Dual maps of free group morphisms

Thomas Fernique
joint work with Valérie Berthé

LIRMM - Univ. Montpellier 2 (France)

Qing Hua Univ. (Beijing - China)
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Morphism of the free group F_d (here, $d = 3$):

$$\sigma \; : \; \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 3^{-1}1 \\ 3 & \mapsto & 3^{-1}2 \end{cases}$$

For example: $\sigma(1^{-1}312) = \sigma(1)^{-1}\sigma(3)\sigma(1)\sigma(2) = 3^{-2}21$.

Incidence matrix: $(M_\sigma)_{ij} = |\sigma(i)|_j - |\sigma(i)|_{j^{-1}}$. Here:

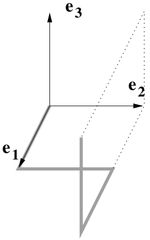
$$M_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

$\vec{y} \in \mathbb{Z}^3$, $j \in \{1, \ldots, d\}$ $\rightsquigarrow$ *segment* $[\vec{y}, \vec{y} + \vec{e}_j]$, denoted by $(\vec{y}, j)$.

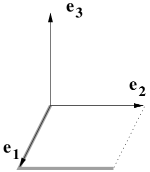
$u \in F_d$ $\rightsquigarrow$ formal sum of segments $\gamma(u)$:

$$\gamma(u_1^{\varepsilon_1} \ldots u_k^{\varepsilon_k}) = \sum_{i=1}^{k} \varepsilon_i . (\vec{y}_i, u_i),$$

where $y_i = \sum_{j<i} \varepsilon_j \vec{e}_{u_j}$ if $\varepsilon_i > 0$, $y_i = \sum_{j<i} \varepsilon_j \vec{e}_{u_j} - \vec{e}_i$ otherwise.



γ(1213)

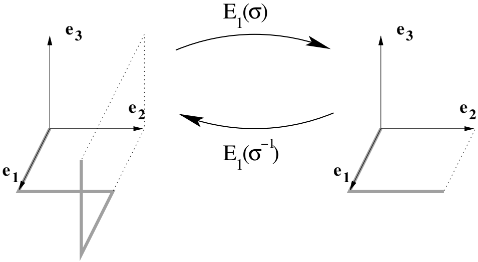


γ(12)

σ over $F_d \rightsquigarrow E_1(\sigma)$ over formal sums of segments:

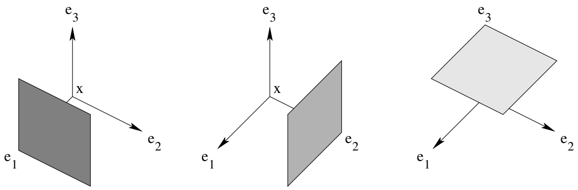
$$E_1(\sigma) \circ \gamma = \gamma \circ \sigma.$$

Explicit formula, with $E_1(\sigma)(\lambda.(\vec{y}, j)) = M_\sigma \vec{y} + \lambda.E_1(\sigma)(\vec{0}, j)$.



γ(1213) γ(12)

$(\vec{e}_1, \ldots, \vec{e}_d)$ basis of $\mathbb{R}^d$. $\vec{x} \in \mathbb{Z}^d$, $i \in \{1, \ldots, d\} \rightsquigarrow$ *face* $(\vec{x}, i^*)$:



Duality segment-face: $[(\vec{y}, j), (\vec{x}, i^*)] = 1$ iff $\vec{x} = \vec{y}$ and $i = j$.
Linear map $E_1(\sigma) \rightsquigarrow$ dual map $E_1^*(\sigma)$:

$$[E_1(\sigma)(\vec{y}, j), (\vec{x}, i^*)] = [(\vec{y}, j), E_1(\sigma)^*(\vec{x}, i^*)].$$

Explicit formula for *unimodular* f. g. morphisms, with:

$$E_1^*(\sigma)(\lambda.(\vec{x}, i^*)) = M_\sigma^{-1}\vec{x} + \lambda.E_1^*(\sigma)(\vec{0}, i^*).$$

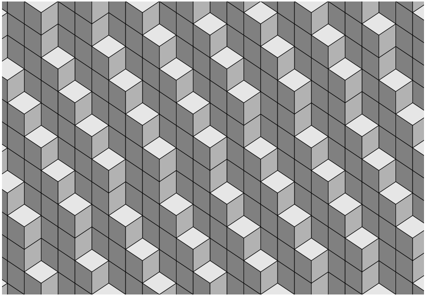
For example, σ previously defined yields:

$$E_1^*(\sigma) \ : \ \begin{cases} (\vec{0}, 1^*) & \mapsto & (\vec{e}_1, 2^*) \\ (\vec{0}, 2^*) & \mapsto & (\vec{e}_1, 3^*) \\ (\vec{0}, 3^*) & \mapsto & (\vec{0}, 1^*) - (\vec{e}_1, 2^*) - (\vec{e}_1, 3^*). \end{cases}$$

Definition

Stepped plane of normal vector $\vec{\alpha} \in \mathbb{R}_+^d \setminus \{0\}$:

$$\mathcal{P}_{\vec{\alpha}} = \{(\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle \leq 0 < \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle\}.$$



Let π be the orthogonal projection along $\vec{u} = \vec{e}_1 + \ldots + \vec{e}_d$.

Proposition

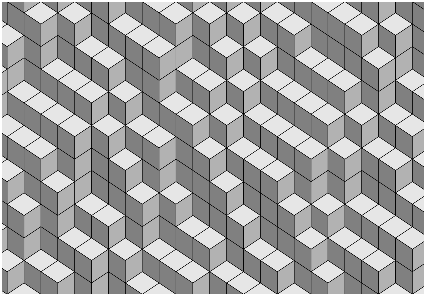
Stepped planes are homeomorphic to $\vec{u}^{\perp}$ by π.

By extension:

Definition [Jamet]

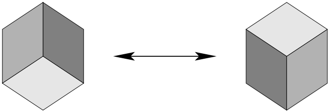
Stepped surfaces : any set of faces homeomorphic to $\vec{u}^{\perp}$ by π.

Stepped surface



A stepped surface.

Projection on $\vec{u}^{\perp}$: lozenge tilings $\rightsquigarrow$ *flip* (mechanical physics):



Flip on a stepped surface $\simeq$ add/remove a unit hypercube

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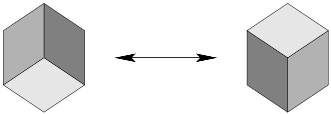


Flip on a stepped surface $\simeq$ add/remove a unit hypercube

Theorem [Arnoux-Berthé-F.-Jamet, 2007]

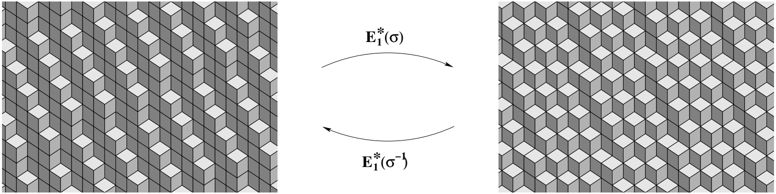
Stepped surface $=$ stepped plane with $\vec{\alpha} \in (0, \infty)^d$ + flips.

$\rightsquigarrow$ two equivalent definitions.

Theorem (Berthé-F. 2007)

For σ unimodular free group morphism and $\vec{\alpha} \in \mathbb{R}^d_+ \backslash \vec{0}$:

$$M_\sigma^\top \vec{\alpha} \in \mathbb{R}^d_+ \;\Rightarrow\; E_1^*(\sigma)(\mathcal{P}_{\vec{\alpha}}) = \mathcal{P}_{M_\sigma^\top \vec{\alpha}}.$$



Note: the action of $E_1^*(\sigma)$ depends only on M_σ (but not on σ).

proof: not so easy (faces cancellations), but note:

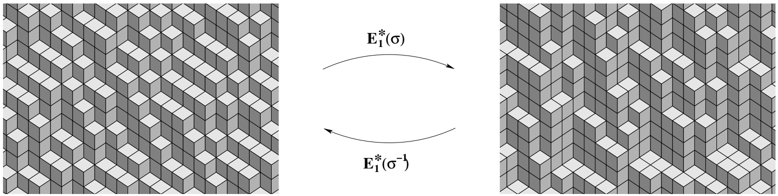
$$\langle \vec{x} | \vec{\alpha} \rangle = 0 \;\Leftrightarrow\; \langle M_\sigma^{-1} \vec{x} | M_\sigma^\top \vec{\alpha} \rangle,$$

that is, $M_\sigma^{-1} P_{\vec{\alpha}} = P_{M_\sigma^\top \vec{\alpha}}$.
Thus, $E_1^*(\sigma)$ can be seen as a *discretization* of M_σ^{-1}. Recall:

$$E_1^*(\sigma)(\lambda.(\vec{x}, i^*)) = M_\sigma^{-1} \vec{x} + \lambda.E_1^*(\sigma)(\vec{0}, i^*).$$

Theorem (Berthé-F. 2007)

For σ unimodular free group morphism: if the image by $E_1^*(\sigma)$ of a stepped surface has faces with weights in $\{0, 1\}$, then it is a stepped surface. This holds, in particular, when $M_\sigma \geq 0$.



Note: the action of $E_1^*(\sigma)$ depends only on M_σ (but not on σ).

proof: relies on flips. More precisely, we show:

$$E_1^*(\sigma)(\mathcal{F}_{\vec{x}}) = \mathcal{F}_{M_\sigma^{-1}\vec{x}},$$

where:

$$\mathcal{F}_{\vec{x}} = \sum_{i=1}^{d}(\vec{x}, i^*) - \sum_{i=1}^{d}(\vec{x} - \vec{e}_i, i^*).$$

Then, we write a stepped surface as flips over a stepped plane and we use the previous theorem (for stepped planes).

1. Dual maps of free group morphisms
2. Stepped planes and stepped surfaces
3. Action of dual maps
4. **Brun expansions of stepped planes**
5. Brun expansions of stepped surfaces
6. Application in discrete geometry

Brun map T, defined for $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \backslash \{0\}$:

$$T(\alpha_1, \ldots, \alpha_d) = \left(\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_d}{\alpha_i} \right),$$

where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_\infty\}$. Matrix viewpoint:

$$(1, T(\vec{\alpha}))^\top \propto \begin{pmatrix} 0 & & 1 & \\ & \mathrm{I}_{i-1} & & \\ 1 & & -a & \\ & & & \mathrm{I}_{d-i} \end{pmatrix} (1, \vec{\alpha})^\top$$

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Brun expansion $(a_n, i_n)_{n \geq 0}$ of $\vec{\alpha}$:

$$a_n = \left\lfloor ||T^n(\vec{\alpha})||_\infty^{-1} \right\rfloor \quad \text{and} \quad i_n = \min\{j \mid \langle T^n(\vec{\alpha}) | \vec{e}_j \rangle = ||T^n(\vec{\alpha})||_\infty\}.$$

How to define Brun exp. of given stepped planes (unknown normal vectors) so that $\mathcal{P}_{(1,\vec{\alpha})}$ will have the Brun exp. of $\vec{\alpha}$?

Note: if $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_\infty\}$ and $a = \lfloor 1/\alpha_i \rfloor$ are known:

$$E_1^*(\beta_{a,i})(\mathcal{P}_{(1,\vec{\alpha})}) = \mathcal{P}_{(1,T(\vec{\alpha}))},$$

where $\beta_{a,i}$ has incidence matrix $B_{a,i}$ s.t. $B_{a,i}(1,\vec{\alpha}) = (1, T(\vec{\alpha}))$.

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Determining $(a,i) \rightsquigarrow$ *entries comparisons* and *floor computation*.

entries comparisons:

$(\vec{x}, (i+1)^*), (\vec{x} + \vec{e}_{j+1}, (i+1)^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}$ for some $\vec{x}$ yields $\alpha_i > \alpha_j$.

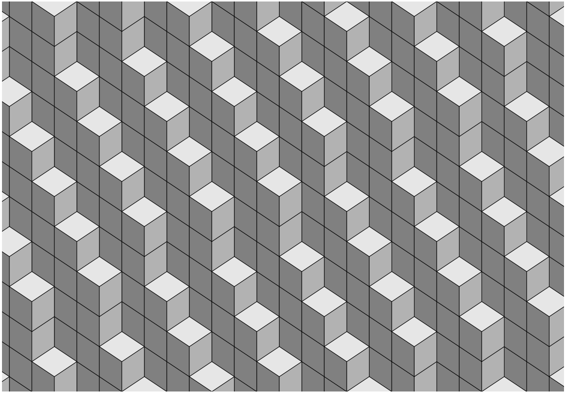
floor computation:

We introduce:

$$a_i(\mathcal{P}) = \max\{a \in \mathbb{N} \mid (\vec{x}, (i+1)^*) \lhd \mathcal{P} \ \Rightarrow \ (\vec{x} - k\vec{e}_{i+1}, 1^*)_{0 \leq k < a} \lhd \mathcal{P}\}.$$

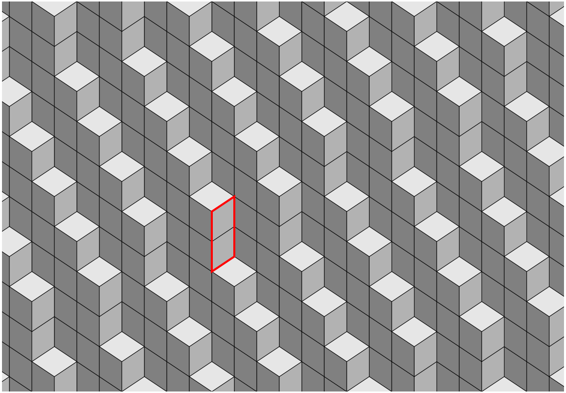
One shows:

$$a_i(\mathcal{P}_{(1,\vec{\alpha})}) = \left\lfloor \frac{1}{\alpha_i} \right\rfloor.$$



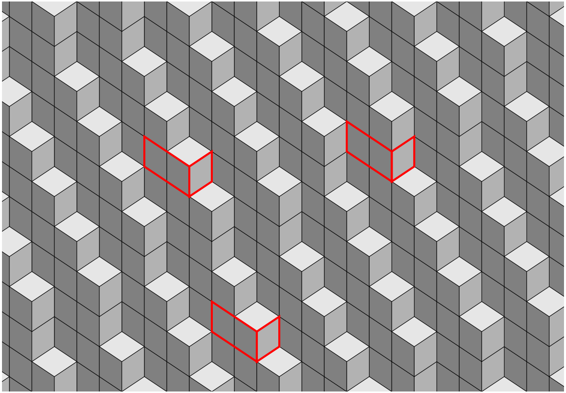
Stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$, with unknown $\vec{\alpha} = (\alpha_1, \alpha_2) \in [0,1]^2 \backslash \{0\}$.

Brun expansion of a stepped plane

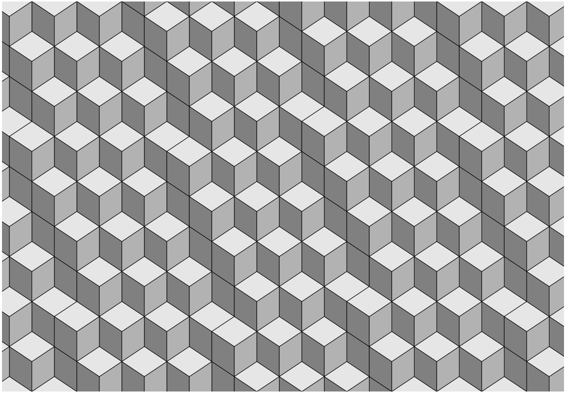


$(\vec{0}, 2^*), (\vec{e}_3, 2^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}$. Thus, $\alpha_1 > \alpha_2$.

Brun expansion of a stepped plane



$$(\vec{x}, 2^*) \lhd \mathcal{P}_{(1,\vec{\alpha})} \;\Rightarrow\; (\vec{x}, 1^*), (\vec{x} - \vec{e}_2, 1^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}.\text{ Thus, } \lfloor 1/\alpha_1 \rfloor = 2.$$



Finally: $\mathcal{P}_{(1,T(\vec{\alpha}))} = E_1^*(\beta_{2,1})(\mathcal{P}_{(1,\vec{\alpha})})$.

Dual maps and “information grabbing” defined for stepped surfaces

$\rightsquigarrow$ natural extension of Brun expansions for stepped surfaces.

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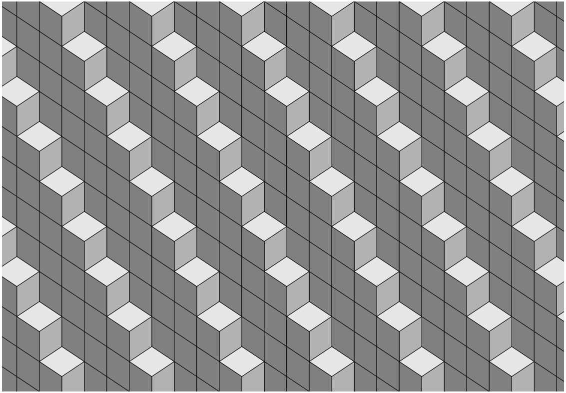
Theorem (Berthé-F. 2007)

Stepped surfaces having the Brun expansion of $\vec{\alpha} \in \mathbb{R}_+^d \backslash \{0\}$ are:

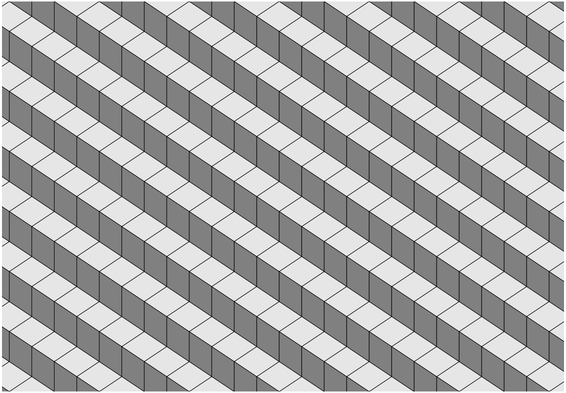
- *the stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$ (finite or infinite expansion);*
- *some stepped surfaces almost equal to $\mathcal{P}_{(1,\vec{\alpha})}$ (idem);*
- *some non-plane stepped surfaces (only finite expansion).*

The stepped plane case (finite or infinite expansion)



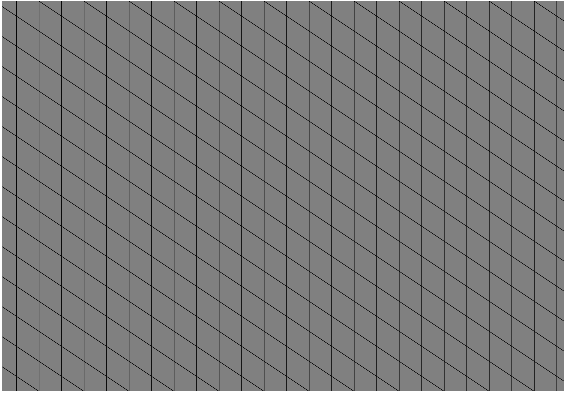
$$(a, i) = (4, 1)$$

The stepped plane case (finite or infinite expansion)



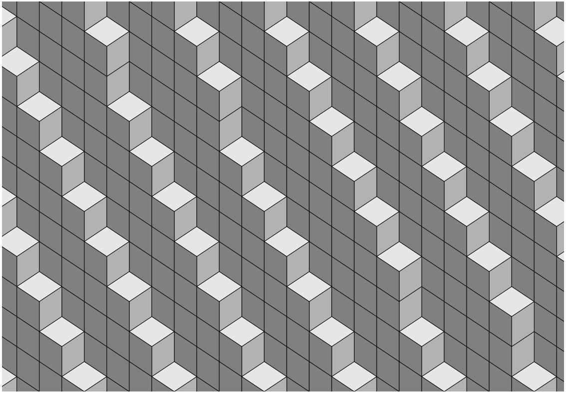
$$(a, i) = (1, 2)$$

The stepped plane case (finite or infinite expansion)



$a = \infty$. Stepped plane recognized.

The stepped quasi-plane case (finite or infinite expansion)



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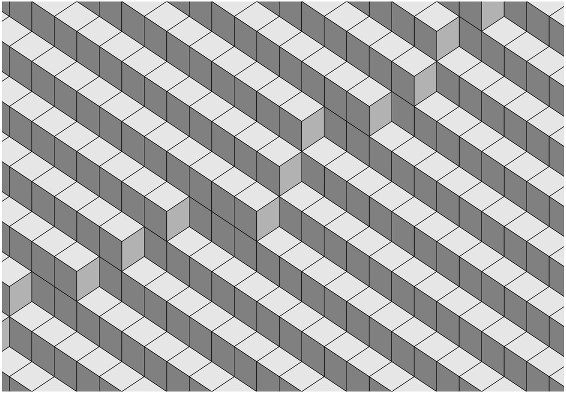
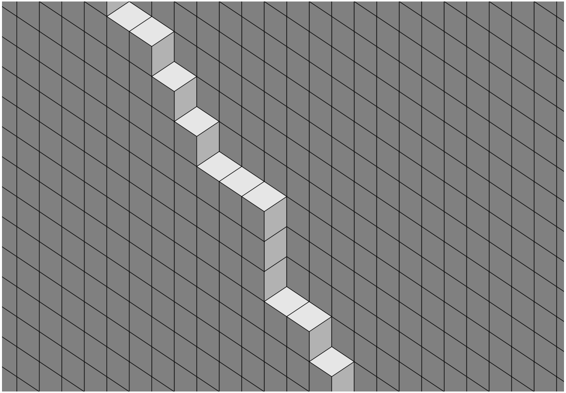
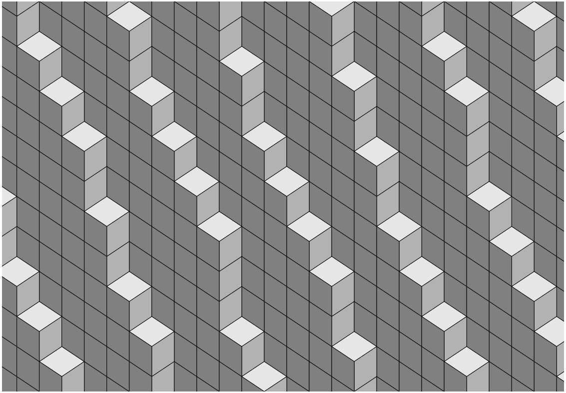


$$(a, i) = (1, 2)$$



$a = \infty$. Not a stepped plane. . . but almost.

The stepped surface case (only finite expansion)



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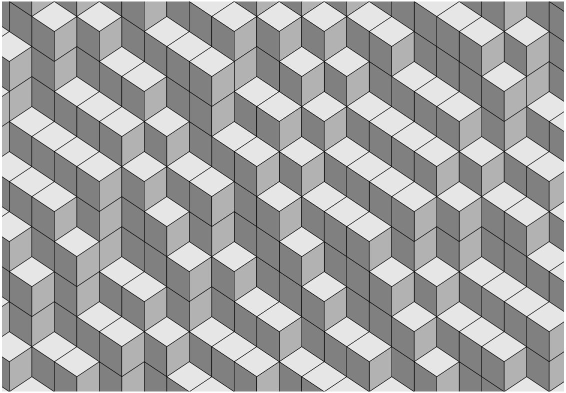

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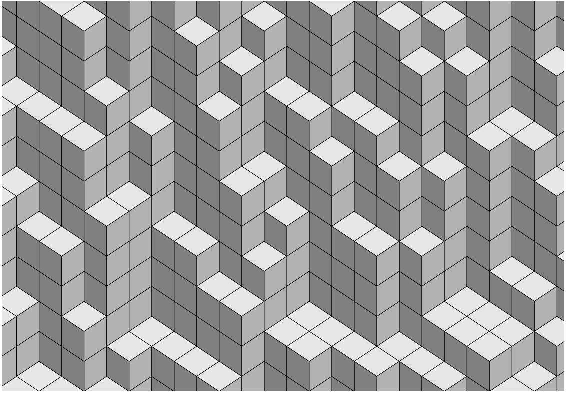

a undefined. Not at all a stepped plane.

Discrete geometry: change from $\mathbb{R}^d$ (maths) to $\mathbb{Z}^d$ (computer).

Vertices of a stepped plane: *standard plane* (widely used).

Challenge: *vectorization* of experimental geometrical data, that is:

- recognize pieces of plane in the data and store parameters;
- reconstruct pieces of plane from stored parameters.

Expand the given stepped surface (suppose finite expansion).

The last obtained stepped surface is plane iff the first one does.

For finite patch, this yields finite expansion but we omit details.

Some boundary problems $\rightsquigarrow$ mixed approach (preimage algorithm).

How to construct $\mathcal{P}_{(1,\vec{\alpha})}$ for given $\vec{\alpha} \in \mathbb{Q}^d$?

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2. compute $\mathcal{D}_{(1,\vec{\alpha})} = E_1^*(\beta_{a_0,i_0}^{-1}) \circ \ldots \circ E_1^*(\beta^{-1} a_N, i_N)(\vec{0}, 1^*)$.
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3. One has $\mathcal{P}_{(1,\vec{\alpha})} = \mathbb{Z} M_N \vec{e}_1 + \mathbb{Z} M_N \vec{e}_2 + \mathcal{D}_{\vec{\alpha}}$, where M_N incidence matrix of $\beta_{a_0,i_0}^{-1} \circ \ldots \circ \beta_{a_N,i_N}^{-1}$.

For example:

$$\vec{\alpha} = \left(\frac{3}{8}, \frac{5}{12}\right) \rightsquigarrow [(2,2),(1,1),(2,2),(4,1),(1,2)] \rightsquigarrow \left(\frac{9}{24}, \frac{10}{24}\right).$$

$\mathcal{D}_{(1,\vec{\alpha})}$ has 24, 9 and 10 faces of, respectively, types 1, 2 and 3.

It tiles periodically $\mathcal{P}_{(1,\vec{\alpha})}$ with periods $(1,4,-6)$ and $(2,-2,-3)$.

Plane generation

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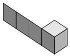

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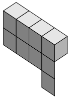

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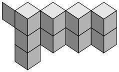

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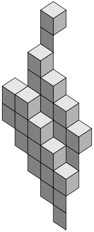

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