Toward an Algebraic Characterization of Substitutive Multidimensional Sturmian Sequences

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SSAO'05

1. Characterizations for words and tilings
   - Substitutions, continued fractions and return words
   - Pseudo-self-similarity and derived Voronoï tilings
2. Hyperplane sequences and generalized substitutions
   - Stepped hyperplanes and associated sequences
   - Generalized substitutions
3. Toward a characterization for hyperplane sequences
   - Multidimensional continued fractions
   - The case of periodic expansions

Alphabet $\mathcal{A} \rightsquigarrow$ words $\mathcal{A}^*$ and sequences $\mathcal{A}^\omega$.
Repetitive sequence: bounded gaps between occurences of a factor.

Substitution: morphism σ of $\mathcal{A}^* \cup \mathcal{A}^\omega$ s.t. $|\sigma^n(i)| \to \infty$ for $i \in \mathcal{A}$.
$\sigma : 1 \mapsto 12, 2 \mapsto 1$:

$$1 \to 12 \to 121 \to 12112 \to 12112121 \to 1211212112112 \to \ldots$$

Fixed-point: $u \in \mathcal{A}^\omega \mid u = \sigma(u)$.
Substitutive sequence: morphic image of a fixed-point.

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Line of $\mathbb{R}^2$ with a slope $\alpha \notin \mathbb{Q} \rightsquigarrow$ Sturmian sequence $u_\alpha \in \{1,2\}^\omega$.

$$\alpha = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1+\cfrac{1}{\ddots}} \quad \rightsquigarrow \quad u_\alpha = 12112121121121\cdots$$

Theorem (Algebraic characterization)

The Sturmian sequence u_α is a substitutive if and only if α has a ultimately periodic continued fraction expansion.

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Repetitive sequence $u \rightsquigarrow$ *locator sets* $\mathcal{L}_n(u) \subset \mathbb{N}$, *return words* $R_n(u) \subset \mathcal{A}^*$ and *derived sequences* $DV_n(u) \in \{1 \ldots |R_n(u)|\}^\omega$.

$$u = \underline{1211}21211211211212112\cdots$$

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Theorem (Durand)

A sequence u is substitutive if and only if $\{DV_n(u), n \in \mathbb{N}\}$ is finite.

Multidimensional generalization of sequences $\rightsquigarrow$ tilings of $\mathbb{R}^n$.
Repetitive tiling: bounded distance between occurences of a patch.

Expansion map: linear map Φ s.t. $|\Phi(x)| = \lambda|x|$, $\lambda > 1$.
Pseudo-self-similar tiling: $\mathcal{T}$ *locally derivable* from $\Phi(\mathcal{T})$.

Repetitive tiling $\mathcal{T} \rightsquigarrow$ *locator sets* $\mathcal{L}_r(\mathcal{T}) \subset \mathbb{R}^n$ and *derived Voronoï* tiling $DV_r(\mathcal{T})$.

(dessin)

Theorem (Solomyak-Priebe)

A non-periodic repetitive tiling $\mathcal{T}$ is pseudo-self-similar if and only if $\{DV_r(\mathcal{T}), r \in \mathbb{R}^+\}/\{\Phi^n, n \in \mathbb{N}\}$ is finite for an expansion map Φ.

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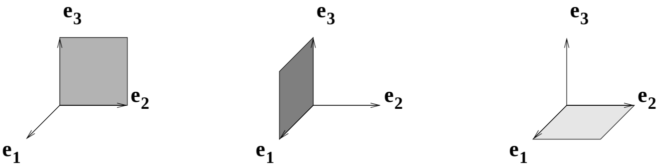
$\mathcal{A} = \{1, \ldots, n\}$ and $(\vec{e}_1, \ldots, \vec{e}_n)$ basis of $\mathbb{R}^n$. Face $(\vec{x}, i^*)$ of $\mathbb{R}^n$:



Definition (Stepped hyperplane)

$$\vec{\alpha} \in \mathbb{R}^n \rightsquigarrow \mathcal{S}_{\vec{\alpha}} = \{(\vec{x}, i^*) \mid 0 \le \langle \vec{x}, \vec{\alpha} \rangle < \langle \vec{e}_i, \vec{\alpha} \rangle\}.$$

$$\mathcal{P}_{\vec{\alpha}} = \{\vec{x} \in \mathbb{R}^n \mid \langle \vec{x}, \vec{\alpha} \rangle = 0\}.$$

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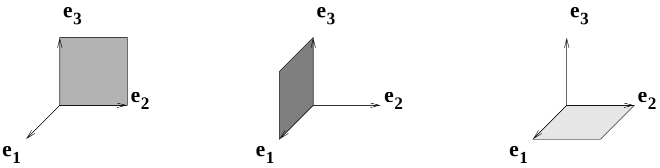

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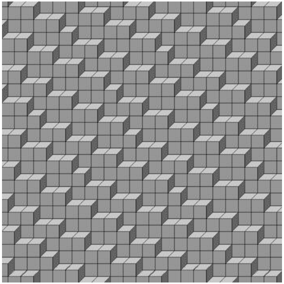
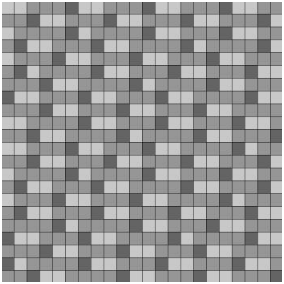
Theorem (Hyperplane sequences)

One can bijectively map the faces of the stepped hyperplane $\mathcal{S}_{\vec{\alpha}}$ to the letters of a $(n-1)$-dimensional sequence $\mathcal{U}_{\vec{\alpha}}$ over $\{1, \ldots, n\}$.

Coordinates of $\vec{\alpha}$ linearly independent over $\mathbb{Q} \Rightarrow \mathcal{U}_{\vec{\alpha}}$ *Sturmian*.



$\longleftrightarrow$



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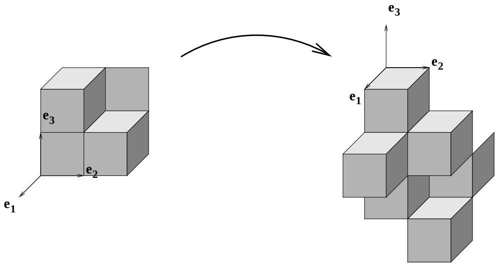
σ unimodular on $\mathcal{A} = \{1, \dots, n\} \rightsquigarrow \Theta(\sigma)$ on faces:

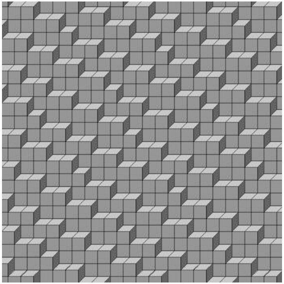
$$\Theta(\sigma)(\vec{x}, i^*) = M_\sigma^{-1}\vec{x} + \sum_{j \in \mathcal{A}} \sum_{s | \sigma(j) = p \cdot i \cdot s} (\vec{f}(s), j^*),$$

where $(M_\sigma)_{i,j} = |\sigma(j)|_i$ and $\vec{f}(u) = {}^t(|u|_1, |u|_2, |u|_3)$.

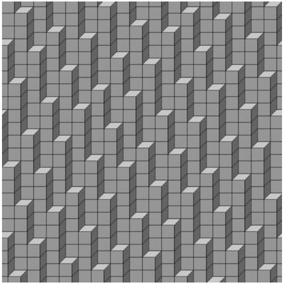


Theorem (Action of $\Theta(\sigma)$)

$$\vec{\alpha}' \propto {}^t M_\sigma \vec{\alpha} \Rightarrow \Theta^*(\sigma)(\mathcal{S}_{\vec{\alpha}}) = \mathcal{S}_{\vec{\alpha}'}.$$



$\longmapsto$



$$M_\sigma^{-1}\mathcal{P}_{\vec{\alpha}} = \mathcal{P}_{\vec{\alpha}'} \rightsquigarrow \Theta^*(\sigma) \text{ ``discretization'' of } M_\sigma^{-1}.$$

Modified Jacobi-Perron for $\vec{\alpha} \in [0,1)^n$:

$$\begin{aligned} \vec{\alpha} &= [(a_1,\varepsilon_1),\ldots,(a_k,\varepsilon_k),[\vec{\alpha}_k]] \\ &= [(a_1,\varepsilon_1),(a_2,\varepsilon_2),\ldots] \end{aligned}$$

where $a_i \in \mathbb{N}$, $\varepsilon_i \in \{1,\ldots,n\}$ and $\vec{\alpha_k} \in [0,1)^n$.
Matrix viewpoint:

$$\vec{\alpha}_{k-1} = \eta_k {}^t M_{\sigma_{(a_k,\varepsilon_k)}} \vec{\alpha}_k,$$

where $\eta_k \in \mathbb{R}$ and $\sigma_{(a_k,\varepsilon_k)}$ unimodular substitution on $\{1,\ldots,n\}$.

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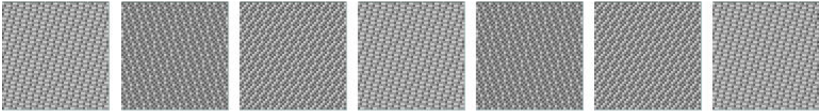
By theorem (Action of $\Theta(\sigma)$):

$$\Theta(\sigma_{(a_k,\varepsilon_k)})(\mathcal{S}_{\vec{\alpha}_k}) = \mathcal{S}_{\vec{\alpha}_{k-1}}.$$

Then, $\Theta(\sigma\sigma') = \Theta(\sigma')\Theta(\sigma)$ yields:

$$\Theta(\sigma_{(a_k,\varepsilon_k)} \ldots \sigma_{(a_1,\varepsilon_1)})(\mathcal{S}_{\vec{\alpha}_k}) = \mathcal{S}_{\vec{\alpha}}.$$



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$\vec{\alpha}_p = \vec{\alpha} \rightsquigarrow$ periodic expansion $\vec{\alpha} = [(a_1, \varepsilon_1), \ldots, (a_p, \varepsilon_p), [\vec{\alpha}]]$:

$$\Theta(\sigma_{(a_p,\varepsilon_p)} \cdots \sigma_{(a_1,\varepsilon_1)})(\underbrace{\mathcal{S}_{\vec{\alpha}_p}}_{\mathcal{S}_{\vec{\alpha}}}) = \mathcal{S}_{\vec{\alpha}}.$$

Theorem

The multidimensional Sturmian sequence $\mathcal{U}_{\vec{\alpha}}$ is a fixed-point if $\vec{\alpha}$ has a periodic multidimensional continued fraction expansion.

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