Local Rules for Planar Tilings

Thomas Fernique

CNRS & Univ. Paris 13

Outline

Outline

Subshifts

Consider bi-infinite words over a finite alphabet $\mathcal{A}$.

- Subshift: the words avoiding a set $\mathcal{F}$ of *forbidden* finite words.
- Subshift of finite type (SFT): $\mathcal{F}$ can be chosen to be finite.
- Sofic subshift: letter-to-letter image of an SFT.

Subshifts

Consider bi-infinite words over a finite alphabet $\mathcal{A}$.

- Subshift: the words avoiding a set $\mathcal{F}$ of *forbidden* finite words.
- Subshift of finite type (SFT): $\mathcal{F}$ can be chosen to be finite.
- Sofic subshift: letter-to-letter image of an SFT.

Examples: the words over $\mathcal{A} = \{a, b\}$ with

- alternating a's and b's?
- at most one b?
- exactly one b?
- runs of b's all of the same length?

Subshifts

Consider bi-infinite words over a finite alphabet $\mathcal{A}$.

- Subshift: the words avoiding a set $\mathcal{F}$ of *forbidden* finite words.
- Subshift of finite type (SFT): $\mathcal{F}$ can be chosen to be finite.
- Sofic subshift: letter-to-letter image of an SFT.

Examples: the words over $\mathcal{A} = \{a, b\}$ with

- alternating a's and b's? — SFT
- at most one b? — Sofic not SFT
- exactly one b? — Not a subshift
- runs of b's all of the same length? — Subshift not sofic

Aperiodicity and quasiperiodicty

Aperiodic word: no invariance by translation.

Example: a random bi-infinite word.

Aperiodicity and quasiperiodicty

Aperiodic word: no invariance by translation.

Example: a random bi-infinite word.

Quasiperiodic word: each pattern reoccurs uniformly.

Example: the *digitization* of an irrational line of the plane.

Aperiodicity and quasiperiodicty

Aperiodic word: no invariance by translation.

Example: a random bi-infinite word.

Quasiperiodic word: each pattern reoccurs uniformly.

Example: the *digitization* of an irrational line of the plane.

Claim: there is no sofic subshift containing only aperiodic words.

Higher dimensions

Bi-infinite word over two letters $\sim$ tiling of the line by two intervals.

Higher dimensions

Bi-infinite word over two letters $\sim$ tiling of the line by two intervals.

Higher dimensional generalization:

letter	$\leftrightarrow$	tile
bi-infinite word	$\leftrightarrow$	tiling of $\mathbb{R}^d$
forbidden word	$\leftrightarrow$	forbidden pattern
subshift	$\leftrightarrow$	tiling space

Finite type, sofic, aperiodicity, quasiperiodicity etc. easily extend.

Higher dimensions

Bi-infinite word over two letters $\sim$ tiling of the line by two intervals.

Higher dimensional generalization:

letter	$\leftrightarrow$	tile
bi-infinite word	$\leftrightarrow$	tiling of $\mathbb{R}^d$
forbidden word	$\leftrightarrow$	forbidden pattern
subshift	$\leftrightarrow$	tiling space

Finite type, sofic, aperiodicity, quasiperiodicity etc. easily extend.

Theorem (Berger 1964)

There is a 2-dim. sofic tiling space with only aperiodic tilings.

Higher dimensions

Bi-infinite word over two letters $\sim$ tiling of the line by two intervals.

Higher dimensional generalization:

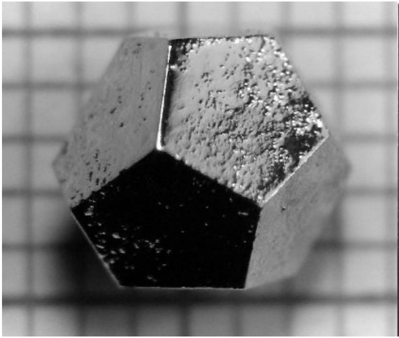
letter	$\leftrightarrow$	tile
bi-infinite word	$\leftrightarrow$	tiling of $\mathbb{R}^d$
forbidden word	$\leftrightarrow$	forbidden pattern
subshift	$\leftrightarrow$	tiling space

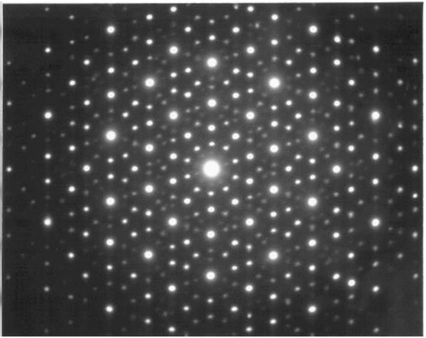
Finite type, sofic, aperiodicity, quasiperiodicity etc. easily extend.

Theorem (Berger 1964)

There is a 2-dim. sofic tiling space with only quasiperiodic tilings.

Motivation





Model *quasicrystals* by sofic quasiperiodic tilings.

Outline

Planar $n \to d$ tilings

Vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^d \rightsquigarrow$ tiles $T_{i_1,\ldots,i_d} := \{\sum \lambda_{i_k} \vec{v}_{i_k} \mid \lambda_{i_k} \in [0,1]\}$.

Definition ($n \to d$ tiling)

A $n \to d$ *tiling* is a "face-to-face" tiling of $\mathbb{R}^d$ by such tiles.

It naturally *lifts* onto a d-dim. "stepped surface" of $\mathbb{R}^n$ via $\vec{v}_i \mapsto \vec{e}_i$.

Planar $n \to d$ tilings

Vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^d \rightsquigarrow$ tiles $T_{i_1,\ldots,i_d} := \{\sum \lambda_{i_k} \vec{v}_{i_k} \mid \lambda_{i_k} \in [0,1]\}$.

Definition ($n \to d$ tiling)

A $n \to d$ *tiling* is a "face-to-face" tiling of $\mathbb{R}^d$ by such tiles.

It naturally *lifts* onto a d-dim. "stepped surface" of $\mathbb{R}^n$ via $\vec{v}_i \mapsto \vec{e}_i$.

Definition (Planar tiling)

A $n \to d$ tiling is said to be *planar* if it lifts into $E + [0,t]^n$, where

- E is a d-plane of $\mathbb{R}^n$ called the *slope*;
- $t \geq 1$ is chosen to be minimal and called the *thickness*.

Examples

A rhombille tiling in Saint-Étienne de Marmoutier (Alsace).

Examples

Michael Baake's place & chocolates: Ammann-Beenker tilings.

Examples

A homemade Penrose tiling (oak & pinewood).

Grassmann coordinates

Definition (Grassmann coordinates)

The *Grassmann (projective) coordinates* of a d-plane of $\mathbb{R}^n$ are the $d \times d$ minors $G_{i_1 \dots i_d}$ of a matrix whose columns generate this plane.

Grassmann coordinates

Definition (Grassmann coordinates)

The *Grassmann (projective) coordinates* of a d-plane of $\mathbb{R}^n$ are the $d \times d$ minors $G_{i_1 \dots i_d}$ of a matrix whose columns generate this plane.

They are the non-zero $\binom{n}{d}$-tuples satisfying the *Plücker relations*

$$G_{i_1,\dots,i_d} G_{j_1,\dots,j_d} = \sum_{1 \le p \le d} \underbrace{G_{i_1,\dots,i_d} G_{j_1,\dots,j_d}}_{\text{swap } i_p \text{ and } j_q}.$$

Grassmann coordinates

Definition (Grassmann coordinates)

The *Grassmann (projective) coordinates* of a d-plane of $\mathbb{R}^n$ are the $d \times d$ minors $G_{i_1 \dots i_d}$ of a matrix whose columns generate this plane.

They are the non-zero $\binom{n}{d}$-tuples satisfying the *Plücker relations*

$$G_{i_1,\dots,i_d} G_{j_1,\dots,j_d} = \sum_{1 \leq p \leq d} \underbrace{G_{i_1,\dots,i_d} G_{j_1,\dots,j_d}}_{\text{swap } i_p \text{ and } j_q}.$$

The frequency of $T_{i_1 \dots i_d}$ in a planar tiling is proportional to $|G_{i_1 \dots i_d}|$.

Local rules

Definition (Local rules)

A d-plane E of $\mathbb{R}^n$ admits *local rules* if there is $t \geq 1$ and a finite set of patterns s. t. the set of $n \to d$ tilings with no such pattern

- contains at least one planar tiling of slope E and thickness 1;
- contains only planar tilings of slope E and thickness at most t.

Local rules

Definition (Local rules)

A d-plane E of $\mathbb{R}^n$ admits *local rules* if there is $t \geq 1$ and a finite set of patterns s. t. the set of $n \to d$ tilings with no such pattern

- contains at least one planar tiling of slope E and thickness 1;
- contains only planar tilings of slope E and thickness at most t.

Local rules are *strong* if $t = 1$, *weak* otherwise (Levitov 1988).

Local rules

Definition (Local rules)

A d-plane E of $\mathbb{R}^n$ admits *local rules* if there is $t \geq 1$ and a finite set of patterns s. t. the set of $n \to d$ tilings with no such pattern

- contains at least one planar tiling of slope E and thickness 1;
- contains only planar tilings of slope E and thickness at most t.

Local rules are *strong* if $t = 1$, *weak* otherwise (Levitov 1988).

Local rules are *colored* if tiles come in (finitely many) different colors.

Outline

Algebraic obstruction

Theorem (Le 1995)

If a slope admits weak uncolored local rules, then it is algebraic.

Algebraic obstruction

Theorem (Le 1995)

If a slope admits weak uncolored local rules, then it is algebraic.

Theorem (Bédaride-F. 2017)

Effective characterization in the $4 \to 2$ *case.*

Algebraic obstruction

Theorem (Le 1995)

If a slope admits weak uncolored local rules, then it is algebraic.

Theorem (Bédaride-F. 2017)

Effective characterization in the $4 \to 2$ *case.*

Effective characterization in the $n \to d$ case:

- achieved under *planarity* and *genericity* assumptions;
- work in progress otherwise (N-folds, relaxed Penrose...).

Window and patterns

Definition (Window)

Window of a planar tiling of slope E: orth. proj. of $[0,1]^n$ on $E^{\perp}$.

Window and patterns

Definition (Window)

Window of a planar tiling of slope E: orth. proj. of $[0,1]^n$ on $E^\perp$.

Provide a complementary insight on planar tilings:

Tiling		Window
vertex	$\leftrightarrow$	projected integer point

Window and patterns

Definition (Window)

Window of a planar tiling of slope E: orth. proj. of $[0,1]^n$ on $E^\perp$.

Provide a complementary insight on planar tilings:

Tiling		Window
vertex	$\leftrightarrow$	projected integer point
all the vertices	$\leftrightarrow$	dense points (*genericity*)

Window and patterns

Definition (Window)

Window of a planar tiling of slope E: orth. proj. of $[0,1]^n$ on $E^\perp$.

Provide a complementary insight on planar tilings:

Tiling		Window
vertex	$\leftrightarrow$	projected integer point
all the vertices	$\leftrightarrow$	dense points (*genericity*)
pointed pattern	$\leftrightarrow$	convex subregion

Window and patterns

Definition (Window)

Window of a planar tiling of slope E: orth. proj. of $[0,1]^n$ on $E^\perp$.

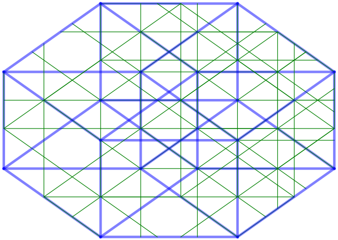
Provide a complementary insight on planar tilings:

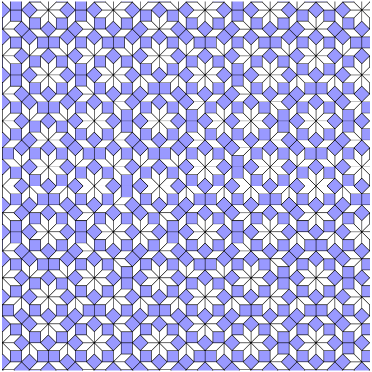
Tiling		Window
vertex	$\leftrightarrow$	projected integer point
all the vertices	$\leftrightarrow$	dense points (*genericity*)
pointed pattern	$\leftrightarrow$	convex subregion
pointed tiling	$\leftrightarrow$	point

Coincidences and slope

Definition (Coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.

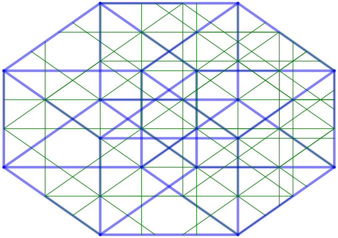


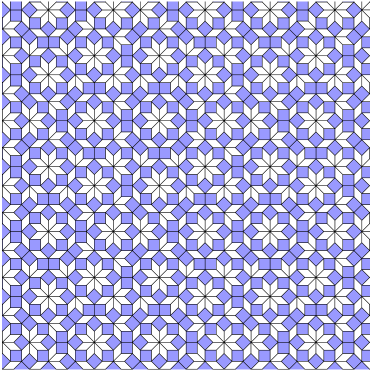


Coincidences and slope

Definition (Coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.

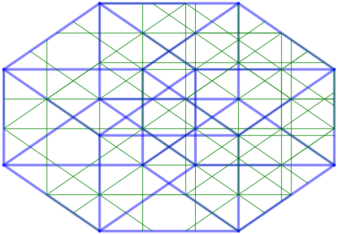


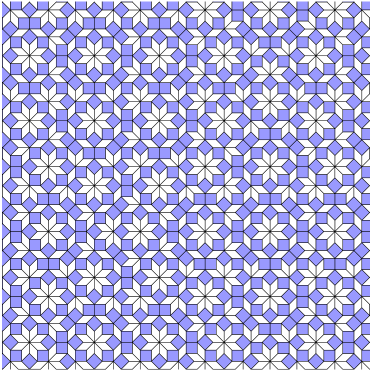


Coincidences and slope

Definition (Coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.

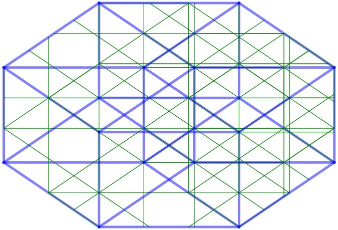


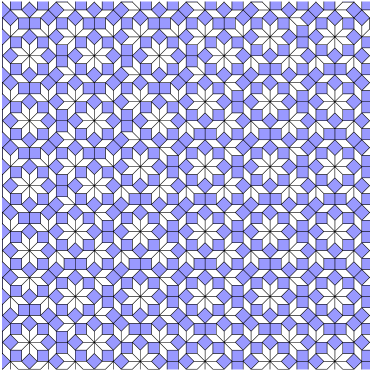


Coincidences and slope

Definition (Coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.

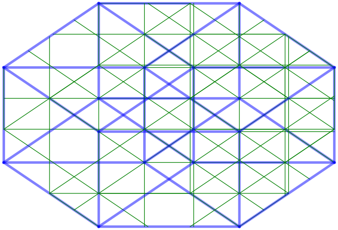


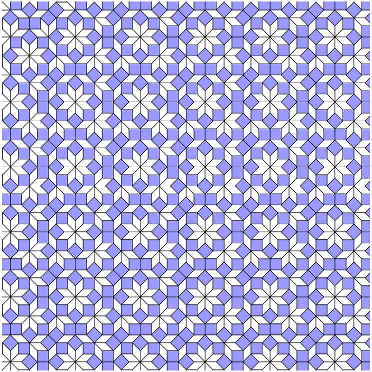


Coincidences and slope

Definition (Coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.

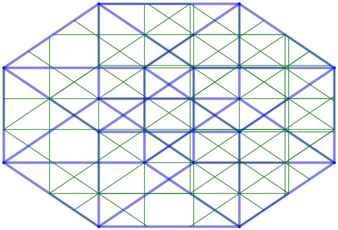


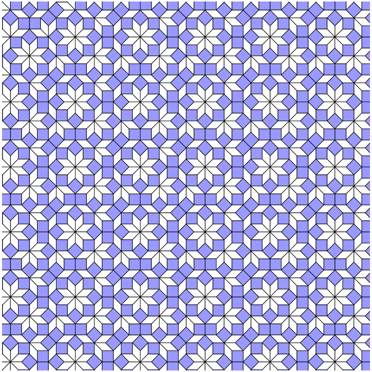


Coincidences and slope

Definition (Coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.



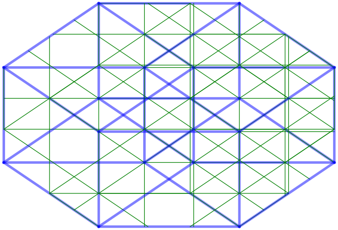


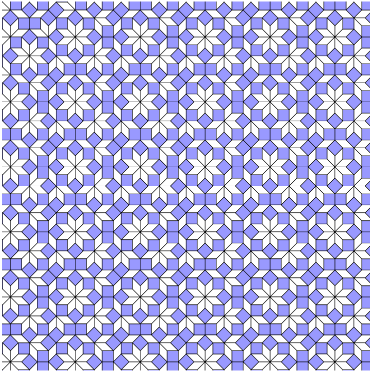
A slope with uncolored local rules is characterized by coincidences.

Coincidences and slope

Definition (Coincidence)

A *coincidence* of a planar $n \to d$ tiling, this is $n-d+1$ unit faces of $\mathbb{Z}^n$ of dim. $n-d-1$ with a common intersection in the window.





A coincidence $\leftrightarrow$ an algebraic equation on Grassmann coordinates.

Subperiods and planarity

Definition (Subperiod)

A *subperiod* of a d-plane is a vector with $d+1$ integer entries.

- Corresponds to a coincidence where two parallel faces overlap.

Subperiods and planarity

Definition (Subperiod)

A *subperiod* of a d-plane is a vector with $d+1$ integer entries.

- Corresponds to a coincidence where two parallel faces overlap.
- Suitably projected in $\mathbb{R}^{d+1}$: period of planar $d+1 \to d$ tiling.

Subperiods and planarity

Definition (Subperiod)

A *subperiod* of a d-plane is a vector with $d+1$ integer entries.

- Corresponds to a coincidence where two parallel faces overlap.
- Suitably projected in $\mathbb{R}^{d+1}$: period of planar $d+1 \to d$ tiling.
- Integer points are dense in a line parallel to a window's facet.

Subperiods and planarity

Definition (Subperiod)

A *subperiod* of a d-plane is a vector with $d+1$ integer entries.

- Corresponds to a coincidence where two parallel faces overlap.
- Suitably projected in $\mathbb{R}^{d+1}$: period of planar $d+1 \to d$ tiling.
- Integer points are dense in a line parallel to a window's facet.
- Line of points exits or enters the window $\leftrightarrow$ "worm" of flips.

Subperiods and planarity

Definition (Subperiod)

A *subperiod* of a d-plane is a vector with $d+1$ integer entries.

- Corresponds to a coincidence where two parallel faces overlap.
- Suitably projected in $\mathbb{R}^{d+1}$: period of planar $d+1 \to d$ tiling.
- Integer points are dense in a line parallel to a window's facet.
- Line of points exits or enters the window $\leftrightarrow$ "worm" of flips.
- Forcing planarity by forbidden patterns requires subperiods...

Outline

Computability obstruction ($\leftrightarrow$)

Theorem (F.-Sablik, 2012–2017)

A slope admits weak colored local rules iff it is computable.

Computability obstruction ($\leftrightarrow$)

Theorem (F.-Sablik, 2012–2017)

A slope admits weak colored local rules iff it is computable.

Includes algebraic slopes (*i.e.*, all the previously known cases).

Computability obstruction ($\leftrightarrow$)

Theorem (F.-Sablik, 2012–2017)

A slope admits weak colored local rules iff it is computable.

Includes algebraic slopes (*i.e.*, all the previously known cases).

Holds for *effectively closed* sets of slopes (*e.g.*, *all* the slopes).

Trial and error ($\rightarrow$)

Consider colored tiles and a set of forbidden colored patterns.

Trial and error ($\rightarrow$)

Consider colored tiles and a set of forbidden colored patterns.

Valid pattern of radius r $\rightarrow$ *candidate* slope within precision ε.

Trial and error ($\rightarrow$)

Consider colored tiles and a set of forbidden colored patterns.

Valid pattern of radius r $\rightarrow$ *candidate* slope within precision ε.

Wrong candidate $\leftrightarrow$ pattern which cannot be indefinitely extended.

Trial and error ($\rightarrow$)

Consider colored tiles and a set of forbidden colored patterns.

Valid pattern of radius r $\rightarrow$ *candidate* slope within precision ε.

Wrong candidate $\leftrightarrow$ pattern which cannot be indefinitely extended.

Algorithm to compute the slope within precision ε:

1. adjust r to have precision $\varepsilon/2$;
2. form all the valid patterns of radius r;
3. try to extend each pattern indefinitely (in parallel);
4. stop when the remaining candidates all agree.

Subaction ($\leftarrow$)

Theorem (Aubrun-Sablik 2013)

Any 1*-dim.* effective *subshift of can be obtained as the lines of a* 2*-dim. sofic subshift (*i.e., *a tiling by Wang tiles).*

Effective: a Turing machine enumerates the forbidden words.

Subaction ($\leftarrow$)

Theorem (Aubrun-Sablik 2013)

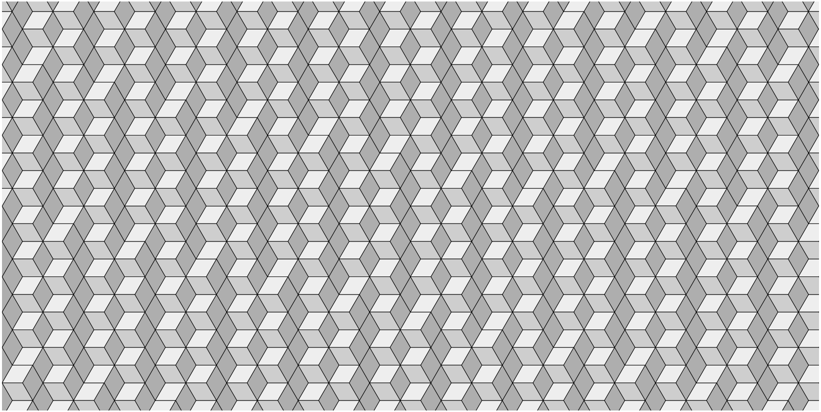
Any 1*-dim.* effective *subshift of can be obtained as the lines of a* 2*-dim. sofic subshift (*i.e., *a tiling by Wang tiles).*

Effective: a Turing machine enumerates the forbidden words.

Idea of the proof:

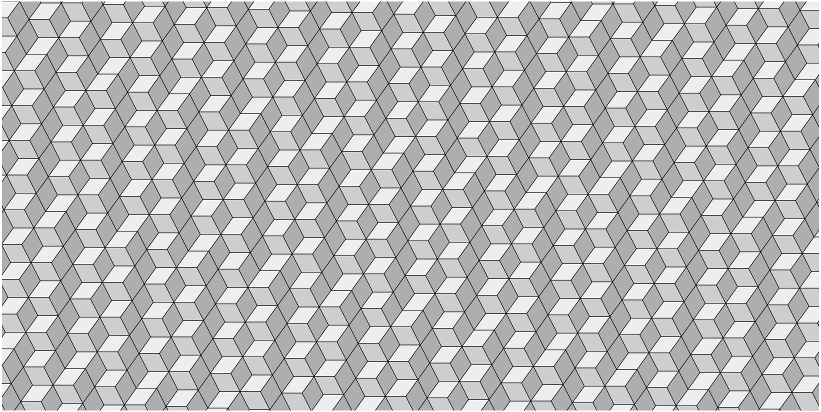
- repeat the same infinite word on every line;
- run Turing machines which enumerate forbidden words;
- do it everywhere and synchronize this!

Quasisturmian words (←)



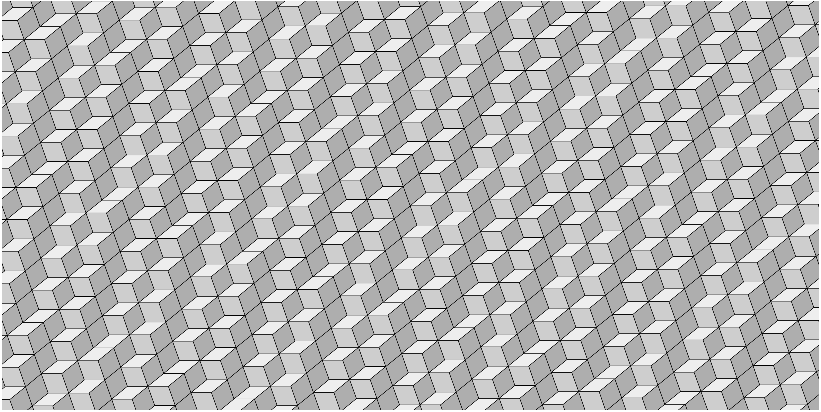
A planar tiling can be seen as several intertwined 2-dim. subshifts.

Quasisturmian words ($\leftarrow$)



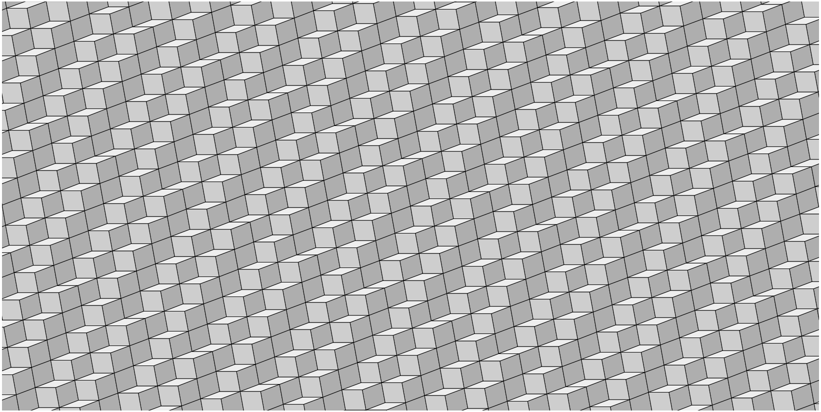
A planar tiling can be seen as several intertwined 2-dim. subshifts.

Quasisturmian words ($\leftarrow$)



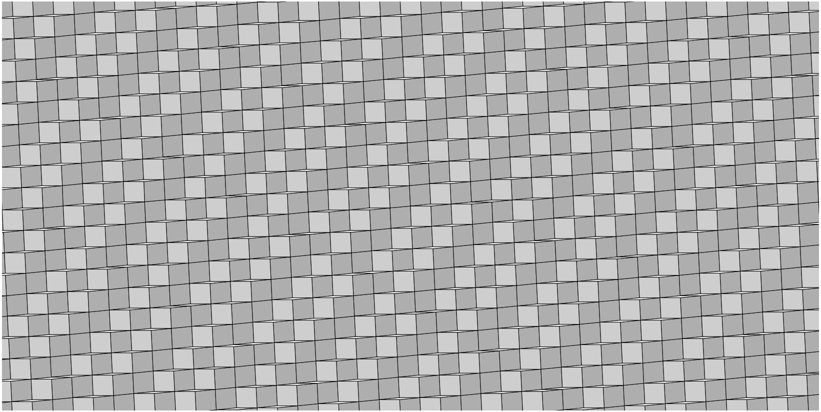
A planar tiling can be seen as several intertwined 2-dim. subshifts.

Quasisturmian words ($\leftarrow$)



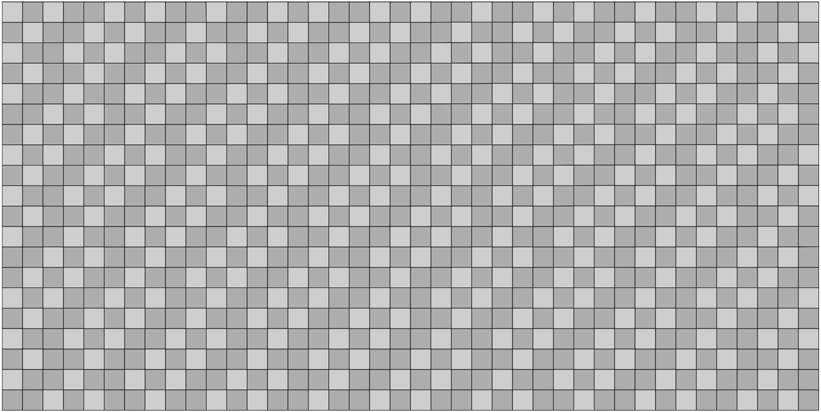
A planar tiling can be seen as several intertwined 2-dim. subshifts.

Quasisturmian words ($\leftarrow$)



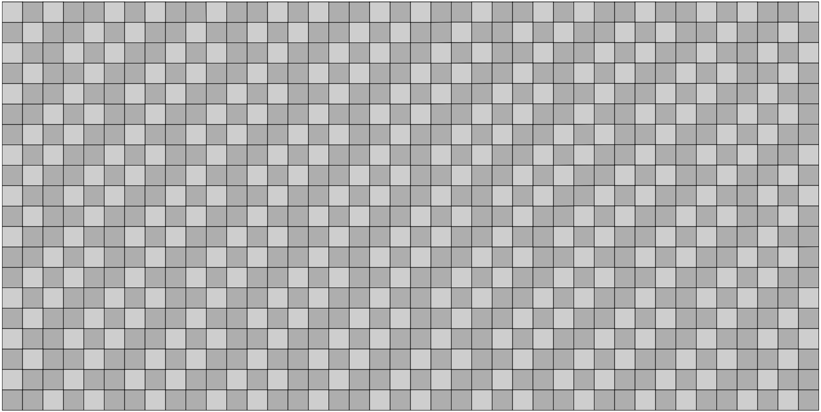
A planar tiling can be seen as several intertwined 2-dim. subshifts.

Quasisturmian words ($\leftarrow$)



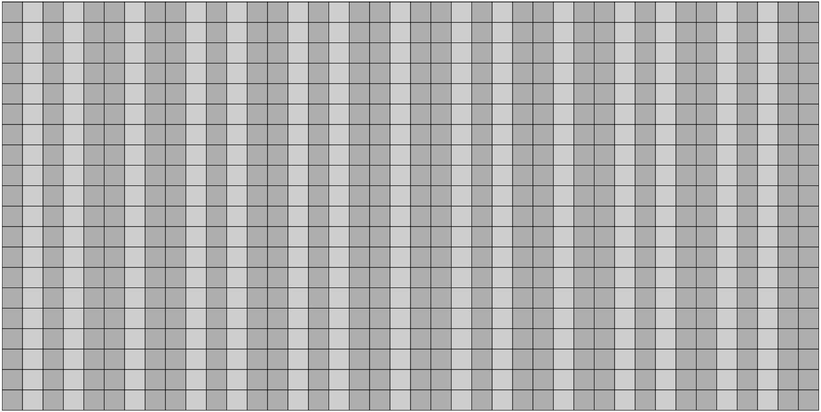
A planar tiling can be seen as several intertwined 2-dim. subshifts.

Quasisturmian words ($\leftarrow$)



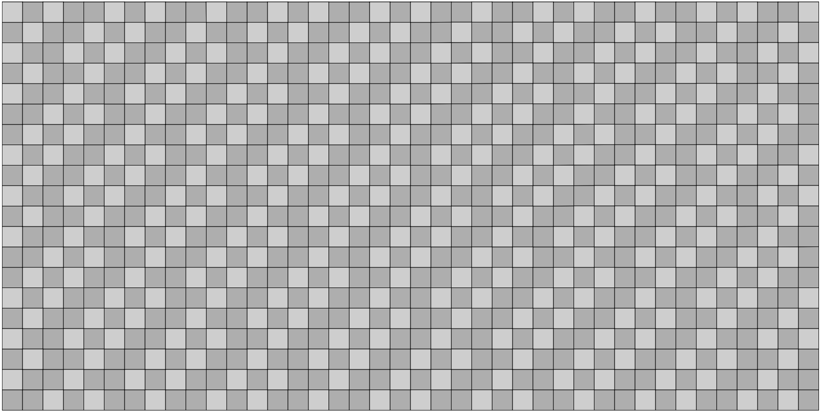
Lines are Sturmian words, but is the 2-dim. subshift sofic?

Quasisturmian words ($\leftarrow$)



The subaction theorem ensures that the one with equal lines does.

Quasisturmian words ($\leftarrow$)



Bounded fluctuations $\Rightarrow$ sofic subshift containing the original one.

References (1/2)

L. S. Levitov, *Local rules for quasicrystals*, Comm. Math. Phys. **119** (1988)

T. Q. T. Le, *Local rules for quasiperiodic tilings*, NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci. **489** (1995)

N. Aubrun and M. Sablik, *Simulation of effective subshifts by two-dim. subshifts of finite type*, Acta Appl. Math. **126** (2013).

References (2/2)

With Nicolas Bédaride:

- *When periodicities enforce aperiodicity*, Comm. Math. Phys. **335** (2015)
- *No weak local rules for the 4p-fold tilings*, Disc. Comput. Geom. **54** (2015)
- *Weak local rules for octagonal tilings*, Israel J. Math. **222** (2017)

With Mathieu Sablik:

- *Weak colored local rules for planar tilings*, arXiv:1603.09485.