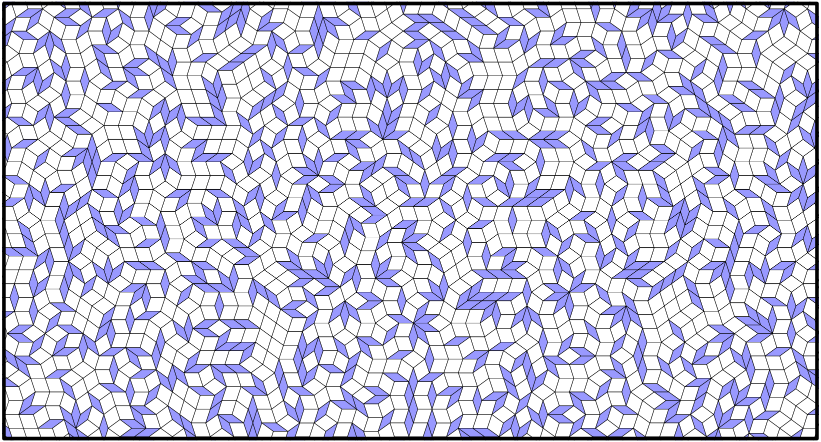
Material with sharp diffraction peaks but non-periodic symmetry.
How short range interactions do stabilize non-periodic crystals?

Planar rhombus tilings



n pairwise non-colinear vectors of $\mathbb{R}^2 \rightsquigarrow$ tilings of $\mathbb{R}^2$ by $\binom{n}{2}$ rhombi.

Planar rhombus tilings



Lift: homeomorphism which maps tiles on 2-faces of unit n-cubes.

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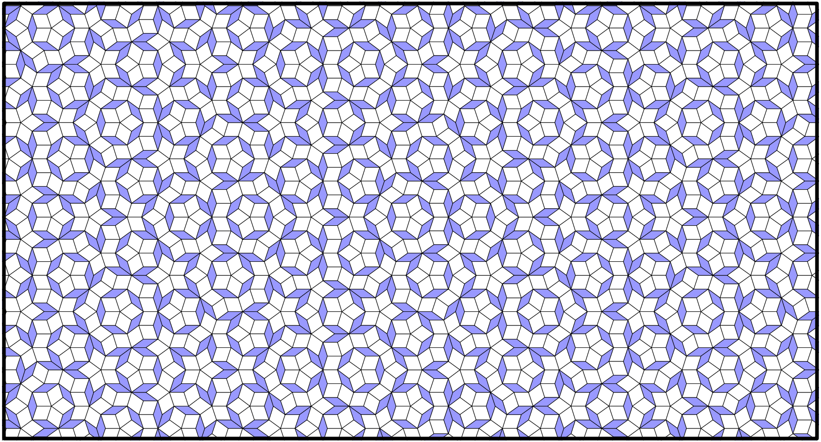

Planar: lift in $E + [0, t]^n$, where E is the *slope* and t the *thickness*.

Finite type slopes

Definition

A slope E has *finite type* if there is finitely many finite patterns s.t.

- slope E and thickness 1 tilings have no such patterns;
- tilings with no such patterns are planar with slope E.

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Theorem (Levitov'88, Le'95, F.-Bédaride'13)

A slope E has finite type *iff it is characterized by its* subperiods.

This allows only algebraic slopes but can yield simple patterns.

Sofic slopes

Definition

A slope E is *sofic* if there is finitely many finite *colored* patterns s.t.

- slope E and thickness 1 colored tilings can avoid such patterns;
- colored tilings with no such patterns are planar with slope E.

Sofic slopes

Definition

A slope E is *sofic* if there are finitely many *colored* rhombi s.t.

- colors can match in planar tilings of slope E and thickness 1;
- rhombus tilings where colors match are planar with slope E.

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A slope E is *sofic* if there are finitely many *colored* rhombi s.t.

- colors can match in planar tilings of slope E and thickness 1;
- rhombus tilings where colors match are planar with slope E.

Theorem (F.-Sablik'12)

A slope E is sofic iff it is computable.

This goes far beyond algebraic slopes but yields huge tile sets.

Substitutive tilings

Definition

A tiling is *substitutive* if it is invariant by a *group/deflate process*.

If the group process is unambiguous, then the tiling is not periodic.

Theorem (Mozes'90, Goodmann-Strauss'95, F.-Ollinger'10)

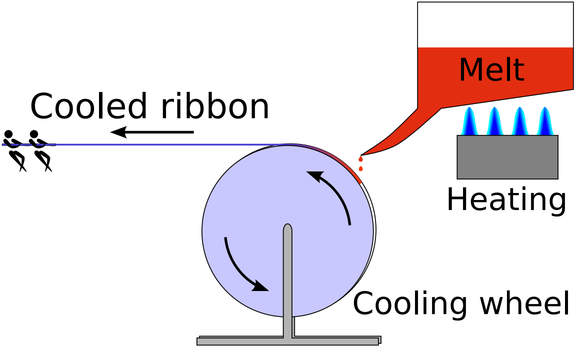
Substitutive rhombus tilings (generally) have sofic slopes.

This allows only algebraic slopes but holds beyond rhombus tilings.

Some open questions

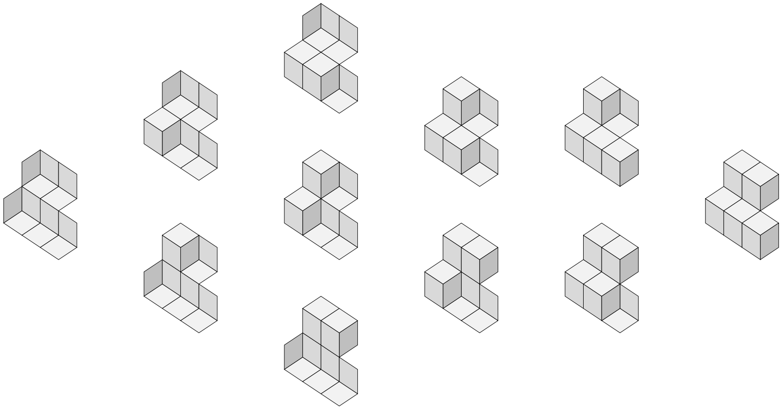
- What can be said on the thickness? When is it minimal?
- Which slopes can be enforced with k colored rhombi?
- How to *assemble* sofic or finite type tilings?

Quasicrystals (Shechtman quenching)



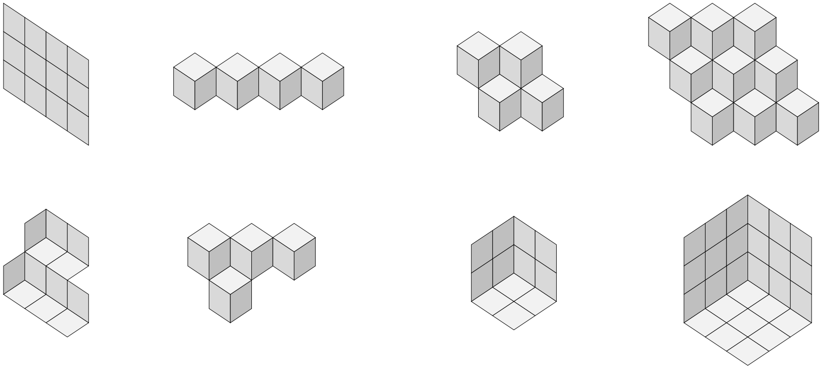
Entropy maximization supersede *energy* minimization.

Tiling space and entropy



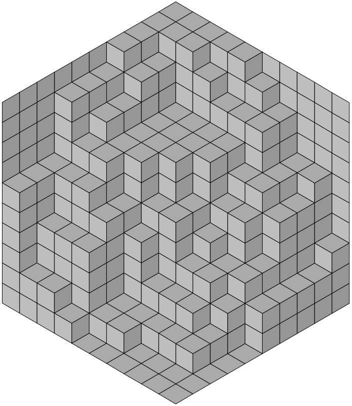
Tiling space: set of the tilings of a given domain.

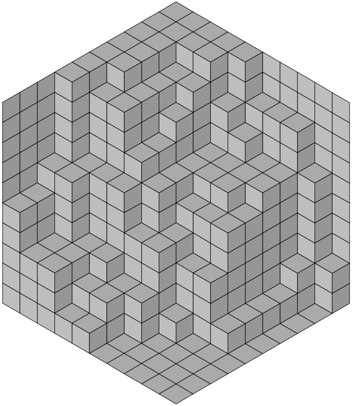
Tiling space and entropy



Tiling entropy: $s := \mathsf{log}(\text{size of the tiling space})/(\text{nb. of tiles})$.

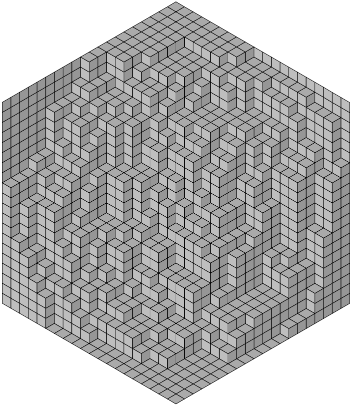
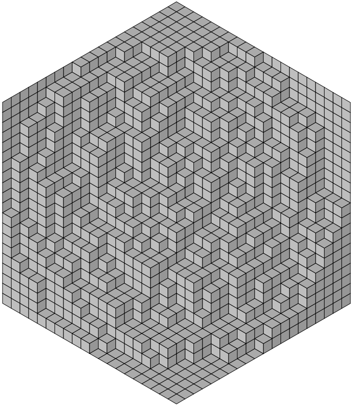
Typical aspect





Do random tilings a.s. satisfy some properties when the size grows?

Typical aspect

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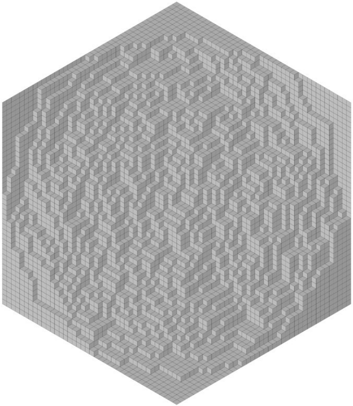

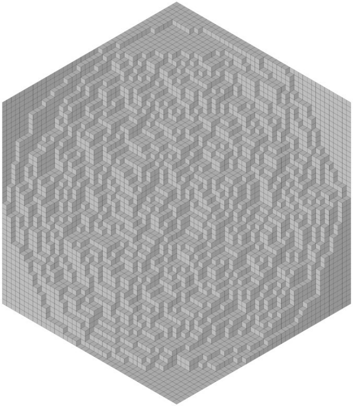

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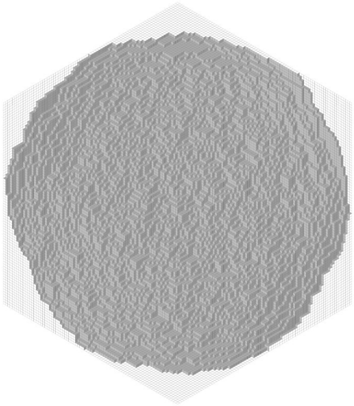

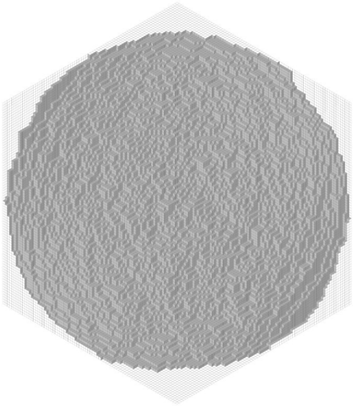

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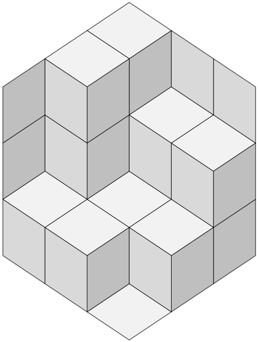
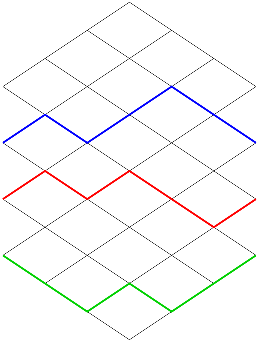
Dimers: Lindström-Gessel-Viennot lemma, 1985





Dimer tiling $\leftrightarrow$ non-intersecting path family.

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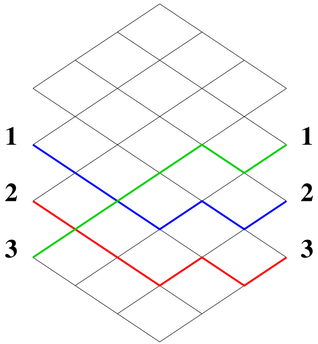

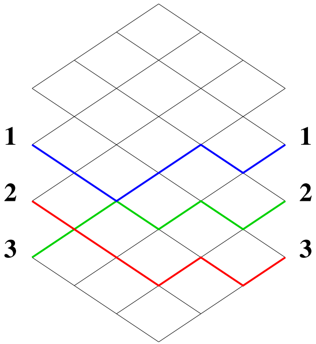

Path family $\leftrightarrow$ permutation. What does count $\det(\#\text{paths } i \leftrightarrow j)$?

Dimers: Cohn-Kenyon-Propp variational principle, 2001

Height function of a dimer tiling: distance to $x+y+z=0$.

Theorem

Let $R \subset \mathbb{R}^2$ be bounded by a piecewise smooth simple closed curve. If, for $n \geq 0$, R_n is a tileable domain which approximates nR, then

$$\lim_{n\to\infty} s(R_n) = \sup_h \frac{1}{|R|} \iint_R \mathrm{ent}\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) dx\, dy,$$

where $\mathrm{ent} : \mathbb{R}^2 \to \mathbb{R}$ and h is any 2-Lipschitz real function on R.

Moreover, the normalized R_n's random height functions converge in probability (exponentially fast) towards the integral-maximizing h.

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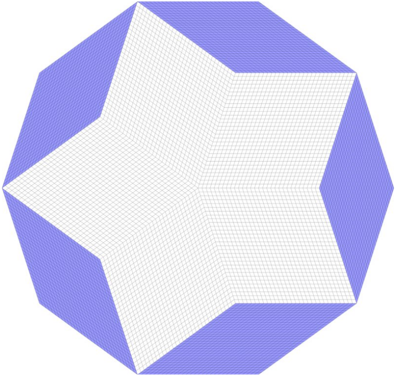
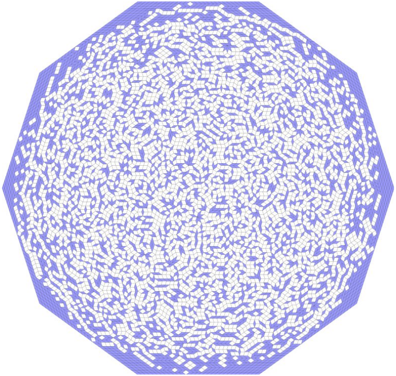
Frozen boundaries form algebraic curves (Kenyon-Okounkov, 2005).

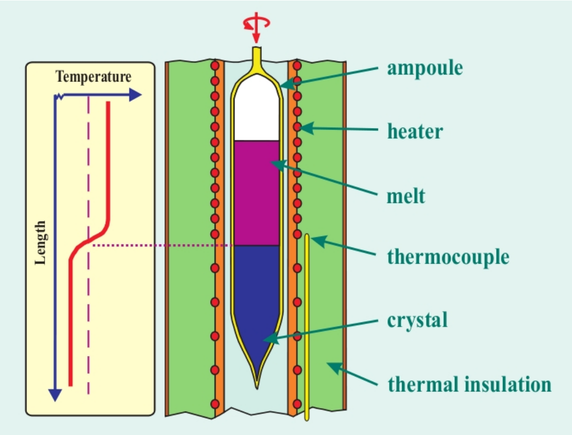
Some open questions





Counting rhombus tilings? Maximal entropy? Typical properties?

Quasicrystals (Bridgman-Stockbarger technique)



From entropy maximization to energy minimization.

Gibbs distribution

Definition (Gibbs measure at temperature T)

Probability of the system being in state x: $\frac{1}{Z(T)}\exp(-E(x)/T)$.

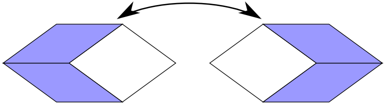
Energy of a finite type/sofic tiling: number of forbidden patterns.

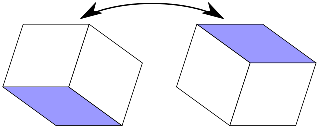
The Gibbs distribution is thus

- concentrated on perfect tiling at $T = 0$;
- uniform at $T = \infty$ (random tilings).

What inbetween? Which dynamics for tilings?

Flips





Theorem (Kenyon'94)

Rhombus tilings of a simply connected finite domain are flip-linked.

The "Metropolis" Markov chain is thus ergodic for $T > 0$:

- draw uniformly at random a vertex of the current tiling;
- if a flip is possible, then do it with probability $\exp(-\Delta E/T)$.

Its stationary distribution is the Gibbs distribution.

Mixing time?

Markov chain mixing time

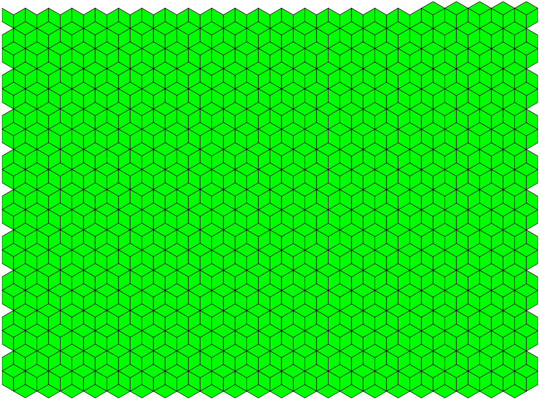
Markov chain $P=(p_{ij})_{ij}$: goes from $i\in S$ to $j\in S$ with proba. p_{ij}.
Example: random walk on the web.

Stationary distribution π: $\pi_j=\sum_{i\in S}\pi_i p_{ij}=\lim_{t\infty}P^t(\cdot\,,j)$.
Example: Google's original PageRank.

Total variation: $\delta(\mu,\nu)=\max_{A\subset S}|\mu(A)-\nu(A)|$.

Mixing time τ: $\min_{t\geq 0}\max_{i\in S}\delta(P^t(i,\cdot\,),\pi)\leq\frac{1}{4}$.
Example: how many time shall you shuffle your Rubik's cube?

A simple dimer case



Forbid adjacent identical rombi. Fix a maximal entropy domain.

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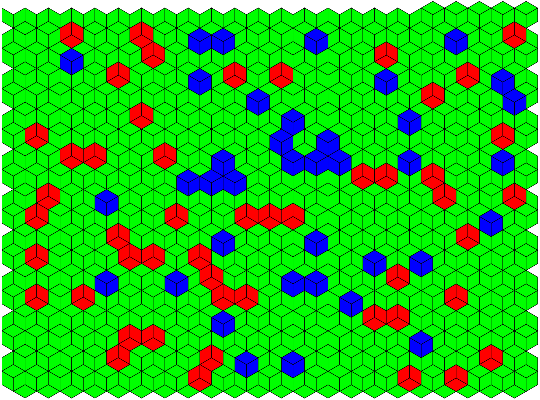

Theorem (Caputo-Martinelli-Toninelli, 2011)

At $T = \infty$ (uniform stationary distribution), $\tau = \mathcal{O}(n^4 \log(n)^{12})$.

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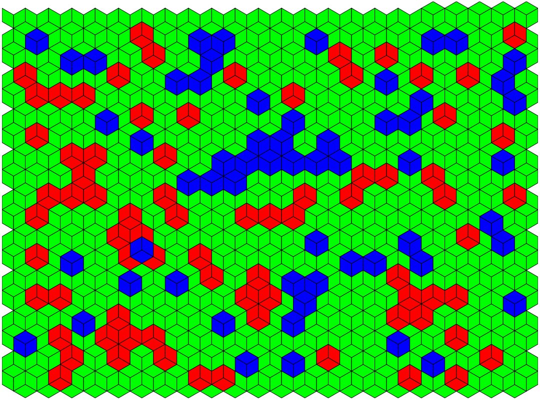

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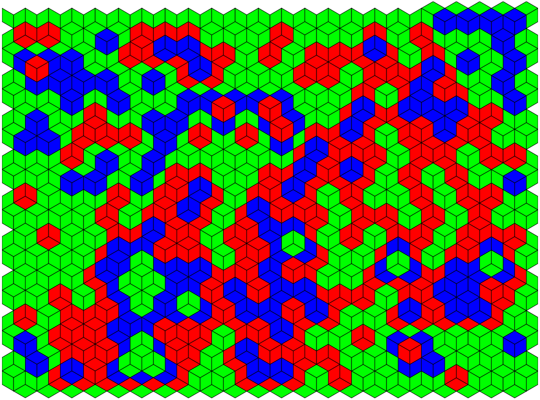

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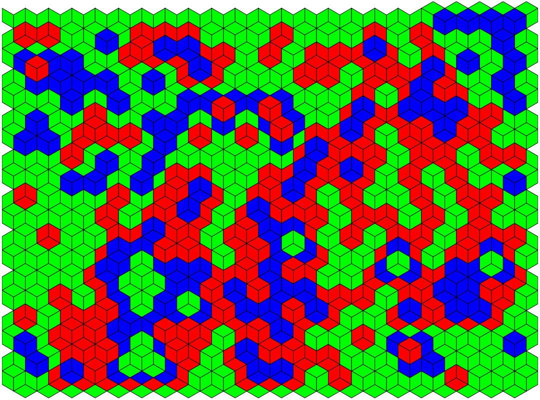

Theorem (F.-Regnault, 2010)

At $T = 0$ (Dirac stationary distribution), $\tau = \mathcal{O}(n^2\sqrt{n})$.

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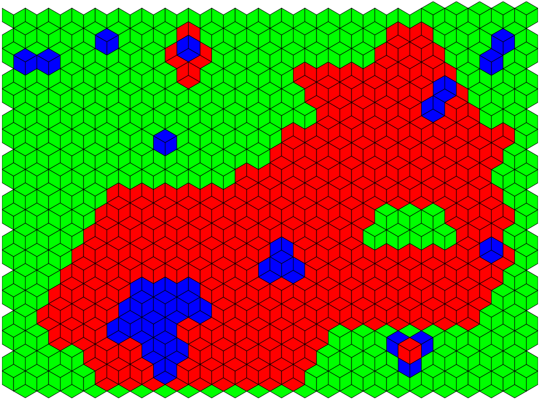

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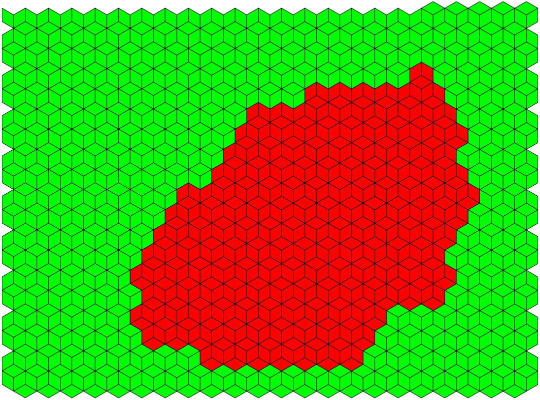

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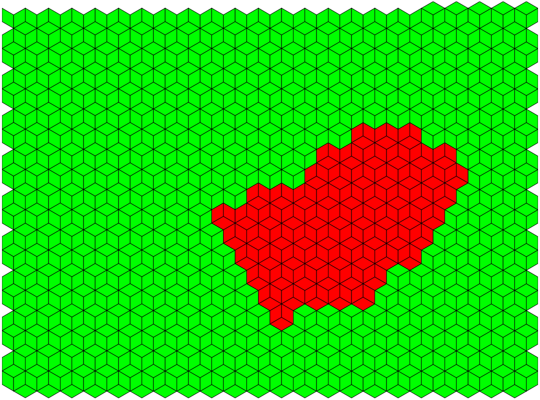

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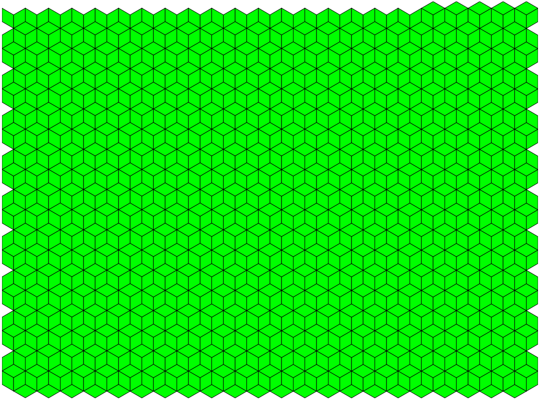

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Some open questions

- Mixing time of the previous example at given T?
- Mixing time for other rhombus tilings (*e.g.*, Penrose)?
- What if T varies (simulated annealing)? Optimal schedule?

Thank you for your attention.

Slides of a longer exposition (12h), with main references accessible:
http://lipn.univ-paris13.fr/~fernique/qc

Project QuasiCool (Ph.D. position open at Univ. Paris 13):
http://lipn.univ-paris13.fr/~fernique/quasicool/