Sturmian Sequences

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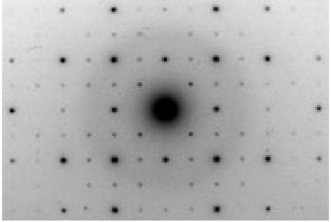
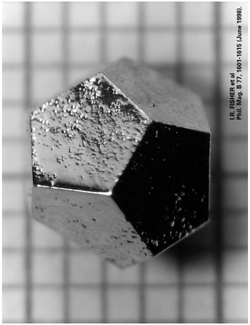
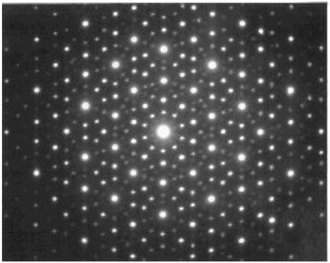
1. Aperiodic order
2. Sturmian sequences
   - Cut and project
   - Substitutions
   - Continued fractions
3. Minimal complexity
   - 1D sequences
   - nD sequences

Periodicity and aperiodicity

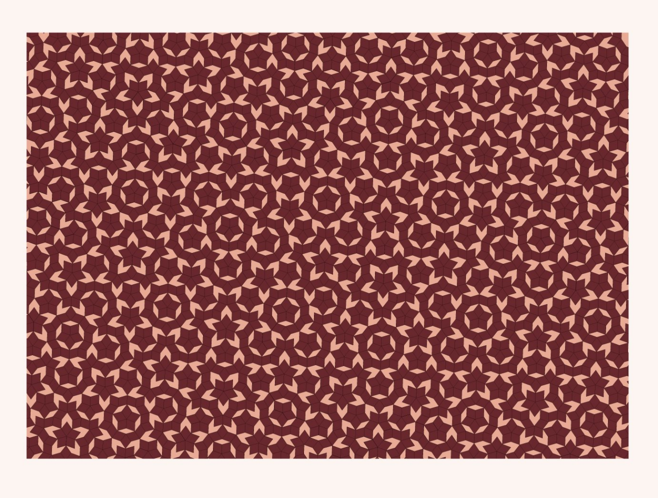


Quasicrystals (Dan Shechtman, 1982)





Not so new (Penrose, 1972, Bohr, 1925)



function of quasiperiodicity/recurrence

Definition

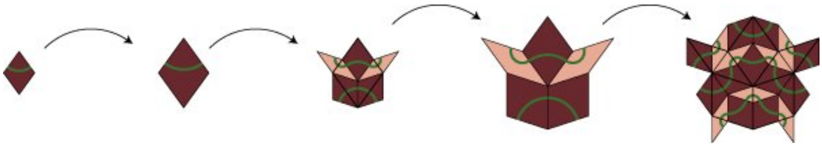
Set $A \rightsquigarrow f_A : \mathbb{R}^+ \to \mathbb{R}^+$ s.t. :

$$\exists x \in A \mid P \subset A \cap B(x,r) \quad \Rightarrow \quad \forall y \in A, \quad P \subset A \cap B(y, f_A(r)).$$

Generation : substitutions

Tiling : partition into a finite number (up to translation/rotation) of polygonal tiles.

Substitution : inflation of tiles + dissection into new tiles.



Cut & Project (Meyer, 1972, Katz et Duneau, 1985)

- directive space D, projective space P s.t. $\mathbb{R}^d = P \times D$;
- window $\Omega \subset P$ (bounded open set) ;
- strip $S(\Omega) = D \times \Omega$;
- projection π on P along D,

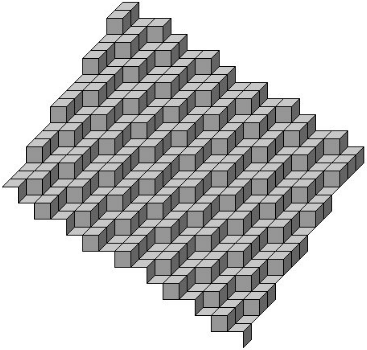
$\rightsquigarrow$ set $\pi(\mathbb{Z}^d \cap S(\Omega))$ or tiling projecting the faces of $\mathbb{Z}^d \cap S(\Omega))$.

Penrose : $d = 5$, $\dim(P) = 2$.

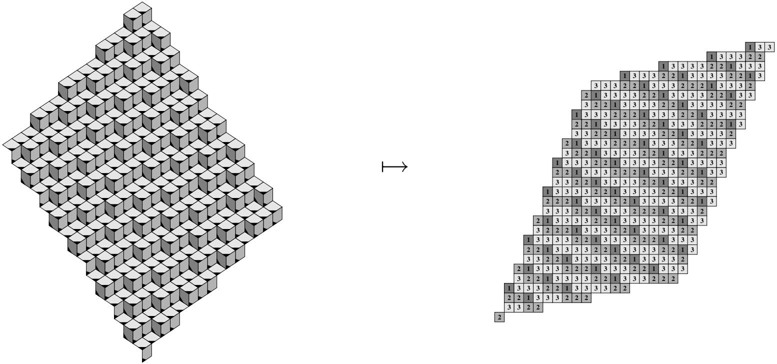
Cut & Project in codimension 1 : *stepped plane*

$\vec{\alpha} \in \mathbb{R}_+^n \rightsquigarrow \mathcal{S}_{\vec{\alpha}}$.

Entries of $\vec{\alpha}$ linearly independent over $\mathbb{Q}$: Sturmian plane.



Stepped plane $\rightsquigarrow$ tiling $\rightsquigarrow$ sequence



In dimension 1 : classic Sturmian sequences.

Substitution : morphism σ of $\mathcal{A}^*$ s.t. $|\sigma^n(i)| \to \infty$ for $i \in \mathcal{A}$.

$\sigma : 1 \mapsto 12, 2 \mapsto 1$:

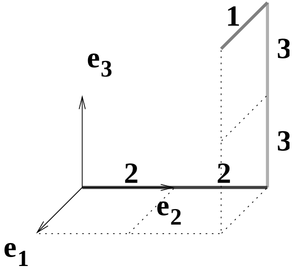
$$1 \to 12 \to 121 \to 12112 \to 12112121 \to \ldots$$

σ extended to $\mathcal{A}^\omega \rightsquigarrow$ fixed-point : $u \in \mathcal{A}^\omega \mid u = \sigma(u)$.

$$\lim_{n\to\infty} \sigma^n(1) = 12112121121121\cdots = u_\alpha = \sigma(u_\alpha).$$

$\mathcal{A} = \{1, 2, 3\}$ and $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ canonical basis of $\mathbb{R}^3$.

$u \in \mathcal{A}^* \rightsquigarrow$ broken line of segments $[\vec{x}, \vec{x} + \vec{e}_i] = (\vec{x}, i)$, $\vec{x} \in \mathbb{N}^3$:

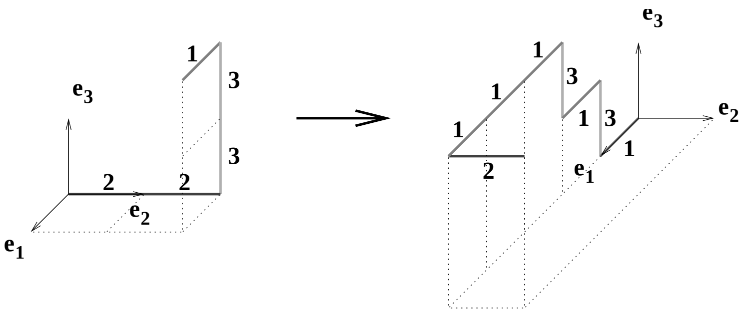


σ on $\mathcal{A} \rightsquigarrow$ linear map $\Theta(\sigma)$ on segments :

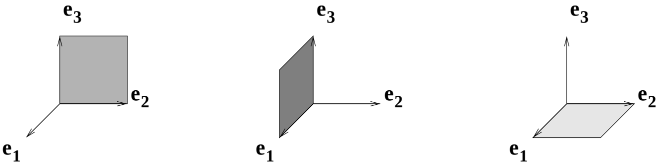
$$\Theta(\sigma) \ : \ (\vec{x}, i) \mapsto M_\sigma \vec{x} + \sum_{p|\sigma(i)=p\cdot j\cdot s} (\vec{f}(p), j),$$

where $(M_\sigma)_{i,j} = |\sigma(j)|_i$ and $\vec{f}(u) = (|u|_1, \ldots, |u|_n)$.

$$\sigma : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 13 \\ 3 \mapsto 1 \end{cases}, \quad M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma(22331) = 13131112.$$



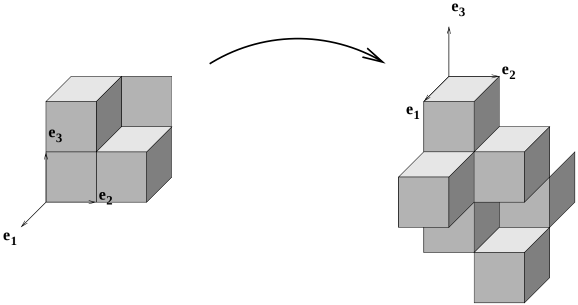
Segment $(\vec{x}, i) \rightsquigarrow$ dual face $(\vec{x}, i^*)$:



If $\det(M_\sigma) = \pm 1$: linear map $\Theta(\sigma) \rightsquigarrow$ dual map $\Theta^*(\sigma)$:

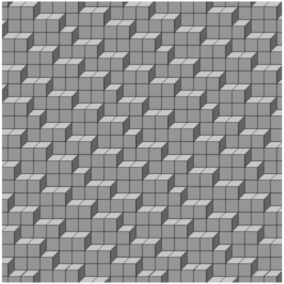
$$\Theta^*(\sigma)(\vec{x}, i^*) = M_\sigma^{-1}\vec{x} + \sum_{j \in \mathcal{A}} \sum_{s | \sigma(j) = p \cdot i \cdot s} (\vec{f}(s), j^*).$$

$\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$:

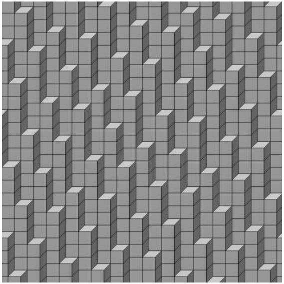


Theorem

$$\Theta^*(\sigma)(\mathcal{S}_{\vec{\alpha}}) = \mathcal{S}_{{}^t M_\sigma \vec{\alpha}}.$$



$\longmapsto$



$$M_\sigma^{-1}\mathcal{P}_{\alpha,\beta} = \mathcal{P}_{\alpha',\beta'} \rightsquigarrow \Theta^*(\sigma) \text{ “discretization” of } M_\sigma^{-1}.$$

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A necessary condition to be substitutive

Proposition

If $\mathcal{S}_{\vec{\alpha}}$ is a fixed point of a substitution $\Theta^*(\sigma)$, then the entries of $\vec{\alpha}$ belong to $\mathbb{Q}(\lambda)$, where λ is an algebraic number of d° $\leq n$.

$$\begin{aligned} \Theta^*(\sigma)(\mathcal{S}_{\vec{\alpha}}) = \mathcal{S}_{\vec{\alpha}} \quad &\Rightarrow \quad \mathcal{S}_{{}^tM_\sigma\vec{\alpha}} = \mathcal{S}_{\vec{\alpha}} \\ &\Rightarrow \quad {}^tM_\sigma\vec{\alpha} = \lambda\vec{\alpha}. \end{aligned}$$

A sufficient condition to be substitutive

If there exists unimodular matrices M_i s.t. :

$$\vec{\alpha} = \lambda M_1 \times M_2 \times \ldots \times M_p \vec{\alpha},$$

then $\mathcal{S}_{\vec{\alpha}}$ is substitutive.

The case $n = 2$: continued fractions

Gauss map : $T : \alpha \mapsto \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor = \frac{1}{\alpha} - a$.
Matrix viewpoint :

$$\alpha \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ T(\alpha) \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}.$$

So, if $(T^n(\alpha))_n$ is periodic, then $\mathcal{S}_{(1,\alpha)}$ is substitutif.

Lagrange : α quadratic has a periodic expansion $(T^n(\alpha))_n$. Thus :

Theorem

$\mathcal{S}_{(1,\alpha)}$ substitutif iff α has a periodic continued fraction expansion.

The case $n > 2$: Brun's expansion

$\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in [0,1)^n \backslash \{0\}$:

$$T(\alpha_1, \ldots, \alpha_n) = \left(\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_n}{\alpha_i}\right),$$

where $\alpha_i = \max \alpha_j$.

$$a(\vec{\alpha}) = \left\lfloor \frac{1}{\max \alpha_j} \right\rfloor \qquad \text{and} \qquad \varepsilon(\vec{\alpha}) = \min\{i \mid \alpha_i = \max_j \alpha_j\}.$$

$\rightsquigarrow$ Brun expansion $(a_n, \varepsilon_n) = (a(T^n(\vec{\alpha})), \varepsilon(T^n(\vec{\alpha})))$

Matrix viewpoint : $\alpha_\varepsilon A_{a,\varepsilon}{}^t(1, T(\vec{\alpha})) = {}^t(1, \vec{\alpha})$, where :

$$A_{a,\varepsilon} = \begin{pmatrix} a & & 1 & \\ & I_{\varepsilon-1} & & \\ 1 & & 0 & \\ & & & I_{n-\varepsilon} \end{pmatrix},$$

Theorem

If $\vec{\alpha}$ has a periodic Brun expansion, then $\mathcal{S}_{{}^t(1,\vec{\alpha})}$ is substitutif.

Lagrange ?

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For $u = (u_i)_{i \in \mathbb{Z}}$, $p_u(n)$: number of different factor of length n in u.

Theorem

p_u bounded iff $\exists n \mid p_u(n) \leq n$ iff u is periodic.

$\forall n,\ p_u(n) = n + 1$: u aperiodic sequence of minimal complexity.

Theorem

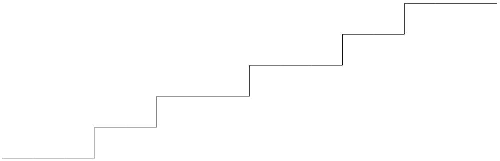
Let u s.t. $\forall n,\ p_u(n) = n + 1$. Three cases :

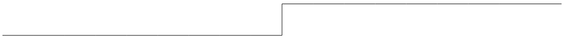
1. *u is sturmian,*
2. $\exists \sigma \in < l, r, e > \ |\ u = \sigma({}^{\omega}010^{\omega})$,
3. $\exists \sigma \in < l, r, e > \ |\ u = \sigma({}^{\omega}01^{\omega})$,

where :

$$l : \left\{ \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 01 \end{array} \right. , \qquad r : \left\{ \begin{array}{l} 0 \mapsto 0 \\ 1 \mapsto 10 \end{array} \right. , \qquad e : \left\{ \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array} \right. .$$

Geometric viewpoint





nD sequence : $u = (u_{i_1,\ldots,i_n})_{i_j \in \mathbb{Z}}$.

$A \subset \mathbb{Z}^n \rightsquigarrow A$-complexity : $p_u(A)$ denotes the number of different factors of shape A in u.

rectangular complexity : $A = \{(i,j) \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$. One writes $p_u(m,n)$.

Question : u periodic iff $p_u(A) \leq ?$

nD Sequence u : r-periodic, $0 \leq r \leq n$.

proposition

u n-periodic iff $p_u(m_1,\ldots,m_n)$ bounded.

Limit 1-periodicity/aperiodicity ?

Conjecture (Nivat, 1997)

For a 2D sequence u :

$$\exists(m_0, n_0) \mid p_u(m_0, n_0) \leq m_0 n_0 \quad \Rightarrow \quad u \text{ periodic.}$$

- sequences of complexity $mn + 1$: characterized 2D sequences over $\{0, 1\}$;
- proved for $p_u(m_0, n_0) \leq \frac{1}{16} m_0 n_0$;
- converse is false (rectangle = bad shape ?) ;
- extensions for $d > 2$ do not hold.

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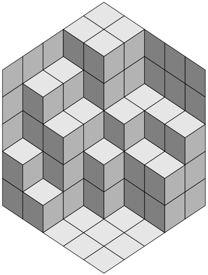
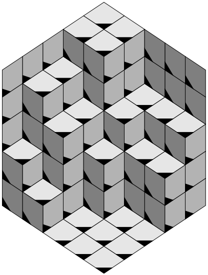
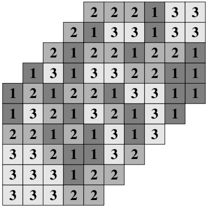
Geometric viewpoint

sequence 1D over $\{0,1\} \rightsquigarrow$ "curve" of the plane.
sequence 2D over $\{0,1\} \rightsquigarrow$?

"stepped surface" can be encoded by 2D sequences over $\{1,2,3\}$







Stepped planes are stepped surfaces. So :

Conjecture

Sturmian planes = aperiodic nD sequences of minimal complexity *among* stepped surfaces ?

Proposition

If this conjecture holds, then the Nivat conjecture also holds.

Converse ?

Proposition

If u is a 2D sequence which encodes a sturmian stepped plane, then $p_u(m,n) = mn + m + n$.

Characterization of the 2D sequence of complexity $mn + m + n$?

Notice that $\Theta^*(\sigma)$ acts on stepped surfaces as well.