Dual maps of free group morphisms

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Dual maps 0000		Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces	Application



- 2 Stepped planes and stepped surfaces
- 3 Action of dual maps
- 4 Brun expansions of stepped planes
- 5 Brun expansions of stepped surfaces
- 6 Application in discrete geometry

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Dual maps	Stp. planes & surfaces	Action of dual maps	Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application

Morphism of the free group F_d (here, d = 3):

$$\sigma : \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 3^{-1}1 \\ 3 & \mapsto & 3^{-1}2 \end{cases}$$

For example: $\sigma(1^{-1}312) = \sigma(1)^{-1}\sigma(3)\sigma(1)\sigma(2) = 3^{-2}21$. Incidence matrix: $(M_{\sigma})_{ij} = |\sigma(i)|_j - |\sigma(i)|_{j^{-1}}$. Here:

$$M_{\sigma}=\left(egin{array}{ccc} 0 & 1 & 0 \ 0 & 0 & 1 \ 1 & -1 & -1 \end{array}
ight).$$

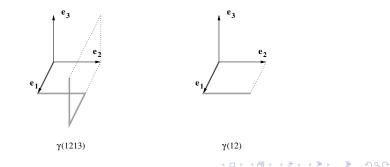
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 $\vec{y} \in \mathbb{Z}^3$, $j \in \{1, \dots, d\} \rightsquigarrow segment [\vec{y}, \vec{y} + \vec{e}_j]$, denoted by (\vec{y}, j) . $u \in F_d \rightsquigarrow$ formal sum of segments $\gamma(u)$:

$$\gamma(u_1^{\varepsilon_1}\ldots u_k^{\varepsilon_k})=\sum_{i=1}^k \varepsilon_i.(\vec{y}_i,u_i),$$

where $y_i = \sum_{j < i} \varepsilon_j \vec{e}_{u_j}$ if $\varepsilon_i > 0$, $y_i = \sum_{j < i} \varepsilon_j \vec{e}_{u_j} - \vec{e}_i$ otherwise.

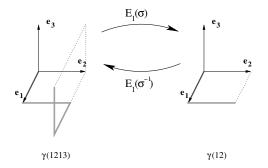


Dual maps 0●00	Action of dual maps 0000	Brun exp. of stp. surfaces	Application 0000
Linear map			

 σ over $F_d \rightsquigarrow E_1(\sigma)$ over formal sums of segments:

 $E_1(\sigma)\circ\gamma=\gamma\circ\sigma.$

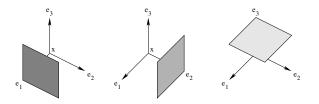
Explicit formula, with $E_1(\sigma)(\lambda.(\vec{y},j)) = M_{\sigma}\vec{y} + \lambda.E_1(\sigma)(\vec{0},j)$.



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Dual maps ○○●○	Action of dual maps	Brun exp. of stp. surfaces	Application 0000
Dual map			

 $(\vec{e}_1, \ldots, \vec{e}_d)$ basis of \mathbb{R}^d . $\vec{x} \in \mathbb{Z}^d$, $i \in \{1, \ldots, d\} \rightsquigarrow face (\vec{x}, i^*)$:



Duality segment-face: $[(\vec{y}, j), (\vec{x}, i^*)] = 1$ iff $\vec{x} = \vec{y}$ and i = j. Linear map $E_1(\sigma) \rightsquigarrow$ dual map $E_1^*(\sigma)$:

$$[E_1(\sigma)(\vec{y},j),(\vec{x},i^*)] = [(\vec{y},j),E_1(\sigma)^*(\vec{x},i^*)].$$

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Dual maps ○○○●	Action of dual maps	Brun exp. of stp. surfaces	Application 0000
Dual map			

Explicit formula for unimodular f. g. morphisms, with:

$$E_1^*(\sigma)(\lambda.(\vec{x},i^*)) = M_{\sigma}^{-1}\vec{x} + \lambda.E_1^*(\sigma)(\vec{0},i^*).$$

For example, σ previously defined yields:

$$E_1^*(\sigma) : \begin{cases} (\vec{0}, 1^*) & \mapsto & (\vec{e}_1, 2^*) \\ (\vec{0}, 2^*) & \mapsto & (\vec{e}_1, 3^*) \\ (\vec{0}, 3^*) & \mapsto & (\vec{0}, 1^*) - (\vec{e}_1, 2^*) - (\vec{e}_1, 3^*). \end{cases}$$

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1 Dual maps of free group morphisms

2 Stepped planes and stepped surfaces

Action of dual maps

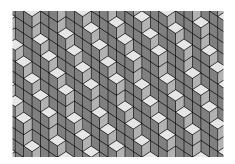
- 4 Brun expansions of stepped planes
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Dual maps 0000		Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application 0000
Stepped plar	ne			

Definition

Stepped plane of normal vector $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \{0\}$:

$$\mathcal{P}_{\vec{\alpha}} = \{ (\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle \leq 0 < \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle \}.$$



Dual maps 0000		Action of dual maps	Brun exp. of stp. surfaces	Application 0000
Stepped surf	face			

Let π be the orthogonal projection along $\vec{u} = \vec{e}_1 + \ldots + \vec{e}_d$.



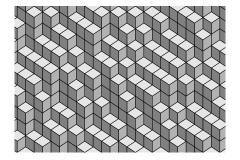
By extension:

Definition [Jamet]

Stepped surfaces : any set of faces homeomorphic to \vec{u}^{\perp} by π .

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Stepped surf	face			



A stepped surface.

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Dual maps 0000		Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application 0000
Flip				

Projection on \vec{u}^{\perp} : lozenge tilings \rightsquigarrow flip (mechanical physics):



Flip on a stepped surface \simeq add/remove a unit hypercube



Dual maps 0000		Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application 0000
Flip				

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Flip on a stepped surface \simeq add/remove a unit hypercube

Theorem [Arnoux-Berthé-F.-Jamet, 2007]

Stepped surface = stepped plane with $\vec{\alpha} \in (0,\infty)^d$ + flips.

 \rightsquigarrow two equivalent definitions.

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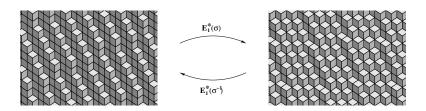
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Action over	stepped planes				

Theorem (Berthé-F. 2007)

For σ unimodular free group morphism and $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \vec{0}$:

$$M_{\sigma}^{\top}\vec{lpha}\in\mathbb{R}^{d}_{+}\ \Rightarrow\ E_{1}^{*}(\sigma)(\mathcal{P}_{\vec{lpha}})=\mathcal{P}_{M_{\sigma}^{\top}\vec{lpha}}.$$



Note: the action of $E_1^*(\sigma)$ depends only on M_{σ} (but not on σ).

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Dual maps 0000		Action of dual maps ○●○○	Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces	Application 0000
Action over	stepped planes				

proof: not so easy (faces cancellations), but note:

$$\langle \vec{x} | \vec{\alpha} \rangle = 0 \iff \langle M_{\sigma}^{-1} \vec{x} | M_{\sigma}^{\top} \vec{\alpha} \rangle,$$

that is, $M_{\sigma}^{-1}P_{\vec{\alpha}} = P_{M_{\sigma}^{\top}\vec{\alpha}}$. Thus, $E_1^*(\sigma)$ can be seen as a *discretization* of M_{σ}^{-1} . Recall:

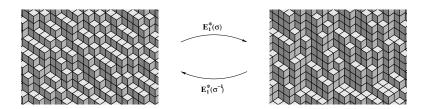
$$E_1^*(\sigma)(\lambda.(\vec{x},i^*)) = M_{\sigma}^{-1}\vec{x} + \lambda.E_1^*(\sigma)(\vec{0},i^*).$$

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	Stp. planes & surfaces 0000	Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces 0000	Application 0000
Action over	stepped surfaces			

Theorem (Berthé-F. 2007)

For σ unimodular free group morphism: if the image by $E_1^*(\sigma)$ of a stepped surface has faces with weights in $\{0,1\}$, then it is a stepped surface. This holds, in particular, when $M_{\sigma} \geq 0$.



Note: the action of $E_1^*(\sigma)$ depends only on M_{σ} (but not on σ).

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Action over	stepped surfaces			

proof: relies on flips. More precisely, we show:

$$E_1^*(\sigma)(\mathcal{F}_{\vec{x}}) = \mathcal{F}_{M_{\sigma}^{-1}\vec{x}},$$

where:

$$\mathcal{F}_{\vec{x}} = \sum_{i=1}^{d} (\vec{x}, i^*) - \sum_{i=1}^{d} (\vec{x} - \vec{e}_i, i^*).$$

Then, we write a stepped surface as flips over a stepped plane and we use the previous theorem (for stepped planes).

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Brun expans	ion of a vector				

Brun map T, defined for $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \setminus \{0\}$:

$$T(\alpha_1,\ldots,\alpha_d) = \left(\frac{\alpha_1}{\alpha_i},\ldots,\frac{\alpha_{i-1}}{\alpha_i},\frac{1}{\alpha_i} - \left\lfloor \frac{1}{\alpha_i} \right\rfloor,\frac{\alpha_{i+1}}{\alpha_i},\ldots,\frac{\alpha_d}{\alpha_i}\right),$$

where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$. Matrix viewpoint:

$$(1, \mathcal{T}(ec{lpha}))^{ op} \propto \left(egin{array}{ccc} 0 & 1 & & \ & \mathrm{I}_{i-1} & & \ & 1 & & -\mathbf{a} & \ & & & \mathrm{I}_{d-i} \end{array}
ight) (1, ec{lpha})^{ op}$$

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Dual maps		Action of dual maps	Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application
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Brun expansion $(a_n, i_n)_{n\geq 0}$ of $\vec{\alpha}$:

 $a_n = \lfloor ||T^n(\vec{\alpha})||_{\infty}^{-1} \rfloor$ and $i_n = \min\{j \mid \langle T^n(\vec{\alpha})|\vec{e}_j \rangle = ||T^n(\vec{\alpha})||_{\infty}\}.$

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How to define Brun exp. of given stepped planes (unknown normal vectors) so that $\mathcal{P}_{(1,\vec{\alpha})}$ will have the Brun exp. of $\vec{\alpha}$?

Note: if $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_{\infty}\}$ and $a = \lfloor 1/\alpha_i \rfloor$ are known:

$$E_1^*(\beta_{a,i})(\mathcal{P}_{(1,\vec{\alpha})})=\mathcal{P}_{(1,T(\vec{\alpha}))},$$

where $\beta_{a,i}$ has incidence matrix $B_{a,i}$ s.t. $B_{a,i}(1, \vec{\alpha}) = (1, T(\vec{\alpha}))$.

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Determining $(a, i) \rightsquigarrow$ entries comparisons and floor computation.

	Stp. planes & surfaces 0000	Brun exp. of stp. planes ○○●○	Brun exp. of stp. surfaces	Application 0000
Brun expans	sion of a stepped plane			

entries comparisons:

 $(\vec{x}, (i+1)^*), (\vec{x} + \vec{e}_{j+1}, (i+1)^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}$ for some \vec{x} yields $\alpha_i > \alpha_j$.

floor computation:

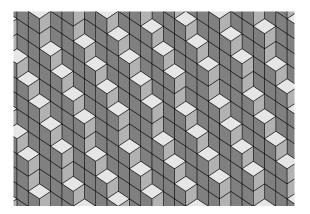
We introduce:

 $a_i(\mathcal{P}) = \max\{a \in \mathbb{N} \mid (\vec{x}, (i+1)^*) \lhd \mathcal{P} \ \Rightarrow \ (\vec{x} - k\vec{e}_{i+1}, 1^*)_{0 \leq k < a} \lhd \mathcal{P}\}.$

One shows:

$$\mathsf{a}_i(\mathcal{P}_{(1,\vec{lpha})}) = \left\lfloor \frac{1}{\alpha_i}
ight
floor.$$

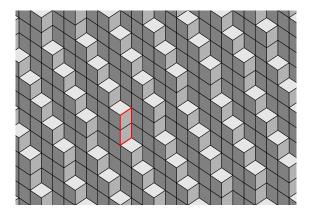
Dual maps 0000		Action of dual maps	Brun exp. of stp. planes ○○○●	Brun exp. of stp. surfaces	Application 0000
Brun expans	ion of a stepped plane				



Stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$, with unknown $\vec{\alpha} = (\alpha_1, \alpha_2) \in [0, 1]^2 \setminus \{0\}$.

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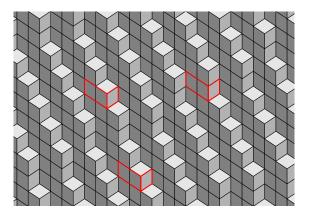
Dual maps		Action of dual maps	Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application
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Brun expans	sion of a stepped plane				



 $(\vec{0}, 2^*), (\vec{e}_3, 2^*) \lhd \mathcal{P}_{(1, \vec{\alpha})}.$ Thus, $\alpha_1 > \alpha_2$.

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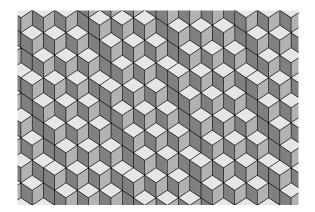
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Brun expans	ion of a stepped plane				



 $(\vec{x}, 2^*) \lhd \mathcal{P}_{(1, \vec{\alpha})} \Rightarrow (\vec{x}, 1^*), (\vec{x} - \vec{e}_2, 1^*) \lhd \mathcal{P}_{(1, \vec{\alpha})}.$ Thus, $\lfloor 1/\alpha_1 \rfloor = 2.$

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Brun expans	ion of a stepped plane				



Finally: $\mathcal{P}_{(1,\mathcal{T}(\vec{\alpha}))} = E_1^*(\beta_{2,1})(\mathcal{P}_{(1,\vec{\alpha})}).$

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Brun expans	sions of stepped surfaces			

Dual maps and "information grabbing" defined for stepped surfaces

 \rightsquigarrow natural extension of Brun expansions for stepped surfaces.

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Brun expans	sions of stepped surfaces			

Dual maps and "information grabbing" defined for stepped surfaces

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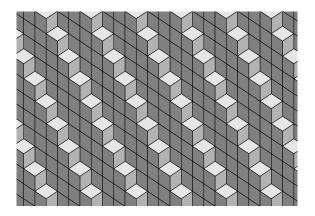
Theorem (Berthé-F. 2007)

Stepped surfaces having the Brun expansion of $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \{0\}$ are:

- the stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$ (finite or infinite expansion);
- some stepped surfaces almost equal to $\mathcal{P}_{(1,\vec{\alpha})}$ (idem);
- some non-plane stepped surfaces (only finite expansion).

Dual maps	Stp. planes & surfaces	Action of dual maps	Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application
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The stepped	plane case (finite or infin	ite expansion)			

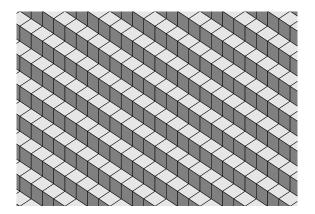
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(a, i) = (4, 1)

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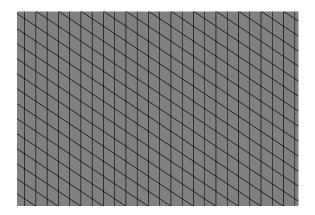
Dual maps 0000		Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application 0000
The stepped	plane case (finite or infir			



(a, i) = (1, 2)

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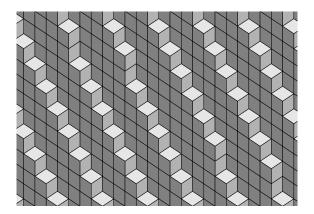
	Stp. planes & surfaces 0000		Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces ○●○○	Application 0000
The stepped	plane case (finite or infin	iite expansion)			



 $a = \infty$. Stepped plane recognized.

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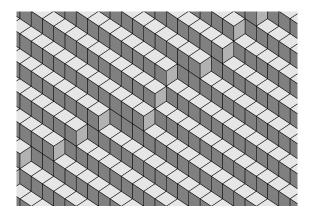
	Stp. planes & surfaces 0000		Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces ○○●○	Application 0000
The stepped	quasi-plane case (finite c	or infinite expansion)			



(a, i) = (4, 1)

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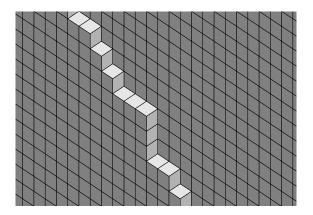
Dual maps 0000			Brun exp. of stp. planes	Brun exp. of stp. surfaces ○○●○	
The stepped	quasi-plane case (finite o	or infinite expansion)			



(a, i) = (1, 2)

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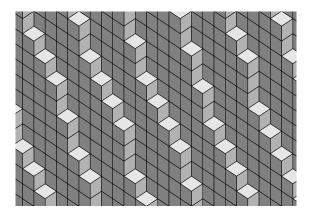
	Stp. planes & surfaces 0000		Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces ○○●○	Application 0000
The stepped	d quasi-plane case (finite c	or infinite expansion)			



 $a = \infty$. Not a stepped plane... but almost.

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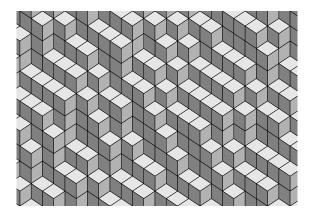
Dual maps 0000			Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces ○○○●	Application 0000
The stepped	surface case (only finite	expansion)			



(a, i) = (4, 1)

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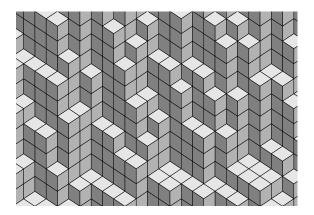
The stepped	surface case (only finite			
	Stp. planes & surfaces	Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application



(a,i)=(1,2)

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The stepped	surface case (only finite			
	Stp. planes & surfaces	Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces ○○○●	Application 0000



a undefined. Not at all a stepped plane.

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Stp. planes & surfaces	Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces	Application

Discrete geometry: change from \mathbb{R}^d (maths) to \mathbb{Z}^d (computer).

Vertices of a stepped plane: standard plane (widely used).

Challenge: vectorization of experimental geometrical data, that is:

• recognize pieces of plane in the data and store parameters;

• reconstruct pieces of plane from stored parameters.

Dual maps 0000		Brun exp. of stp. planes 0000	Brun exp. of stp. surfaces	Application •••••
Plane recogr	nition			

Expand the given stepped surface (suppose finite expansion).

The last obtained stepped surface is plane iff the first one does.

For finite patch, this yields finite expansion but we omit details.

Some boundary problems ~> mixed approach (preimage algorithm).

Dual maps 0000		Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application 0000
Plane genera	ation			

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How to construct $\mathcal{P}_{(1,\vec{\alpha})}$ for given $\vec{\alpha} \in \mathbb{Q}^d$?

• compute the Brun expansion $(a_n, i_n)_{0 \le n \le N}$ of $\vec{\alpha}$;

Dual maps 0000		Brun exp. of stp. planes	Brun exp. of stp. surfaces	Application 0000
Plane genera	ation			

How to construct $\mathcal{P}_{(1,\vec{\alpha})}$ for given $\vec{\alpha} \in \mathbb{Q}^d$?

- compute the Brun expansion $(a_n, i_n)_{0 \le n \le N}$ of $\vec{\alpha}$;
- compute $\mathcal{D}_{(1,\vec{\alpha})} = E_1^*(\beta_{a_0,i_0}^{-1}) \circ \ldots \circ E_1^*(\beta^{-1}a_N,i_N)(\vec{0},1^*).$ Then, $(\vec{0},1^*) \lhd \mathcal{P}_{(1,\vec{\alpha})}$ yields $\mathcal{D}_{(1,\vec{\alpha})} \lhd \mathcal{P}_{(1,\vec{\alpha})}.$

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• One has $\mathcal{P}_{(1,\vec{\alpha})} = \mathbb{Z}M_N\vec{e}_1 + \mathbb{Z}M_N\vec{e}_2 + \mathcal{D}_{\vec{\alpha}}$, where M_N incidence matrix of $\beta_{a_0,i_0}^{-1} \circ \ldots \circ \beta_{a_N,i_N}^{-1}$.

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For example:

$$\vec{\alpha} = \left(\frac{3}{8}, \frac{5}{12}\right) \rightsquigarrow [(2,2), (1,1), (2,2), (4,1), (1,2)] \rightsquigarrow \left(\frac{9}{24}, \frac{10}{24}\right)$$

 $\mathcal{D}_{(1,\vec{\alpha})}$ has 24, 9 and 10 faces of, respectively, types 1, 2 and 3.

It tiles periodically $\mathcal{P}_{(1,\vec{\alpha})}$ with periods (1,4,-6) and (2,-2,-3).

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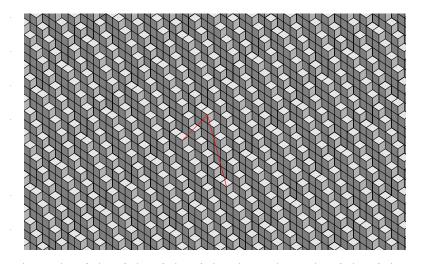




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