Cut and Project Tilings 1: Definitions and Patterns

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Cut and project tilings

Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^d$ and define the tiles:

$$T_{i_1,\ldots,i_d} := \left\{ \sum_{j=1}^{d} \lambda_{i_j} \vec{v}_{i_j} \;\middle|\; 0 \leq \lambda_{i_j} \leq 1 \right\}.$$

Definition ($n \to d$ tiling)

A $n \to d$ tiling is a "face-to-face" tiling of $\mathbb{R}^d$ by such tiles.

It naturally lifts onto a d-dim. surface of $\mathbb{R}^n$ via $\vec{v}_i \mapsto \vec{e}_i$.

Planar tilings

Definition (Planar tiling)

A $n \to d$ tiling is said to be *planar* if it lifts into a tube $E + [0,1]^n$, where E is an affine d-plane of $\mathbb{R}^n$ called the *slope* of the tiling.

n	d	example
2	1	Sturmian words
3	1	Billiard words
3	2	Discrete planes
4	2	Ammann-Beenker tilings
5	2	Penrose tilings
6	3	Icosahedral tilings
$\vdots$	$\vdots$	$\vdots$

Examples

A rhombille tiling in Saint-Étienne de Marmoutier (Alsace).

Examples

Michael Baake's place & my chocolates: Ammann-Beenker tilings.

Examples

My place: a homemade Penrose tiling (oak & pinewood).

Grassmann coordinates

Definition (Grassmann coordinates)

The *Grassmann (projective) coordinates* of a d-plane of $\mathbb{R}^n$ are the $d \times d$ minors $G_{i_1 \ldots i_d}$ of a matrix whose columns generate this plane.

For $n - d = 1$ this is the usual normal vector.

Proposition

The frequency of $T_{i_1 \ldots i_d}$ in a planar tiling is given by $\frac{1}{||\mathbf{G}||_1} |G_{i_1 \ldots i_d}|$.

For Penrose tilings, there are φ fat rhombi for 1 thin rhomb:

$$(G_{ij}) = (\varphi, 1, -1, -\varphi, \varphi, 1, -1, \phi, 1, \varphi).$$

The multigrid method

Definition (Multigrid)

The *multigrid* with shifts $s_1, \ldots, s_n$ in $\mathbb{R}$ and *grid vectors* $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^d$ is the set of n families of equally spaced parallel hyperplanes

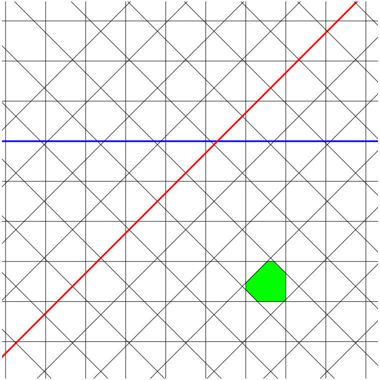
$$H_i := \{\vec{x} \in \mathbb{R}^d \mid \langle \vec{x} | \vec{v}_i \rangle + s_i \in \mathbb{Z}\}.$$

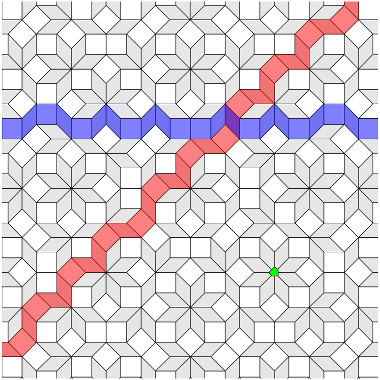
Theorem (De Bruijn, 1981)

The planar tiling with slope $E \subset \mathbb{R}^n$ is the dualization *of the multigrid "drawn" on E by intersecting it, for $1 \leq i \leq n$, with*

$$G_i := \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x} | \vec{e}_i \rangle \in \mathbb{Z}\}.$$

Example





Window and patterns

Definition (window)

The *window* W of a planar tiling of slope $E \subset \mathbb{R}^n$ is the orthogonal projection of $[0,1]^n$ onto $E^\perp$:

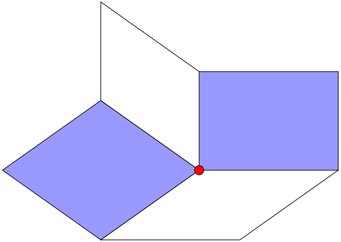
$$W := \pi'([0,1]^n).$$

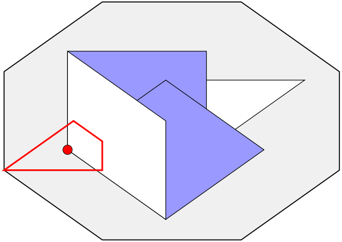
Proposition

To any pointed pattern P corresponds a subregion R of the window in which project the vertices which point this pattern in the tiling:

$$R := \bigcap_{\pi\vec{y} \in \mathcal{V}(P)} \left(W - \pi'(\vec{y} - \vec{x})\right).$$

Examples





Examples