Brun expansions of stepped surfaces

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main result:

Action of dual maps of *free group morphisms* over stepped planes and surfaces (extends the substitutive case). Hidden tool: *flip*.

Application 1 (now): Define Brun expansions of stepped planes and surfaces.

Application 2 (ask for details later): Decide whether a given stepped surface is a stepped plane or not (and for patches too).

Stp. planes & stp. surfaces 0000	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces



- 2 Dual maps of free group morphisms
- 3 Brun expansions of stepped planes
- 4 Brun expansions of stepped surfaces

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces

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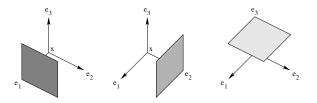
1 Stepped planes and stepped surfaces

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Stp. planes & stp. surfaces ●000	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Stepped plane			

 $(\vec{e}_1, \ldots, \vec{e}_d)$ basis of \mathbb{R}^d . $\vec{x} \in \mathbb{Z}^d$, $i \in \{1, \ldots, d\} \rightsquigarrow face (\vec{x}, i^*)$:



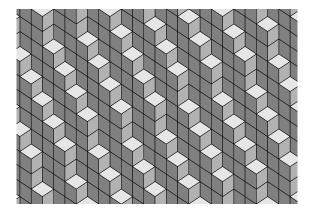
Definition

Stepped plane of normal vector $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \{0\}$:

$$\mathcal{P}_{\vec{\alpha}} = \{ (\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle \leq 0 < \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle \}.$$

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Stp. planes & stp. surfaces ○●○○	Dual maps of f. g. morphisms	Brun expansions of stp. planes 0000	Brun expansions of stp. surfaces
Stepped plane			



A stepped plane.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Stepped surface			

Let π be the orthogonal projection along $\vec{u} = \vec{e}_1 + \ldots + \vec{e}_d$.



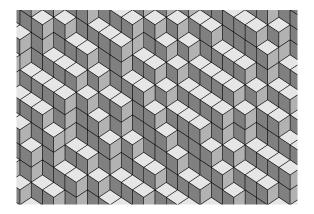
By extension:

Definition [Jamet]

Stepped surfaces : any set of faces homeomorphic to \vec{u}^{\perp} by π .

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Stp. planes & stp. surfaces ○○○●	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Stepped surface			



A stepped surface.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces

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1 Stepped planes and stepped surfaces

2 Dual maps of free group morphisms

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms ●○○○	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Definition			

Morphism of the free group over $\{1, \ldots, d\}$ (here, d = 3):

$$\sigma : \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 3^{-1}1 \\ 3 & \mapsto & 3^{-1}2 \end{cases}$$

For example: $\sigma(1^{-1}312) = \sigma(1)^{-1}\sigma(3)\sigma(1)\sigma(2) = 3^{-2}21$. Incidence matrix: $(M_{\sigma})_{ij} = |\sigma(i)|_j - |\sigma(i)|_{j^{-1}}$. Here:

$$M_{\sigma}=\left(egin{array}{ccc} 0 & 1 & 0 \ 0 & 0 & 1 \ 1 & -1 & -1 \end{array}
ight).$$

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 000	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Definition			

 σ unimodular f. g. morph. \rightsquigarrow dual map $E_1^*(\sigma)$ over weighted faces.

For σ previously defined:

$$E_1^*(\sigma) : \begin{cases} (\vec{0}, 1^*) & \mapsto & (\vec{e}_1, 2^*) \\ (\vec{0}, 2^*) & \mapsto & (\vec{e}_1, 3^*) \\ (\vec{0}, 3^*) & \mapsto & (\vec{0}, 1^*) - (\vec{e}_1, 2^*) - (\vec{e}_1, 3^*). \end{cases}$$

and, for $\lambda \in \mathbb{Z}$, $\vec{x} \in \mathbb{Z}^d$:

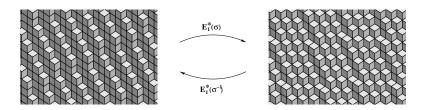
$$E_1^*(\sigma)(\lambda.(\vec{x},i^*)) = M_{\sigma}^{-1}\vec{x} + \lambda.E_1^*(\sigma)(\vec{0},i^*).$$

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Theorem (B. F. 2007)

For σ unimodular free group morphism and $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \vec{0}$:

$$M_{\sigma}^{\top}\vec{\alpha} \in \mathbb{R}^{d}_{+} \Rightarrow E_{1}^{*}(\sigma)(\mathcal{P}_{\vec{\alpha}}) = \mathcal{P}_{M_{\sigma}^{\top}\vec{\alpha}}.$$



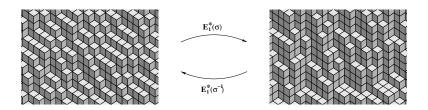
Note: the action of $E_1^*(\sigma)$ depends only on M_{σ} (but not on σ).

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms ○○○●	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Properties			

Theorem (B. F. 2007)

For σ unimodular free group morphism: if the image by $E_1^*(\sigma)$ of a stepped surface has faces with weights in $\{0,1\}$, then it is a stepped surface. This holds, in particular, when $M_{\sigma} \geq 0$.



Note: the action of $E_1^*(\sigma)$ depends only on M_{σ} (but not on σ).

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces

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1 Stepped planes and stepped surfaces

2 Dual maps of free group morphisms

3 Brun expansions of stepped planes

4 Brun expansions of stepped surfaces

Stp. planes & stp. surfaces 0000	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Brun expansion of a vector			

Brun map T, defined for
$$\vec{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d \setminus \{0\}$$
:

$$T(\alpha_1,\ldots,\alpha_d) = \left(\frac{\alpha_1}{\alpha_i},\ldots,\frac{\alpha_{i-1}}{\alpha_i},\frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor,\frac{\alpha_{i+1}}{\alpha_i},\ldots,\frac{\alpha_d}{\alpha_i} \right),$$

where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$. Matrix viewpoint:

$$(1, \mathcal{T}(ec{lpha}))^{ op} \propto \left(egin{array}{ccc} 0 & 1 & & \ & \mathrm{I}_{i-1} & & \ & 1 & & -oldsymbol{a} & \ & 1 & & \mathrm{I}_{d-i} \end{array}
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Stp. planes & stp. surfaces 0000	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$. Matrix viewpoint:

$$(1, T(\vec{\alpha}))^{\top} \propto \begin{pmatrix} 0 & 1 & \\ & \mathrm{I}_{i-1} & & \\ 1 & & -\mathbf{a} & \\ & & & \mathrm{I}_{d-i} \end{pmatrix} (1, \vec{\alpha})^{\top}$$

Brun expansion $(a_n, i_n)_{n\geq 0}$ of $\vec{\alpha}$:

 $a_n = \lfloor ||T^n(\vec{\alpha})||_{\infty}^{-1} \rfloor$ and $i_n = \min\{j \mid \langle T^n(\vec{\alpha})|\vec{e}_j \rangle = ||T^n(\vec{\alpha})||_{\infty}\}.$

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes ○●00	Brun expansions of stp. surfaces
Brun expansion of a stepped	plane		

How to define Brun exp. of given stepped planes (unknown normal vectors) so that $\mathcal{P}_{(1,\vec{\alpha})}$ will have the Brun exp. of $\vec{\alpha}$?

Note: if $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$ and $a = \lfloor 1/\alpha_i \rfloor$ are known:

$$E_1^*(\beta_{a,i})(\mathcal{P}_{(1,\vec{\alpha})})=\mathcal{P}_{(1,\mathcal{T}(\vec{\alpha}))},$$

where $\beta_{a,i}$ has incidence matrix $B_{a,i}$ s.t. $B_{a,i}(1, \vec{\alpha}) = (1, T(\vec{\alpha}))$.

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes ○●○○	Brun expansions of stp. surfaces
Brun expansion of a stepped	plane		

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Determining $(a, i) \rightsquigarrow$ entries comparisons and floor computation.

	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes ○○●○	Brun expansions of stp. surfaces
Brun expansion of a stepped	plane		

entries comparisons:

 $(\vec{x}, (i+1)^*), (\vec{x} + \vec{e}_{j+1}, (i+1)^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}$ for some \vec{x} yields $\alpha_i > \alpha_j$.

floor computation:

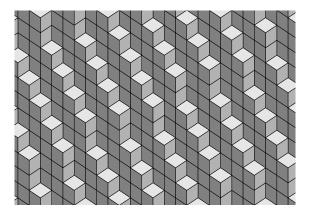
Let us introduce:

 $a_i(\mathcal{P}) = \max\{a \in \mathbb{N} \mid (\vec{x}, (i+1)^*) \lhd \mathcal{P} \ \Rightarrow \ (\vec{x} - k\vec{e}_{i+1}, 1^*)_{0 \leq k < a} \lhd \mathcal{P}\}.$

One shows:

$$\mathsf{a}_i(\mathcal{P}_{(1,\vec{lpha})}) = \left\lfloor \frac{1}{\alpha_i}
ight
floor.$$

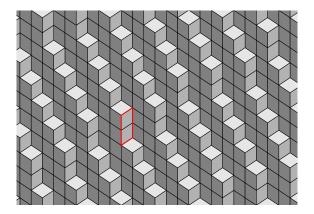
Brun expansion of a stepped	nlane		
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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stn. planes	Brun expansions of stp. surfaces



Stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$, with unknown $\vec{\alpha} = (\alpha_1, \alpha_2) \in [0, 1]^2 \setminus \{0\}$.

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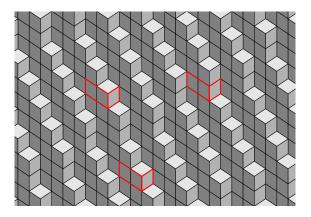
Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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Brun expansion of a stepped	plane		



 $(\vec{0}, 2^*), (\vec{e}_3, 2^*) \lhd \mathcal{P}_{(1, \vec{\alpha})}.$ Thus, $\alpha_1 > \alpha_2$.

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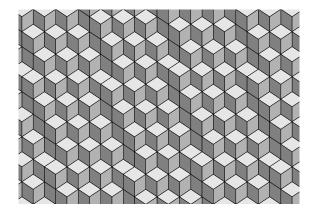
Brun expansion of a stepped i			0000
Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces



 $(\vec{x}, 2^*) \lhd \mathcal{P}_{(1, \vec{\alpha})} \ \Rightarrow \ (\vec{x}, 1^*), (\vec{x} - \vec{e}_2, 1^*) \lhd \mathcal{P}_{(1, \vec{\alpha})}.$ Thus, $\lfloor 1/\alpha_1 \rfloor = 2.$

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Brun expansion of a stepped			0000



Finally: $\mathcal{P}_{(1,\mathcal{T}(\vec{\alpha}))} = E_1^*(\beta_{2,1})(\mathcal{P}_{(1,\vec{\alpha})}).$

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes	Brun expansions of stp. surfaces ●○○○
Brun expansions of stepped	surfaces		

Dual maps and "information grabbing" defined for stepped surfaces

 \rightsquigarrow natural extension of Brun expansions for stepped surfaces.

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
Brun expansions of stepped	surfaces		

Dual maps and "information grabbing" defined for stepped surfaces

 \rightsquigarrow natural extension of Brun expansions for stepped surfaces.

Theorem (B. F. 2007)

Stepped surfaces having the Brun expansion of $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \{0\}$ are:

- the stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$ (finite or infinite expansion);
- some stepped surfaces almost equal to $\mathcal{P}_{(1,\vec{\alpha})}$ (idem);
- some non-plane stepped surfaces (only finite expansion).

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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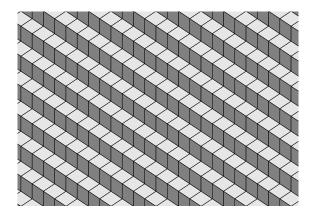
The stepped plane case (finite or infinite expansion)

(a, i) = (4, 1)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
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The stepped plane case (finite or infinite expansion)



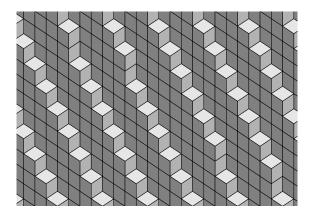
(a,i)=(1,2)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces	
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The stepped plane case (finite or infinite expansion)				

 $a = \infty$. Stepped plane recognized.

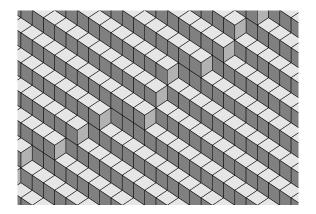
	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces ○○●○	
The stepped quasi-plane case (finite or infinite expansion)				



(a, i) = (4, 1)

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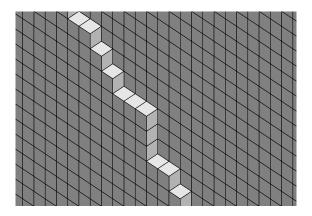
Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces $\circ \circ \bullet \circ$
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The stepped quasi-plane case	e (finite or infinite expansion)		



(a,i)=(1,2)

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Stp. planes & stp. surfaces	Dual maps of f. g. morphisms 0000	Brun expansions of stp. planes 0000	Brun expansions of stp. surfaces $\circ \circ \bullet \circ$
The stepped quasi-plane cas	e (finite or infinite expansion)		



 $a = \infty$. Not a stepped plane... but almost.

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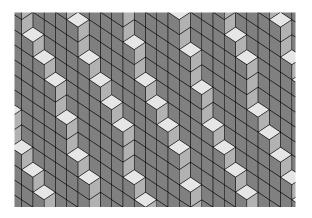
Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun ex

Brun expansions of stp. planes

Brun expansions of stp. surfaces $\circ \circ \circ \bullet$

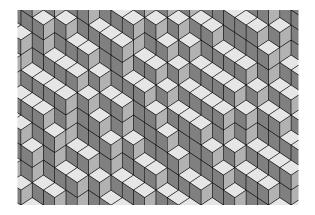
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The stepped surface case (only finite expansion)



(a, i) = (4, 1)

Stp. planes & stp. surfaces	Dual maps of f. g. morphisms	Brun expansions of stp. planes	Brun expansions of stp. surfaces
The stepped surface case (or		0000	



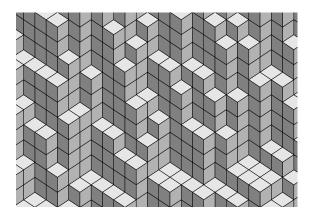
(a,i)=(1,2)

Stp. planes &	stp. surfaces	Dual maps of f. g. morphisms	Bru
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Brun expansions of stp. planes

Brun expansions of stp. surfaces $\circ \circ \circ \bullet$

The stepped surface case (only finite expansion)



a undefined. Not at all a stepped plane.

Where is "digital plane recognition"?

Finite expansions \rightsquigarrow stepped planes recognized from the last obtained stepped surface.

Patches (finite subset of stepped surfaces) \rightsquigarrow finite expansions...