The Penrose Tilings Revisited

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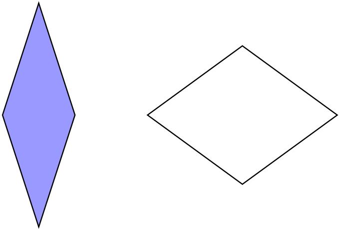
Pentagonal tilings



Thin and fat rhombi form so-called pentagonal tilings.

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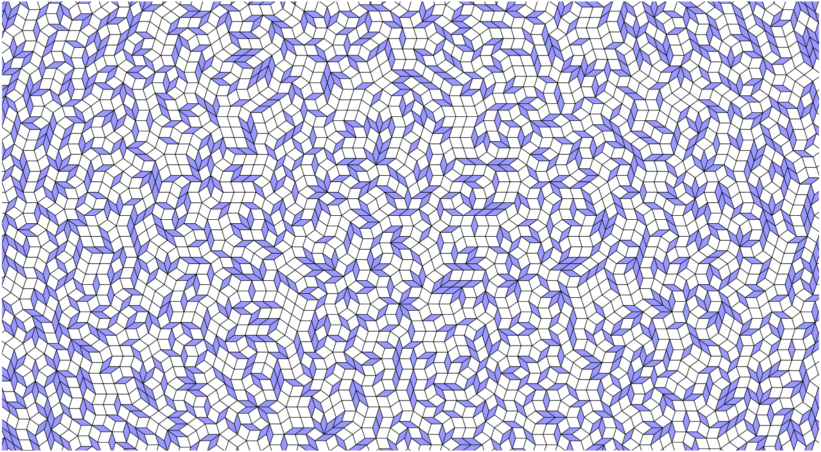

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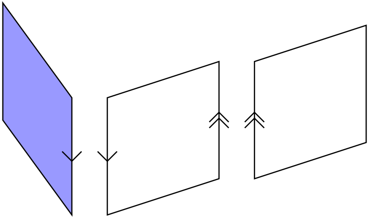

Edges can be arrowed to constrain the way tiles can match.

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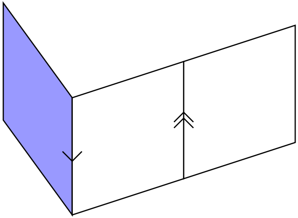

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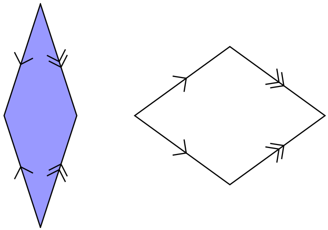
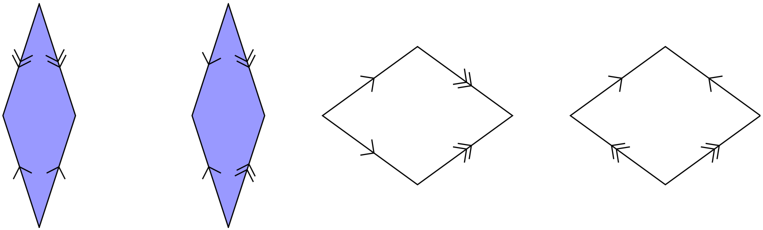


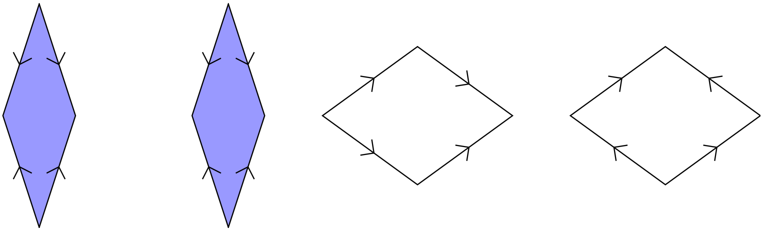
Ammann and Penrose introduced two arrowed tiles in the 70's.

Arrowed tiles



Pavlovitch and Kléman added two new arrowed tiles in 1985.

Arrowed tiles



What if we simplify arrows, reducing to a three tile set?

Arrowed tilings

Theorem (De Bruijn, 1981)
AP-tilings digitize the slope $(\varphi, 1, -1, -\varphi, \varphi, 1, -1, \varphi, 1, \varphi)$.

Theorem (Socolar, 1990)
PK-tilings digitize the slope $(\varphi, 1, -1, -\varphi, \varphi, 1, -1, \varphi, 1, \varphi)$.

Theorem
BF-tilings digitize the slopes $(x, 1, \text{-}1, \text{-}y, \frac{x+1}{y}, 1, \text{-}1, \frac{x+y+1}{xy}, 1, \frac{y+1}{x})$.

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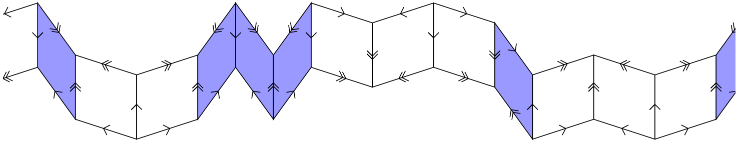
Corollary

PK-tilings are the BF-tilings of maximal thin rhombi density.

Rhombus alternance



Consider the arrows crossed by travelling a *stripe* of AP-tiles.

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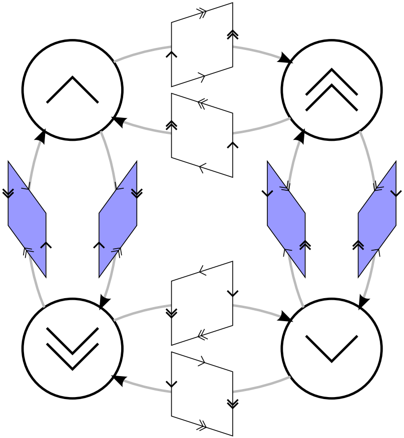

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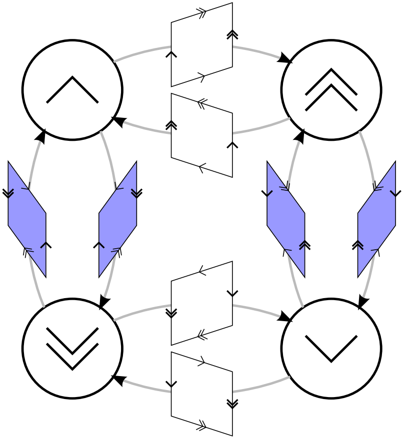

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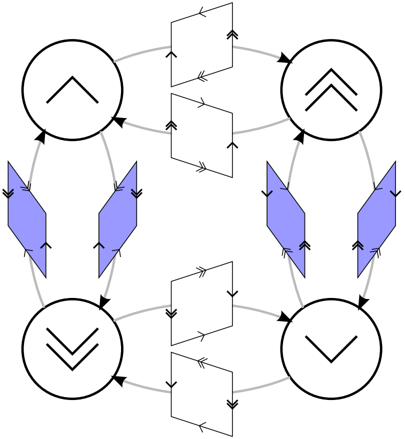
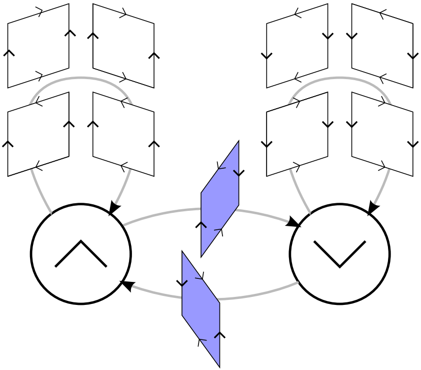
The bi-infinite sequence they form is a path of a finite automaton.
In particular, each rhombus type alternates in two orientations.

Rhombus alternance



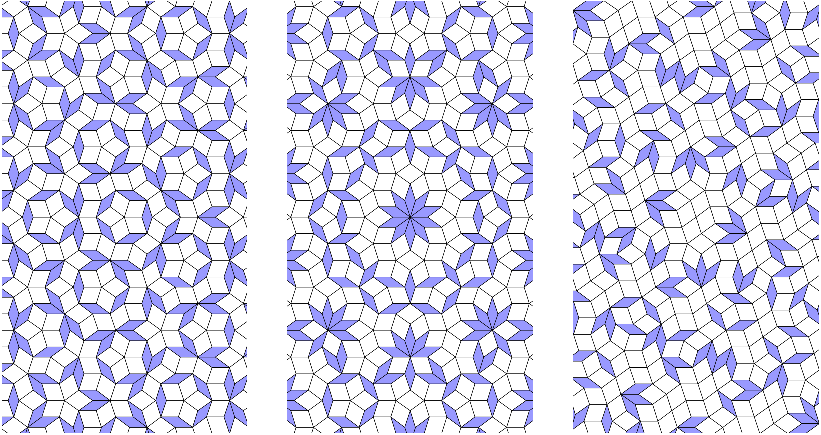
PK-tiles extend AP-tiles by allowing stripes to freely cross:
PK-tilings are exactly the tilings whose rhombi alternate.

Rhombus alternance



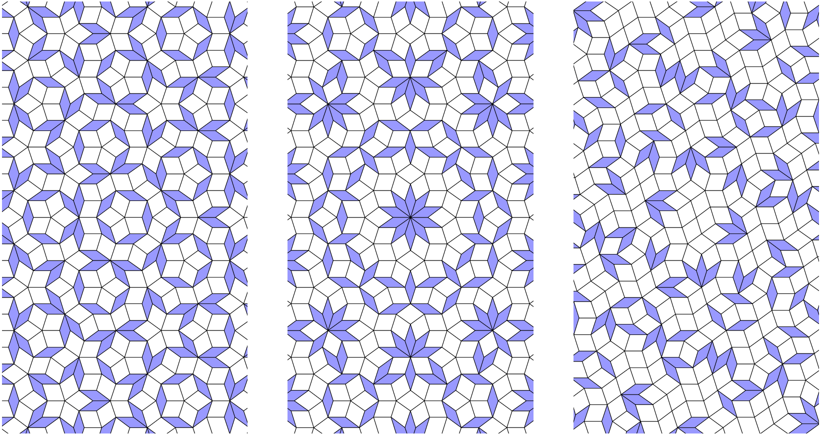
BF-tiles mimic PK-tiles but enforce only thin rhombus alternance.

Planar pentagonal tilings



Lift: homeomorphism from rhombi to 2-faces of unit cubes of $\mathbb{R}^5$.
Planar: lift in $E + [0, t]^5$, where E is the slope and t the thickness.

Shadows and subperiods



Shadow: orthogonal projection of the lift along two basis vector.

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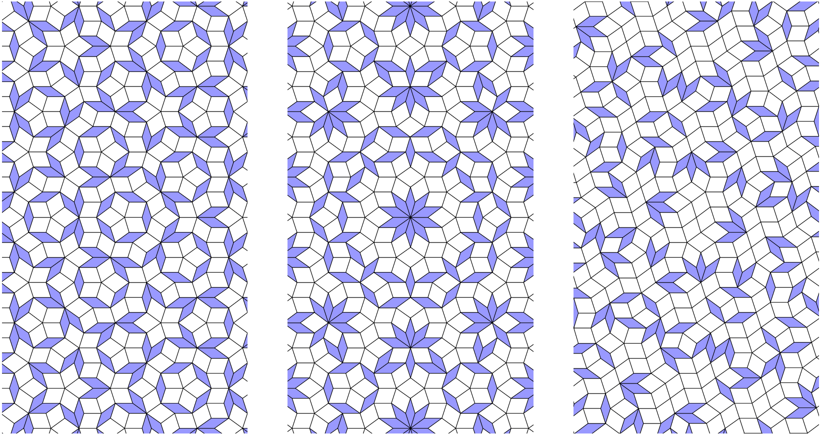

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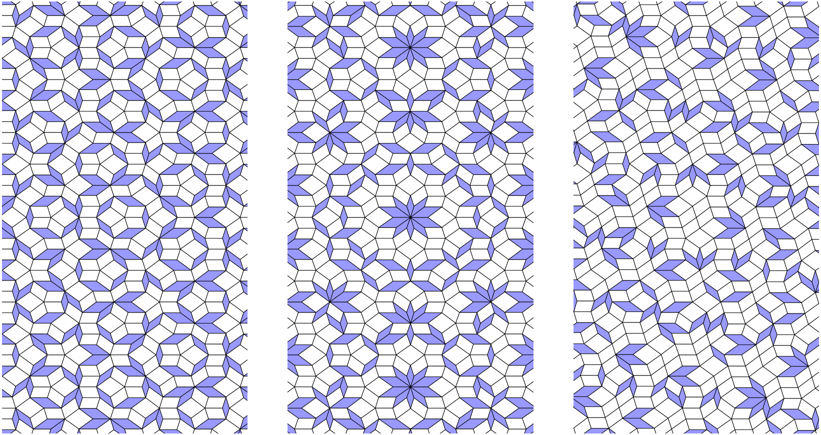

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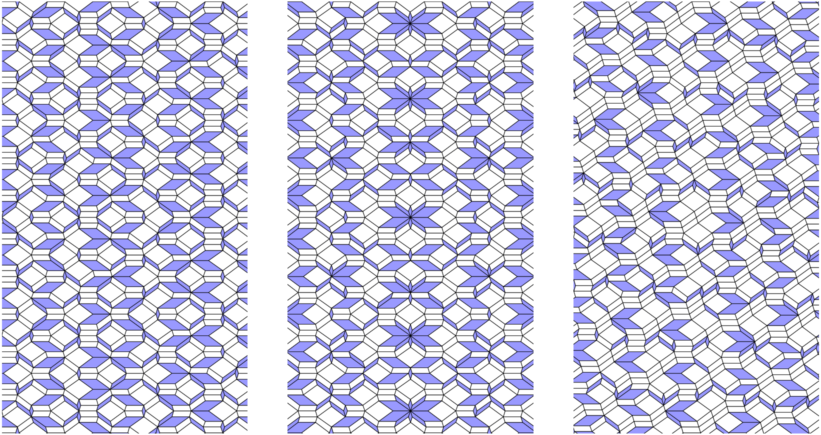

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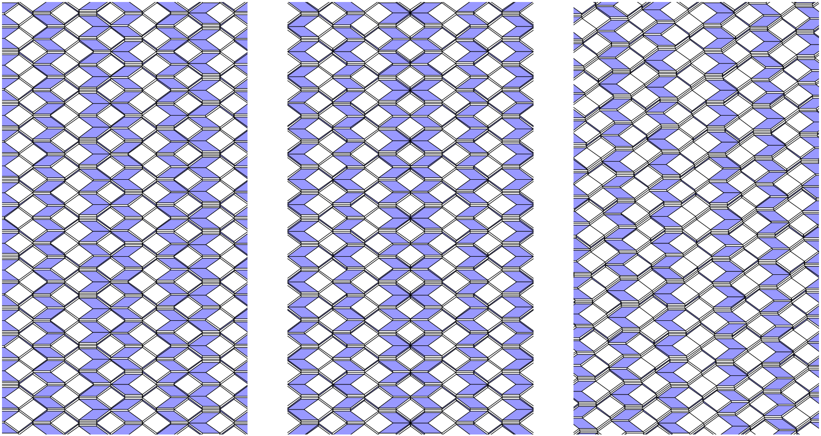

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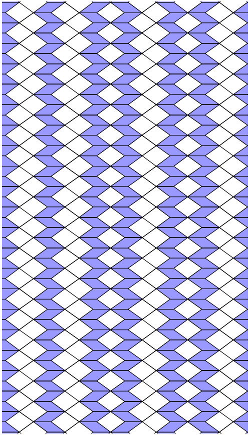 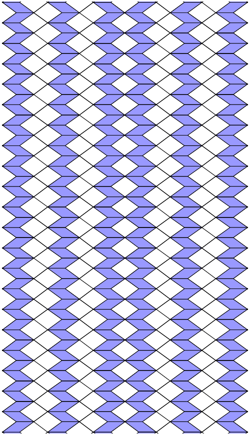 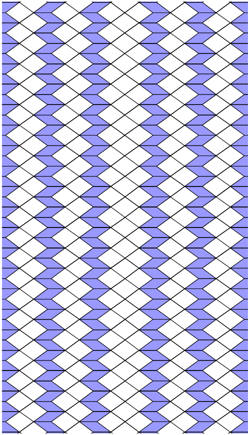

Shadow: orthogonal projection of the lift along two basis vector.
Subperiod: shadow period. Rhombus alternances force simple ones.

Grassmann-Plücker coordinates

Definition (Grassmann-Plücker)

The plane $\mathbb{R}\vec{u} + \mathbb{R}\vec{v}$ has GP-coordinates $(G_{ij})_{i<j} = (u_i v_j - u_j v_i)_{i<j}$.

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Proposition

Tile frequencies of a planar tiling are given by its GP-coordinates.

Example

AP/PK-tilings have a ratio of φ fat rhombi for 1 thin rhombus.
This is the maximal ratio that can be achieved by a BF-tiling.

Linear and quadratic relations

Proposition

Whenever a planar tiling admits $p\vec{e}_i + q\vec{e}_j + r\vec{e}_k$ as a subperiod, the GP-coordinates of its slope satisfy $pG_{jk} - qG_{ik} + rG_{ij} = 0$.

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Example

Subperiods forced by BF-tiles yield $G_{13} = G_{41} = G_{24} = G_{52} = G_{35}$.
AP/PK-tiles yield, in addition $G_{12} = G_{51} = G_{45} = G_{34} = G_{23}$.

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Example

BF-tiles yield all the slopes $(x, 1, \text{-}1, \text{-}y, \frac{x+1}{y}, 1, \text{-}1, \frac{x+y+1}{xy}, 1, \frac{y+1}{x})$, while AP/PK-tiles yield the slope $(\varphi, 1, -1, -\varphi, \varphi, 1, -1, \phi, 1, \varphi)$.

Planarity

Lemma

Subperiods forced by BF-tiles (thus AP/PK-tiles) enforce planarity.

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Proof sketch:

1. Let $\mathcal{S}$ be the lift of a BF-tiling.
2. Let $E = (\varphi, 1, -1, -\varphi, \varphi, 1, -1, \phi, 1, \varphi)$, E' its algebraic conjugate and $\vec{u} = \sum_i \vec{e}_i$. One has $\mathbb{R}^5 = E \oplus E' \oplus \mathbb{R}\vec{u}$.
3. Let $\vec{p}_1, \vec{p}_2, \vec{p}_3$ be subperiods corresp. to projections π_1, π_2, π_3, and $\vec{q}_i \in E$, $\vec{q}_i' \in E'$ s.t. $\pi_i(\vec{q}_i) = \pi_i(\vec{q}_i') = \vec{p}_i$, for $i = 1, 2, 3$.
4. $\mathcal{S} = \{\lambda\vec{q}_1 + \mu\vec{q}_2 + z_1(\lambda,\mu)\vec{q}_1' + z_2(\lambda,\mu)\vec{q}_2' + z(\lambda,\mu)\vec{u} \mid \lambda, \mu \in \mathbb{R}\}$.
5. $\pi_1(\mathcal{S})$ $\vec{p}_1$-periodic $\rightsquigarrow$ $z_2(\lambda,\mu) \simeq z_2(\mu)$ and $z(\lambda,\mu) \simeq z(\mu)$.
6. $\pi_2(\mathcal{S})$ $\vec{p}_2$-periodic $\rightsquigarrow$ $z_1(\lambda,\mu) \simeq z_1(\lambda)$ and $z(\mu) \simeq z(\lambda) \simeq cte$.
7. $\pi_3(\mathcal{S})$ $\vec{p}_3$-periodic $\rightsquigarrow$ $\varphi z_2(\mu) + z_1(\lambda) \simeq h(\varphi\mu + \lambda)$ $\rightsquigarrow$ h linear.

Remark

BF-tilings force thin rhombus alternance, which forces slopes

$$E_{x,y} = \left(x, 1, \text{-}1, \text{-}y, \frac{x+1}{y}, 1, \text{-}1, \frac{x+y+1}{xy}, 1, \frac{y+1}{x}\right).$$

But arrowed tilings form *closed* sets. What about *limit* cases?

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$$\lim_{n\to\infty} E_{\frac{1}{n},\frac{1}{n}} = (0,0,0,0,0,0,0,1,0,0).$$

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$$\lim_{n\to\infty} E_{n,n^2} = (0,0,0,1,0,0,0,0,0,0).$$

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BF-tilings with degenerated alternance are not necessarily planar!
But one can limit the number of consecutive fat rhombi in a stripe.

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What about the thickness?