Combinatorial Substitutions and Sofic Tilings

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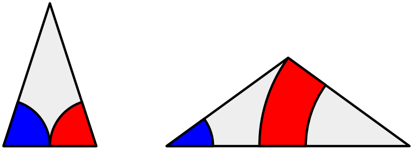
1 Introductory example

2 General case

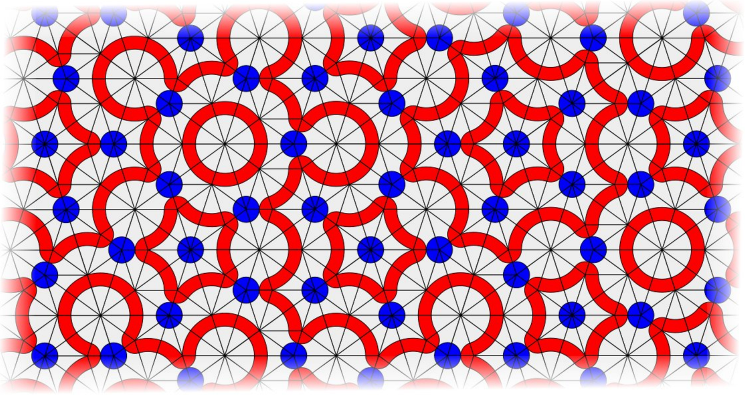
3 Constructive proof

Penrose tiles



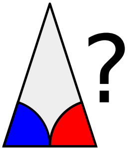
Tiles: thin and fat decorated triangles (up to isometries of $\mathbb{R}^2$).

Penrose tiles



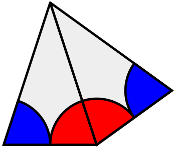
Tiling: covering of $\mathbb{R}^2$ without overlap, with matching decorations.

Macro-tiles



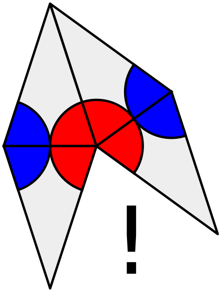
Consider a thin tile in a tiling: what is its *red neighbor*?

Macro-tiles



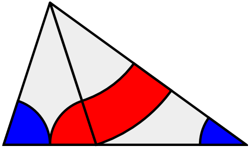
Does a thin tile suits?

Macro-tiles



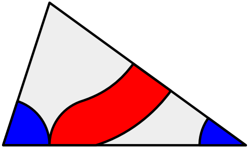
No, because symmetric tiles yield an uncompletable vertex.

Macro-tiles



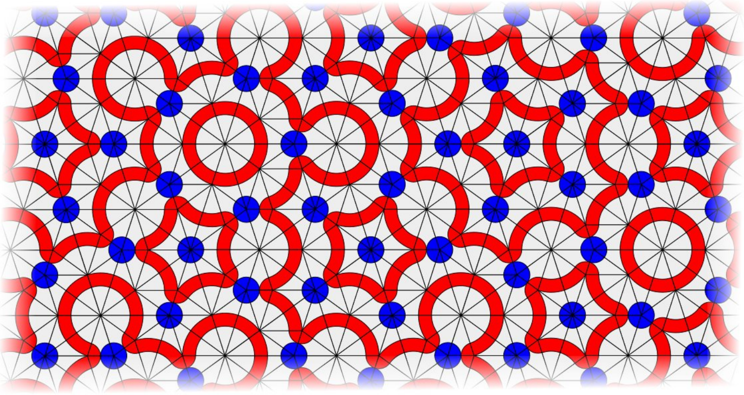
The red neighbor of a thin tile is thus always a fat tile.

Macro-tiles



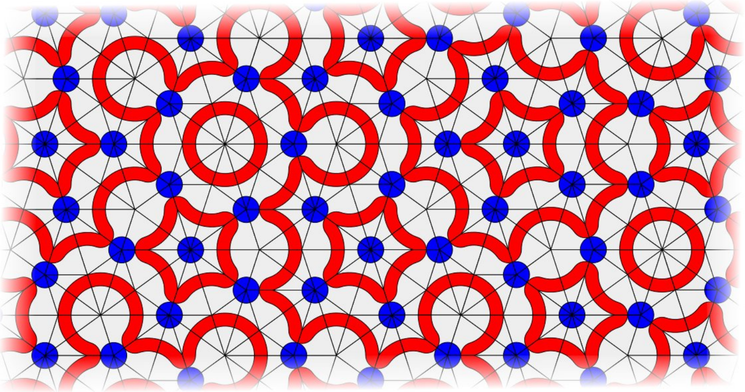
We group them into a *thin macro-tile*.

Macro-tiles



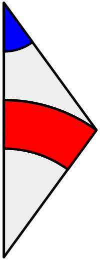
Hence, any tiling by fat and thin tiles. . .

Macro-tiles



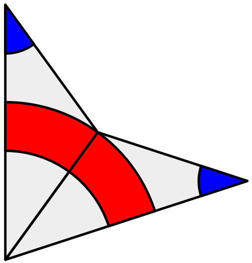
. . . can be (uniquely) seen as a tiling by fat tiles and thin macro-tiles.

Macro-tiles



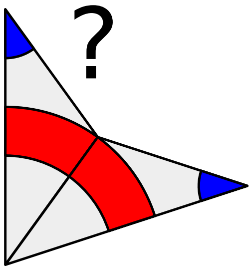
Then, consider a free (*i.e.*, ungrouped) fat tile.

Macro-tiles



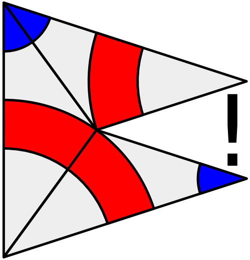
Since it has not been grouped, its red neighbor is fat.

Macro-tiles



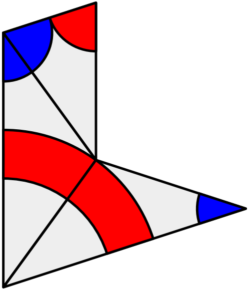
What is its *blue neighbor*?

Macro-tiles



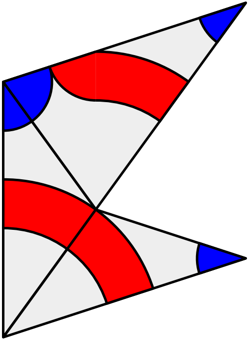
A fat tile would yield an uncompletable vertex.

Macro-tiles



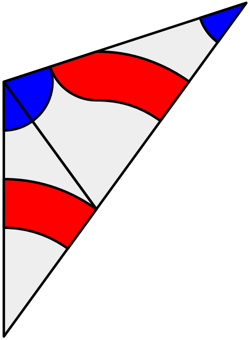
We thus have a thin tile. . .

Macro-tiles



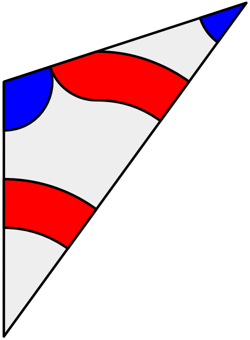
...formerly grouped into a thin macro-tile.

Macro-tiles



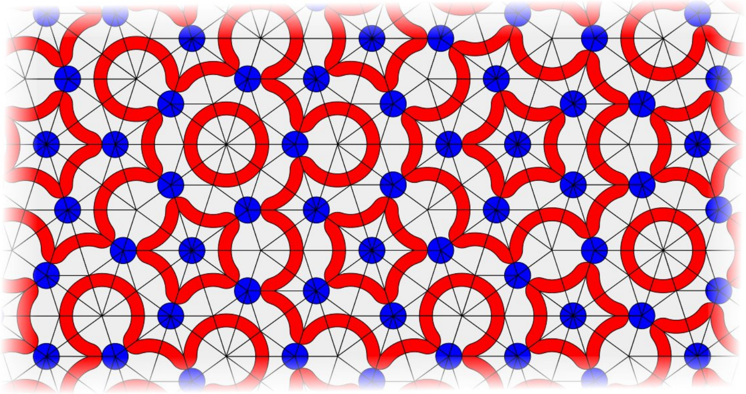
The blue neighbor of a free fat tile is thus always a thin macro-tile.

Macro-tiles



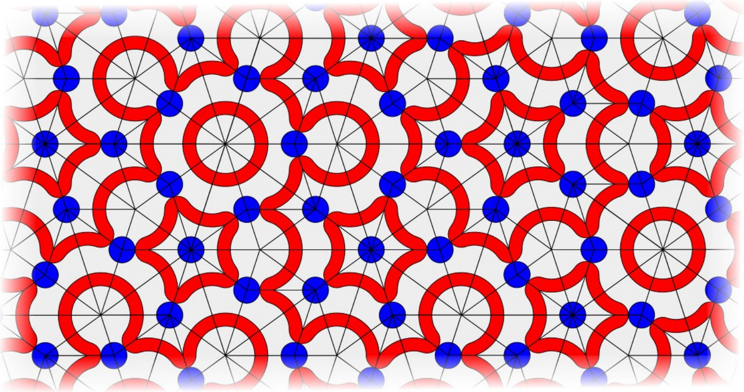
We group them into a *fat macro-tile*.

Macro-tiles



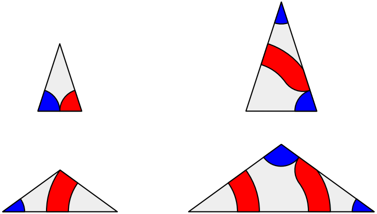
Hence, any tiling by fat tiles and thin macro-tiles. . .

Macro-tiles



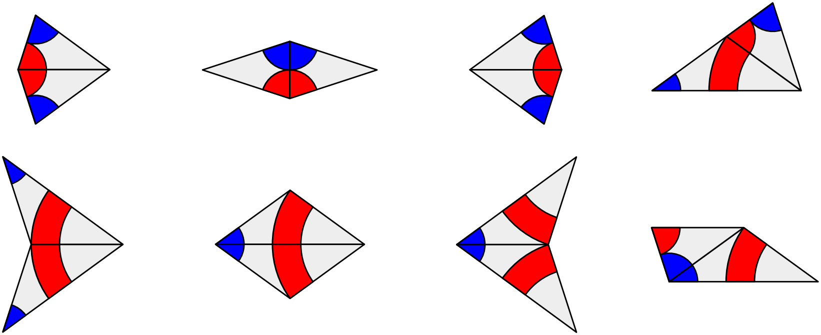
...can be (uniquely) seen as a tiling by fat and thin macro-tiles.

Self-simulation



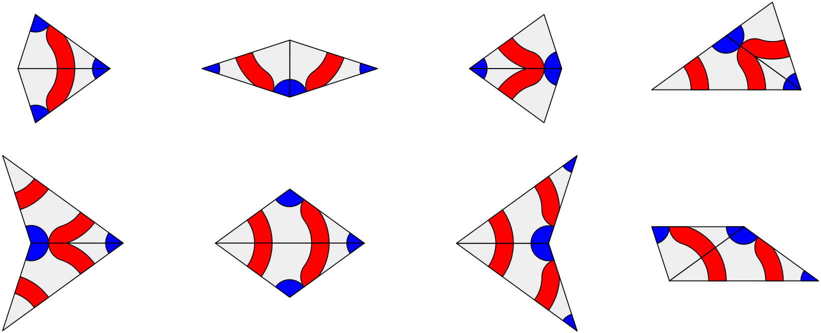
The trick: tiles and macro-tiles have the same combinatorics!

Self-simulation



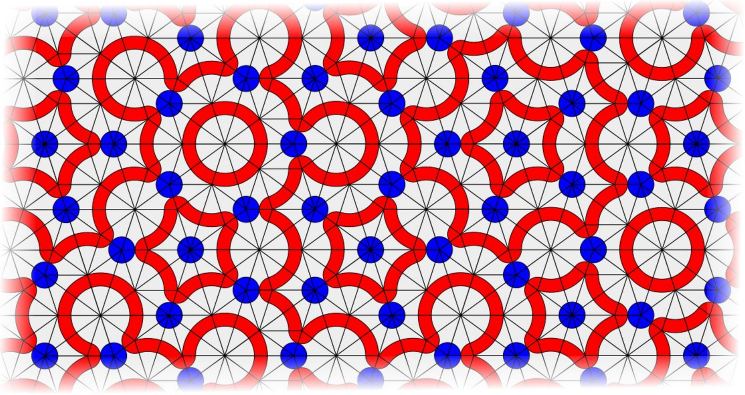
Two tiles match along an edge iff...

Self-simulation



the corresponding macro-tiles match along the corresponding edge.

Limit set



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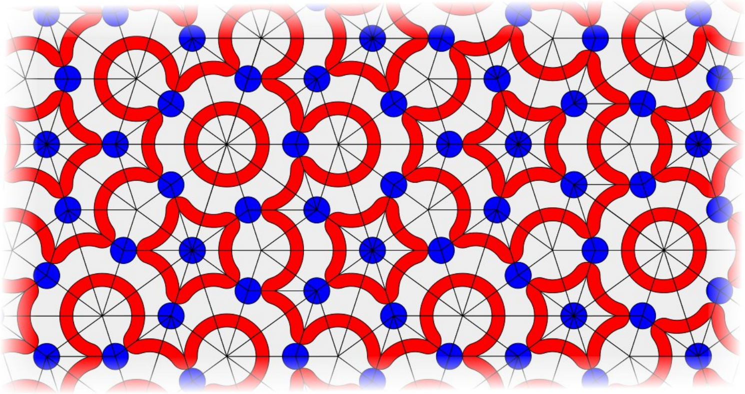
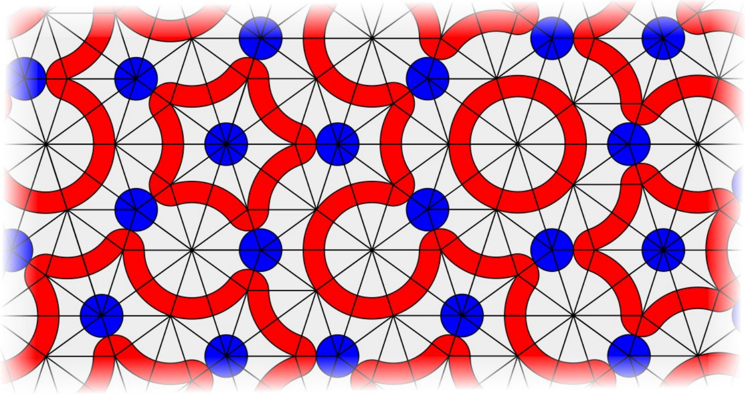


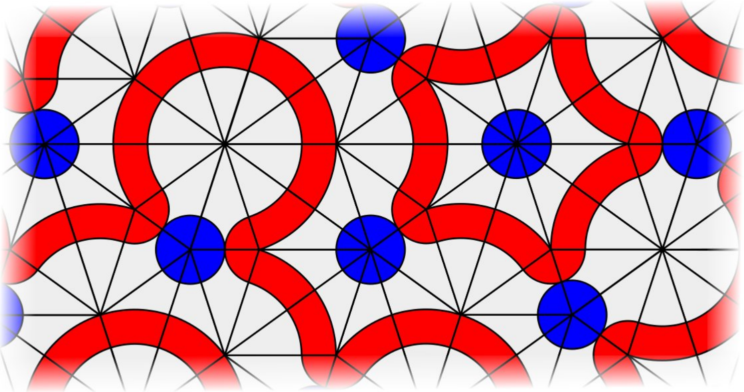
Hence, any tiling can be (uniquely) seen as a *macro-tiling*. . .

Limit set



...which can be transformed back into a tiling.

Limit set

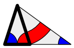


This can be indefinitely iterated (notably preventing periodicity).

Penrose tilings

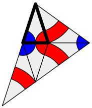
Conversely, consider a single tile, say thin.

Penrose tilings



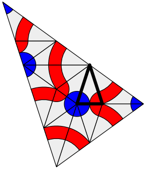
We can group it into a thin macro-tile. . .

Penrose tilings



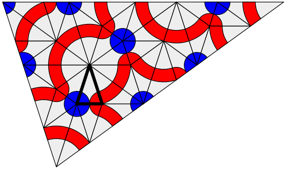
. . . which can be grouped into a thin macro-macro-tile, and so on.

Penrose tilings



This yields a spiral-growing increasing sequence of partial tilings.

Penrose tilings



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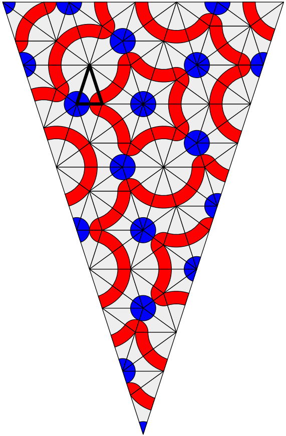

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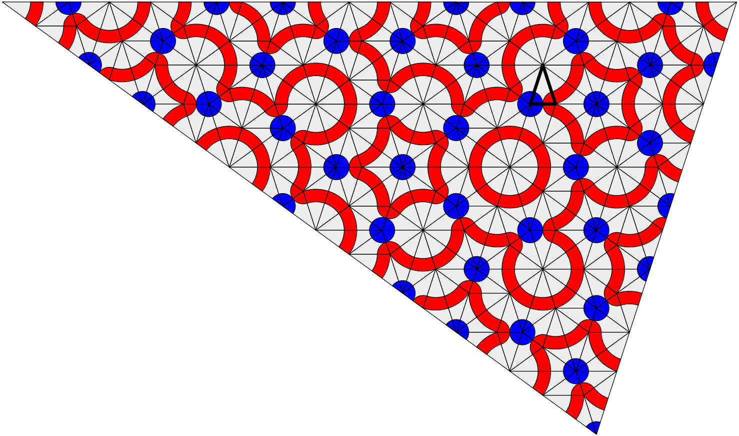

The union of which defines (up to multiplicity) a complete tiling.

A generic result

We here provide a general way to find such tricky tiles.

Formally, we constructively prove:

Theorem (Fernique & Ollinger, 2010)

The limit set of a good combinatorial substitution is sofic.

Extends and (hopefully) simplifies previous similar results:

- Shahar Mozes (1990);
- Chaim Goodman-Strauss (1998).

Sofic tilings

Decorated tiles and tilings:

- A decorated tile is a polytope of $\mathbb{R}^d$ with finitely many *facets*, on which is defined a real map, called *decoration*;
- Two decorated tiles match if their intersection (if any) is made of entire faces on which decorations are equal;
- A decorated tiling is a covering of $D \subset \mathbb{R}^d$ by pairwise matching decorated tiles. It is complete if $D = \mathbb{R}^d$;
- Decorated tile set τ (+direct isometries) $\rightsquigarrow$ decorated tilings Λ_τ.

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Tiles and tilings: by removing decorations (map π).

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Tiles and tilings: by removing decorations (map π).

Definition (Sofic tiling)

A set of tilings is sofic if it is the image under π of some set Λ_τ, with τ being finite (up to direct isometries).

Combinatorial substitutions

We are inspired by Natalie Priebe-Frank:

Definition (Combinatorial substitution)

A combinatorial substitution is a finite set of rules (P, Q, γ), where P is a tile, Q is a finite connected tiling, and $\gamma : \partial P \to \partial Q$ maps distinct facets on disjoint sets of facets.

The tiling Q is a macro-tile.
If f is the k-th facet of P, then $\gamma(f)$ is the k-th macro-facet of Q.

Combinatorial substitutions

Two macro-tiles match if their intersection contains only entire macro-facets; a macro-tiling is a tiling whose tiles can be partitioned into matching macro-tiles.

Given a combinatorial substitution $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$:

Definition (Preimage)

A tiling T by P_i's is the σ-preimage of a macro-tiling T' by Q_i's if there is a one-to-one correspondence between the tiles of T and the macro-tiles of T' which *preserves the combinatorial structure*.

Definition (Limit set)

The limit set of σ, denoted by Λ_σ, is the set of complete tilings which admit an infinite sequence of σ-preimages.

Good combinatorial substitutions

A good combinatorial substitution is *consistent* and *connecting*:

Definition (Consistency)

A combinatorial substitution $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$ is consistent if any complete macro-tiling by Q_i's admits a σ-preimage.

Definition (Connectivity)

A combinatorial substitution $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$ is connecting if the dual graph of any Q_i has a subgraph N_i, called its network, s.t.

1. N_i is a star connecting macro-facets through ports;
2. removing N_i-edges yields a graph connecting macro-facets;
3. the center of N_i, called central tile, is in the interior of Q_i;

Self-simulation

Let $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$ be a combinatorial substitution.

Definition (Self-simulation)

A decorated tile set τ self-simulates for σ if there are τ-tilings, called τ-*macro-tiles*, and a map ϕ from those to τ, such that

1. $\forall$ τ-macro-tile $\mathcal{Q}$, $\exists i$ t.q. $\pi(\mathcal{Q}) = Q_i$ and $\pi(\phi(\mathcal{Q})) = P_i$;
2. any complete τ-tiling is also a τ-macro-tiling;
3. any τ-macro-tile $\mathcal{Q}$ is *combinatorially equivalent* to $\phi(\mathcal{Q})$.

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Proposition

If σ is consistent and τ self-simulates for σ, then $\pi(\Lambda_\tau) \subset \Lambda_\sigma$.

1. Introductory example

2. General case

3. Constructive proof

Settings

Let $\sigma = \{(P_i, Q_i, \gamma_i)_i\}$ be a good combinatorial substitution.

Tiles and internal facets of the Q_i's: $(T_i)_{1 \leq i \leq n}$ and $(f_j)_{1 \leq j \leq m}$.

$N_\sigma(i, k)$ denotes the index of the k-th facet of T_i if it is internal, or a special value "port", "macro-facet" or "boundary" otherwise.

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τ-tiles will be decorated T_i's. On each facet will be encoded:

- a facet-index (or special value), called macro-index;
- a tile-index (or 0), called parent-index;
- a facet-index (or special value), called neighbor-index.

Clearly, τ is necessarily finite (but possibly huge).

Steps 1–3 of 5

We define τ step by step by allowing/forbidding indices on facets.

Step 1 specifies all the macro-indices:

1. macro-index of the k-th facet of a decorated T_j: $N_\sigma(j, k)$.

Steps 1–3 of 5

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Step 1 specifies all the macro-indices:

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Steps 2–3 specify non-port facets crossed by no network:

2. parent-index of such a facet in a Q_i: 0 if external, any j with $T_j = P_i$ otherwise, with uniform indices within each τ-tile;
3. neighbor-index of such a facet in a k-th macro-facet: $N_\sigma(j,k)$, where j is the parent-index within the corresponding τ-tile; neighbor-index of other such facets: copy of the macro-index.

Steps 4–5 of 5

Steps 4–5 specify parent/neighbor pairs on networks and ports:

4. on the k-th branch (strictly) and k-th port of a macro-tile: any pair not forbidden on the k-th facet of its parent-tile, with uniform pairs within each τ-tile;

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5. on the facets of a central τ-tile $\mathcal{T}$: copy all the pairs of any non-central decorated tile which has as many facets as $\mathcal{T}$ (one says that $\mathcal{T}$ *derives* from this non-central decorated tile).

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This completly defines the decorated tile set τ.

First inclusion: $\pi(\Lambda_\tau) \subset \Lambda_\sigma$

Let $\mathcal{Q}$ be a τ-macro-tile with parent-index j and central τ-tile $\mathcal{T}'$. We define $\phi(\mathcal{Q})$ as the tile T_j endowed with the decorations of $\mathcal{T}'$.

This easily yields that τ self-simulates for σ, whence $\pi(\Lambda_\tau) \subset \Lambda_\sigma$.

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Provided that $\phi(\mathcal{Q}) \in \tau$ holds!

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Let $\mathcal{T}$ be the non-central τ-tile from which derives $\mathcal{T}'$ (step 5):

- if T_j is central, then $\phi(\mathcal{Q})$ also derives from $\mathcal{T}$;
- otherwise, neighbor-indices of $\mathcal{T}$-facets crossed by no network are copies of macro-indices allowed only on a decorated T_j. This yields $\pi(\mathcal{T}) = T_j$, and thus $\phi(\mathcal{Q}) = \mathcal{T}$.

Converse inclusion: $\Lambda_\sigma \subset \pi(\Lambda_\tau)$

Let $P_0 \in \Lambda_\sigma$ and, for any $n \geq 1$, P_n be a preimage of P_{n-1}.

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Let τ' extend τ by allowing a decoration "undefined" on facets.
We fix n and inductively define τ'-tiling $(\mathcal{P}_k)_{n \geq k \geq 0}$ as follows:

- $\mathcal{P}_n$: endow uniformly P_n with "undefined" decorations;
- $\mathcal{P}_{k-1}$: endow each macro-tile of P_{k-1} as in steps 1–3 and copy decorations of the corresponding τ'-tile of $\mathcal{P}_k$ on its network.

The "undefined" decorations appear in the $\mathcal{P}_k$'s on sort of grids whose cells grow when k decreases.

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The "undefined" decorations appear in the $\mathcal{P}_k$'s on sort of grids whose cells grow when k decreases.

Taking $n \to \infty$ yields a τ'-tiling $\mathcal{P}_0$ whose "undefined" decorations appear only on a star or a line – easily tranformed into a τ-tiling. This yields $P_0 = \pi(\mathcal{P}_0) \in \pi(\Lambda_\tau)$, and thus $\Lambda_\sigma \subset \pi(\Lambda_\tau)$.

Bringing all together

Given any good combinatorial substitution σ, we are thus able to defined a finite decorated tile set τ such that

$$\Lambda_\sigma = \pi(\Lambda_\tau).$$

This yields the claimed result:

Theorem (Fernique & Ollinger, 2010)

The limit set of a good combinatorial substitution is sofic.

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Unfortunately τ is rather huge (up to millions of tiles!).
Can we achieve a much lighter generic construction?

Thank you for your attention