Generalized Substitutions and Stepped Surfaces

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Abstract. Generalized substitutions are defined by duality from classic substitutions on words. It is known that they act on aperiodic stepped hyperplanes - a multidimensional extension of sturmian words. In this paper, we show that they also act on stepped surfaces - an extension of stepped hyperplanes which could be considered as a multidimensional extension of two-letter words. We give some prospects of this extension.

Introduction

A finite (resp. infinite) word over an alphabet $\mathcal{A}$ is a finite (resp. infinite) concatenation of letters of $\mathcal{A}$. The set of finite (resp. infinite) words over $\mathcal{A}$ is denoted by $\mathcal{A}^*$ (resp. $\mathcal{A}^{\mathbb{N}}$). A *substitution on words* maps letters of $\mathcal{A}$ to non-empty finite words of $\mathcal{A}^*$, and maps a finite (or infinite) concatenation of letters onto the concatenation of the images of these letters. In other words, a substitution is a non-erasing morphism of the free monoid $\mathcal{A}^*$ (or $\mathcal{A}^{\mathbb{N}}$). A *sturmian word* is a two-letter infinite words which can be defined - among other possibilities - as digitization of an irrational straight line of the plane. Then, a *sturmian morphism* is a substitution on words which maps sturmian words to sturmian words (see e.g. [12], Chap. 2 and the references inside). We are here interested in a multidimensional extension of these notions.

It is rather natural to define a d-dimensional word over $\mathcal{A}$ as a d-dimensional array with coefficients in $\mathcal{A}$, that is, as an element of $\mathcal{A}^{\mathbb{Z}^d}$ (see e.g. [7]). Then, extending the definition of sturmian words as digitizations of irrational straight lines of $\mathbb{R}^2$, *sturmian d-dimensional words* are defined as digitizations of irrational hyperplanes of $\mathbb{R}^{d+1}$ (see [18, 5]). They are the d-dimensional words over a $(d+1)$-letter alphabet which encode the aperiodic stepped hyperplanes of [9].

Defining substitutions on multidimensional words is not straightforward. Duality has been used in [4] to associate to a classic substitution on words a so-called *generalized substitution*. Generalized substitutions map aperiodic stepped hyperplanes to aperiodic stepped hyperplanes and could be considered as a multidimensional extension of sturmian morphims (see [9]). However, it is worth noticing that sturmian morphisms act not only on sturmian words but on any *two-letter* infinite word. By analogy with this, we would like to define d-dimensional $(d+1)$-letter words, including sturmian d-dimensional words and on which act generalized substitutions. The main purpose of this paper is to show that the *stepped surfaces* introduced in [11] (see also [1]) provide such an extension.

Before summing up these multidimensional extensions, let us also mention the notion of *substitutions by local rules*, introduced in [3], which has been proved in [10] to extend the notion of generalized substitutions, and thus could play the role of multidimensional substitutions. So, writing from left to right the unidimensional notions and their extensions, we obtain:

Two-letters words	$\longrightarrow$	Stepped surfaces
Sturmian words	$\longrightarrow$	Stepped hyperplanes
Sturmian morphisms	$\longrightarrow$	Generalized substitutions
Substitutions	$\longrightarrow$	Substitutions by local rules?

The paper is organized as follows. The definition of *stepped hyperplanes* is first recalled in Section 1. Then, we define *stepped surfaces* in Section 2: they are obtained performing simple operations called *flips* on stepped hyperplanes. In Section 3, we restate the definition of *generalized substitutions* and we show that they map stepped surfaces to stepped surfaces: this is the main result of this paper. We conclude this paper giving in Section 4 some prospects concerning a possible extension of the definition of sturmian words as aperiodic two-letter words of minimal complexity.

1 Faces and stepped hyperplanes

Let $(\boldsymbol{e_1},\ldots,\boldsymbol{e_d})$ be the canonical basis of $\mathbb{R}^d$. The *face* $(\boldsymbol{x},i^*)$ is the subset of $\mathbb{R}^d$ defined for $\boldsymbol{x}\in\mathbb{Z}^d$ and $1\leq i\leq d$ by (see Fig. 1):

$$(\boldsymbol{x},i^*)=\{\boldsymbol{x}+\sum_{j\neq i}\lambda_j\boldsymbol{e_j},\ 0\leq\lambda_j\leq 1\}.$$

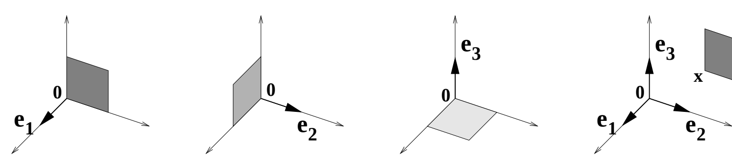

Fig. 1. From left to right, the faces $(\mathbf{0},1^*)$, $(\mathbf{0},2^*)$, $(\mathbf{0},3^*)$ and $(\boldsymbol{x},1^*)$ (in $\mathbb{R}^3$).

The set of faces is denoted by $\mathcal{F}$. Given two sets of faces $\mathcal{E}\subset\mathcal{F}$ and $\mathcal{E}'\subset\mathcal{F}$ (finite or not), let $B(0,r)$ be the largest ball - possibly the whole space - on which $\mathcal{E}$ and $\mathcal{E}'$ are equal (recall that both are subsets of $\mathbb{R}^d$). We set $d(\mathcal{E},\mathcal{E}')=2^{-r}$, and it is easy to check that it defines a distance d over sets of faces. We then use faces to approximate hyperplanes of $\mathbb{R}^d$ (see Fig. 2):

Definition 1. *A* stepped hyperplane *is a set of faces defined by:*

$$\mathcal{P}_{\boldsymbol{\alpha},\rho} = \{(\boldsymbol{x}, i^*) \mid \langle \boldsymbol{x}, \boldsymbol{\alpha} \rangle + \rho < 0 \leq \langle \boldsymbol{x} + \boldsymbol{e_i}, \boldsymbol{\alpha} \rangle + \rho\},$$

where $\rho \in \mathbb{R}$ and $\boldsymbol{\alpha} \in \mathbb{R}^d$ has strictly positive entries.

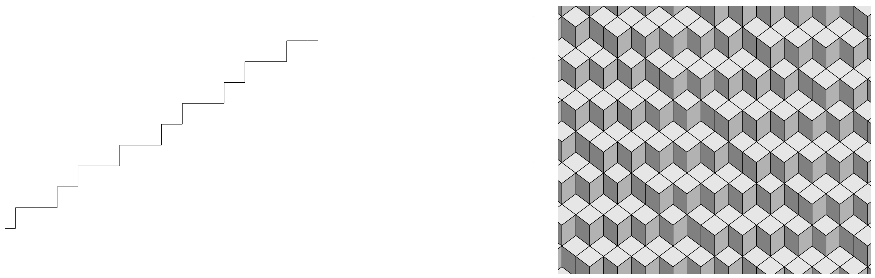

Fig. 2. Stepped hyperplanes for $d = 2$ (left) and $d = 3$ (right).

The case $d = 2$ corresponds to digitizations of straight lines of the plane and is inspired by the notion of *arithmetical discrete line* introduced in [17]. In this case, the faces are just horizontal or vertical unit segments, so that the whole stepped hyperplane (which is in fact a "stepped line") can be easily encoded over a two-letter words. In particular, when the slope of the line is irrational, this is a way to define sturmian words (see e.g. [12], Chap. 2 or the original paper [13]). It turns out that stepped hyperplanes can in fact be encoded by words in any dimension d (see [3] for $d = 3$ or [9] for any d). Indeed one has:

Proposition 1. *Let $v : \mathcal{F} \to \mathbb{Z}^d$ and $\pi : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ be the maps defined by:*

$$v(\boldsymbol{x}, i^*) = \boldsymbol{x} + \boldsymbol{e_1} + \ldots + \boldsymbol{e_i} \quad \text{and} \quad \pi(x_1, \ldots, x_d) = (x_1 - x_d, \ldots, x_{d-1} - x_d).$$

Then, $\pi \circ v$ maps bijectively the faces of a given stepped hyperplane onto $\mathbb{Z}^{d-1}$.

Thus, stepped hyperplanes of $\mathbb{R}^d$ can be encoded into $(d-1)$-dimensional words over the alphabet $\{1, \ldots, d\}$ by the following map (see Fig. 3):

$$\Psi \; : \; \begin{array}{ccc} \mathcal{F} & \to & \mathbb{Z}^{d-1} \times \{1, \ldots, d\} \\ (\boldsymbol{x}, i^*) & \to & (\pi \circ v(\boldsymbol{x}, i^*), i) \end{array}.$$

Remark 1. The map Ψ is not one-to-one, but "almost": one checks that two stepped hyperplanes are mapped onto the same multidimensional word if and only if they are equal up to a translation of vector $\lambda(\boldsymbol{e_1} + \ldots + \boldsymbol{e_d})$, $\lambda \in \mathbb{R}$. Conversely, it is easily seen that Ψ is not at all onto. For example, when $d = 2$, a word which contains both two consecutive 1's and two consecutive 2's cannot be obtained as the image by Ψ of a stepped hyperplane (or "stepped line").

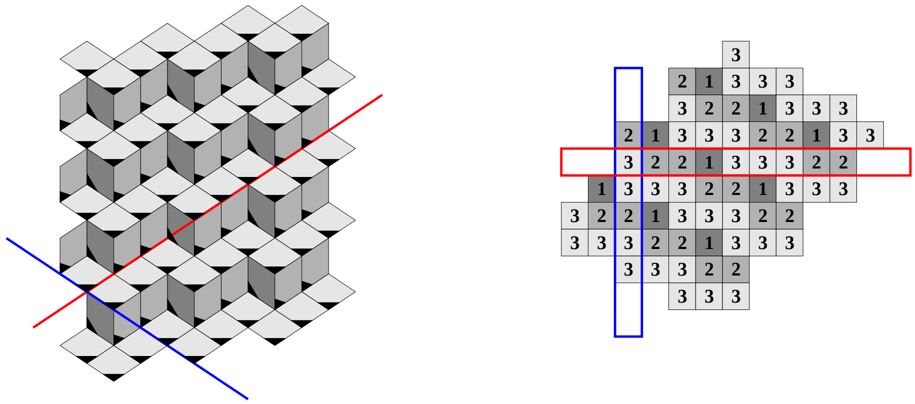

Fig. 3. Right: a subset of a stepped hyperplane; for each face $(\boldsymbol{x}, i^*)$ a black corner shows the vertex $v(\boldsymbol{x}, i^*)$. Left, the image by Ψ of this set of faces. For example, the faces whose blacked corner belongs to one of the two lines on the left are mapped to the corresponding framed letters on the right.

2 Flips and stepped surfaces

We extend here the notion of stepped hyperplane to the more general notion of stepped surface. We first define, for $\boldsymbol{x} \in \mathbb{Z}^d$, two specific sets of faces:

$$\check{c}_{\boldsymbol{x}} = \bigcup_{i=1}^{d} (\boldsymbol{x}, i^*) \qquad \text{and} \qquad \hat{c}_{\boldsymbol{x}} = \bigcup_{i=1}^{d} (\boldsymbol{x} + \boldsymbol{e}_i, i^*),$$

which are then used to define an operation on sets of faces (see Fig. 4, 5):

Definition 2. *A* flip $\varphi_{\boldsymbol{x}}$ *is the map defined for a set of faces* $\mathcal{E}$ *and* $\boldsymbol{x} \in \mathbb{Z}^d$ *by:*

$$\varphi_{\boldsymbol{x}}(\mathcal{E}) = \begin{cases} (\mathcal{E} \backslash \check{c}_{\boldsymbol{x}}) \cup \hat{c}_{\boldsymbol{x}} & \text{if } \check{c}_{\boldsymbol{x}} \subset \mathcal{E}, \\ (\mathcal{E} \backslash \hat{c}_{\boldsymbol{x}}) \cup \check{c}_{\boldsymbol{x}} & \text{if } \hat{c}_{\boldsymbol{x}} \subset \mathcal{E}, \\ \mathcal{E} & \text{otherwise.} \end{cases}$$

It is worth noticing that flips admit a natural interpretation when acting on stepped hyperplanes: considering a stepped hyperplane as towers of unit cubes, perform a flip corresponds to adding or removing a unit cube. By performing such flips, one also obtains sets of faces which are not stepped hyperplanes (see Fig. 4). This leads to define *stepped surfaces.*

Definition 3. *A* stepped surface $\mathcal{S}$ *is a set of faces obtained by performing a finite or infinite sequence of flips onto a stepped hyperplane* $\mathcal{P}$*:*

$$\mathcal{S} = \varphi_{x_N} \circ \ldots \circ \varphi_{x_1}(\mathcal{P}) \qquad \text{or} \qquad \mathcal{S} = \lim_{n \to \infty} \varphi_{x_n} \circ \ldots \circ \varphi_{x_1}(\mathcal{P}).$$

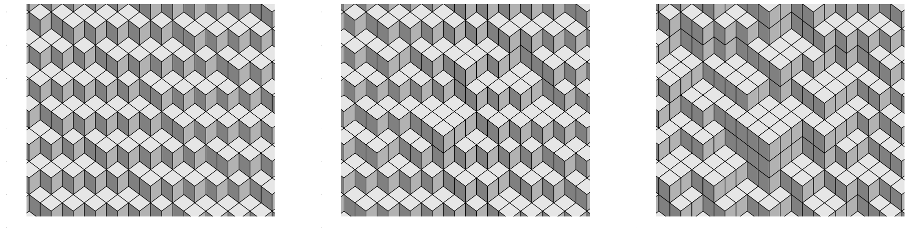

Fig. 4. From left to right: some flips on a stepped hyperplane.

Notice that in the previous definition, the notion of limit follows from the distance defined in Sec. 1. Stepped surfaces share with stepped hyperplanes the property that they can be encoded by multidimensional words. Indeed, one checks:

$$\forall \boldsymbol{x} \in \mathbb{Z}^d, \qquad \pi \circ v(\hat{c}_{\boldsymbol{x}}) = \pi \circ v(\check{c}_{\boldsymbol{x}}),$$

where π and v are the maps defined in Prop. 1. Hence, by performing a flip on a stepped hyperplane, $\pi \circ v$ still maps bijectively the obtained set of faces onto $\mathbb{Z}^{d-1}$. By induction on the number of flips, $\pi \circ v$ maps bijectively the faces of any stepped surface onto $\mathbb{Z}^{d-1}$: this extends Prop. 1 to stepped surfaces. Then, the map Ψ allows to encode stepped surfaces, as well as stepped hyperplanes, by $(d-1)$-dimensional words over $\{1,\ldots,d\}$. Fig. 5 and 6 illustrate this.

Fig. 5. Left: a flip exchanges $\check{c}_{\boldsymbol{x}}$ and $\hat{c}_{\boldsymbol{x}}$. Right: encoded by Ψ, this corresponds to a permutation of letters.

Remark 2. As for stepped hyperplanes (Rem. 1), Ψ is "almost" one-to-one: two stepped surfaces are mapped onto the same multidimensional word if and only if they are equal up to a translation of vector $\lambda(\boldsymbol{e_1} + \ldots + \boldsymbol{e_d})$, $\lambda \in \mathbb{R}$. Moreover, Ψ is onto when $d = 2$. Indeed, any word over $\{1, 2\}$ indexed by $\mathbb{Z}$ can be seen as a sequence of horizontal and vertical unit segments, that is, as a stepped surface (or stepped line). It is however false for $d > 2$ (see e.g. [11]).

3 Generalized substitutions and stepped surfaces

Here, we briefly recall the notion of *generalized substitutions* and we show how they act on stepped hyperplanes and stepped surfaces.

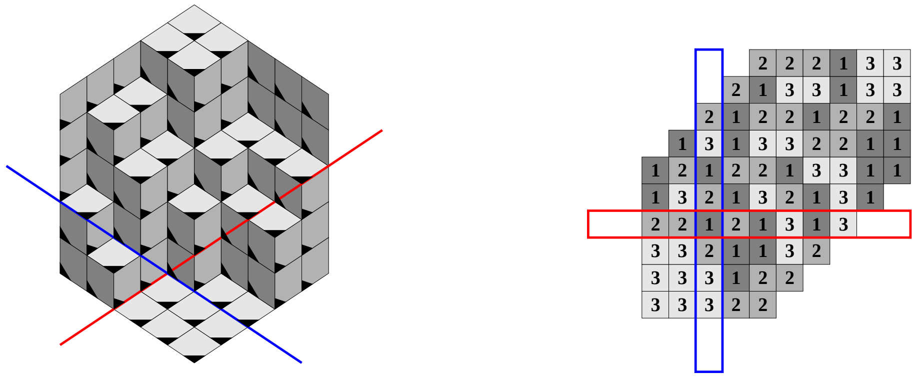

Fig. 6. The analogue of Fig. 3 for a stepped surface.

Let us recall that the *incidence matrix* of a substitution σ over $\{1,\ldots,n\}$ is the $n \times n$ integer matrix whose coefficient at row i and column j is the number of occurences of the letter i in the word $\sigma(j)$. A substitution σ is then said *unimodular* if $\det M_\sigma = \pm 1$. In particular, M_σ^{-1} has *integer* coefficients.

We also denote by $\boldsymbol{f}(u)$ the vector of $\mathbb{Z}^n$ whose i-th coordinate is the number of occurences of the letter i in the word u. In particular, $\boldsymbol{f}(\sigma(i)) = M_\sigma \boldsymbol{e_i}$.

Definition 4 ([4]). *Given a unimodular substitution* σ *over* $\{1,\ldots,n\}$*, the* generalized substitution Θ_σ^* *is defined over (non-empty) sets of faces as follows:*

$$\forall \mathcal{E},\mathcal{E}' \subset \mathcal{F}, \qquad \Theta_\sigma^*(\mathcal{E}\cup\mathcal{E}') = \Theta_\sigma^*(\mathcal{E}) \cup \Theta_\sigma^*(\mathcal{E}'),$$

$$\forall (\boldsymbol{x}, i^*) \in \mathcal{F}, \qquad \Theta_\sigma^*(\{(\boldsymbol{x}, i^*)\}) = \bigcup_{\substack{j,p,s \\ |\sigma(j)=p\cdot i\cdot s}} \left(M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s)), j^*\right).$$

Let us consider for example the substitution $\sigma : 1 \to 13, 2 \to 1, 3 \to 2$. One has:

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad M_\sigma^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and this yields (see Fig. 7):

$$\Theta_\sigma^* \ : \ \begin{array}{l} \{(\boldsymbol{x}, 1^*)\} \mapsto \{(M_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e_1} - \boldsymbol{e_2}, 1^*), (M_\sigma^{-1}\boldsymbol{x}, 2^*)\} \\ \{(\boldsymbol{x}, 2^*)\} \mapsto \{(M_\sigma^{-1}\boldsymbol{x}, 3^*)\} \\ \{(\boldsymbol{x}, 3^*)\} \mapsto \{(M_\sigma^{-1}\boldsymbol{x}, 1^*)\}. \end{array}$$

Notice that one has:

$$\Theta_\sigma^*(\{(\boldsymbol{0}, 1^*)\}) = \{(\boldsymbol{e_1} - \boldsymbol{e_2}, 1^*), (\boldsymbol{0}, 2^*)\} \qquad \text{and} \qquad \Theta_\sigma^*(\{(\boldsymbol{e_3}, 3^*)\}) = \{(\boldsymbol{e_1} - \boldsymbol{e_2}, 1^*)\},$$

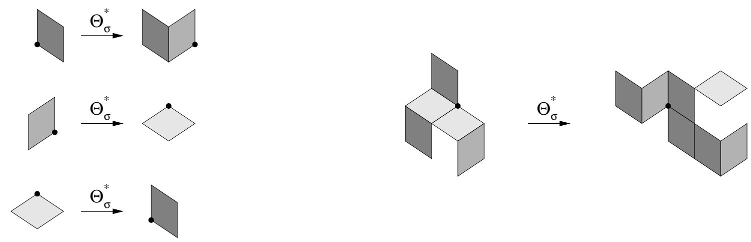

Fig. 7. Action of Θ_σ^* on single faces and on a small set of faces.

that is, $\Theta_\sigma^*(\{(\mathbf{0}, 1^*)\}) \cap \Theta_\sigma^*(\{(\boldsymbol{e_3}, 3^*)\}) \neq \emptyset$. So, a generalized substitution does not necessarily map different faces onto *disjoint* sets of faces. Thus, it is worth stressing when it does:

Definition 5. *A generalized substitution Θ_σ^* is said to map* properly *a set of faces $\mathcal{E}$ onto $\Theta_\sigma^*(\mathcal{E})$ if it maps distinct faces of $\mathcal{E}$ to* disjoint *subsets of $\Theta_\sigma^*(\mathcal{E})$.*

In this regard, stepped hyperplanes are particularly interesting:

Theorem 1 ([9]). *A generalized substitution maps properly a stepped hyperplane onto a stepped hyperplane. More precisely, the generalized substitution Θ_σ^* maps properly the stepped hyperplane $\mathcal{P}_{\boldsymbol{\alpha},\rho}$ onto the stepped hyperplane $\mathcal{P}_{{}^tM_\sigma\boldsymbol{\alpha},\rho}$.*

Let us now consider the case of stepped surfaces. The following lemma links the action of a generalized susbtitution and the action of a flip (see Fig. 8):

Lemma 1. *If a generalized substitution Θ_σ^* maps properly a set of faces $\mathcal{E}$ onto $\Theta_\sigma^*(\mathcal{E})$, then Θ_σ^* maps properly $\varphi_{\boldsymbol{x}}(\mathcal{E})$ onto $\varphi_{M_\sigma^{-1}\boldsymbol{x}}(\Theta_\sigma^*(\mathcal{E}))$, for any $\boldsymbol{x} \in \mathbb{R}^d$ such that $\varphi_{\boldsymbol{x}}(\mathcal{E}) \neq \mathcal{E}$.*

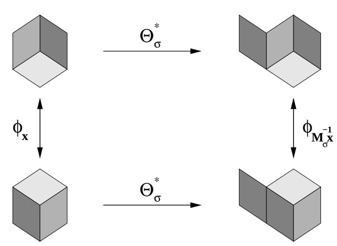

Fig. 8. If two sets of faces differ by a flip $\varphi_{\boldsymbol{x}}$, then their images by Θ_σ^* differ by a flip $\varphi_{M_\sigma^{-1}\boldsymbol{x}}$ (here: $\sigma\ :\ 1 \to 13, 2 \to 1, 3 \to 2$).

The next lemma allows to extend this result to the action of a *sequence* of flips:

Lemma 2. *If a sequence* $(\mathcal{E}_n)$ *of sets of faces admits a limit, then:*

$$\lim_{n\to\infty} \Theta_\sigma^*(\mathcal{E}_n) = \Theta_\sigma^*(\lim_{n\to\infty} \mathcal{E}_n).$$

Proof. (Sketch) The result follows once we proved:

$$d(\Theta_\sigma^*(\mathcal{E}), \Theta_\sigma^*(\mathcal{E}')) \leq 2^{-\rho_\sigma} d(\mathcal{E}, \mathcal{E}'),$$

where ρ_σ denotes the spectral radius of M_σ. □

We are now in a position to prove the following theorem (which can also be stated in terms of multidimensional words thanks to the encoding map Ψ):

Theorem 2. *A generalized substitution maps properly a stepped surface onto a stepped surface.*

Proof. Let us consider a stepped surface $\mathcal{S} = \lim_{n\to\infty} \varphi_{\boldsymbol{x}_n} \circ \ldots \circ \varphi_{\boldsymbol{x}_1}(\mathcal{P})$, where $\mathcal{P}$ is a stepped hyperplane. According to Lem. 1 and 2, one has:

$$\begin{aligned}
\Theta_\sigma^*(\mathcal{S}) &= \Theta_\sigma^* \left(\lim_{n\to\infty} \varphi_{\boldsymbol{x}_n} \circ \ldots \circ \varphi_{\boldsymbol{x}_1}(\mathcal{P}) \right) \\
&= \lim_{n\to\infty} \Theta_\sigma^*(\varphi_{\boldsymbol{x}_n} \circ \ldots \circ \varphi_{\boldsymbol{x}_1}(\mathcal{P})) \\
&= \lim_{n\to\infty} \varphi_{M_\sigma^{-1}\boldsymbol{x}_n} \circ \ldots \circ \varphi_{M_\sigma^{-1}\boldsymbol{x}_1}(\Theta_\sigma^*(\mathcal{P}))
\end{aligned}$$

The result follows, since Th. 1 yields that Θ_σ^* maps properly $\mathcal{P}$ onto the stepped hyperplane $\Theta_\sigma^*(\mathcal{P})$. □

Fig. 9 illustrates this. Th. 2 improves the result of [2], which proves for a *particular* generalized substitution that it maps a stepped surface to a *subset* of a stepped surface. The present improvement mainly relies on the notion of flip, a promising tool newly introduced here in such a context.

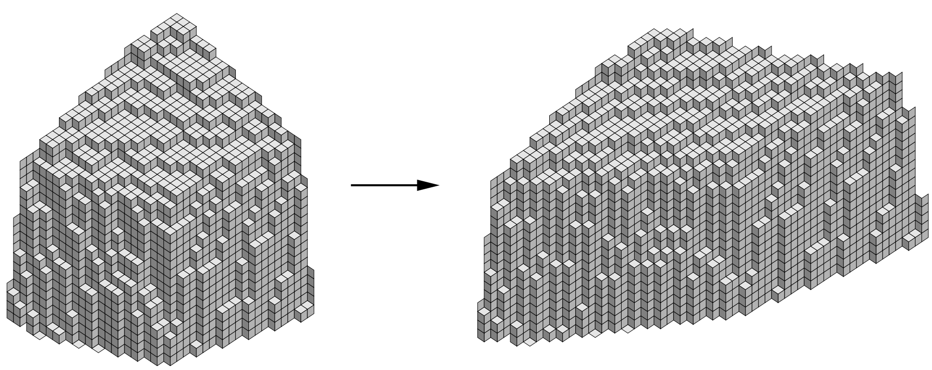

Fig. 9. Generalized substitutions map stepped surfaces onto stepped surfaces.

4 Prospects

It is known that an infinite word is periodic if and only if it admits, for $n_0 \in \mathbb{N}$, no more than n_0 distinct factors of length n_0 (a factor of length k consists in k consecutive letters). Hence, an infinite word with $n+1$ distinct factors of length n for any $n \in \mathbb{N}$ - one says *with complexity* $n+1$ - is aperiodic and there is no infinite word with a smaller complexity. Such words turn out to be exactly the words defined by digitizations of irrational straight half-lines, that is, the sturmian words (see [13]).

A similar result holds for bi-infinite words, that is, 1-dimensional words. Indeed, sturmian 1-dimensional words can be defined equivalently as the digitizations of irrational straight lines or as the *recurrent* aperiodic 1-dimensional words with minimal complexity (recall that a word is recurrent if any of its factors occurs with unbounded indices). Moreover, the *non-recurrent* aperiodic 1-dimensional words with minimal complexity are also characterized (see [8]).

However, no similar result yet exists for d-dimensional words ($d > 1$). Let us mention a conjecture due to M. Nivat in [14], which states that a 2-dimensional word having no more than $m_0 n_0$ distinct rectangular factors of size $m_0 \times n_0$, for at least one couple $(m_0, n_0) \in \mathbb{Z}^2$, is necessarily periodic. If this conjecture would hold true (see e.g. [15, 16] for partial results), the aperiodic 2-dimensional words of "minimal complexity" could be defined as those with rectangular complexity $mn+1$ for any $(m,n) \in \mathbb{Z}^2$. Nevertheless, these 2-dimensional words, characterized in [6], do not correspond to sturmian 2-dimensional words defined by digitizations of real planes. Thus, we do not get in this way a simultaneous multidimensional extension of both definitions by digitizations and complexity of sturmian words.

To get round this, the stepped surfaces studied in this paper could be helpful. Indeed, since sturmian 1-dimensional words can be defined as the recurrent aperiodic *two-letter* words of minimal complexity, considering stepped surfaces as a multidimensional extension of two-letter 1-dimensional words leads naturally to the following conjecture:

Conjecture 1. The recurrent aperiodic stepped surfaces of minimal complexity are the aperiodic stepped hyperplanes.

Notice that we do not have here defined precisely the notions of complexity and recurrence. In particular, it could be worth considering notions specific to stepped surfaces instead of using definitions which hold for general d-dimensional words (as the rectangular complexity). Once these notions are fixed, it would also be interesting to characterize all the stepped surfaces with the same complexity as aperiodic stepped hyperplanes, as done in [8] for sturmian 1-dimensional words, and similarly to the work of [6] relatively to the Nivat's conjecture.

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Appendix

Proof of Lemma 1:
In the one hand, one computes:

$$\begin{aligned}
\Theta^*_\sigma(\check{c}_{\boldsymbol{x}}) &= \bigcup_{i} \Theta^*_\sigma(\{(\boldsymbol{x}, i^*)\}) \\
&= \bigcup_{\substack{j,p,i,s \\ |\sigma(j)=p\cdot i\cdot s}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s)), j^*) \\
&= \bigcup_{\substack{j,p'\neq\varepsilon,s \\ |\sigma(j)=p'\cdot s}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s)), j^*) \\
&= \bigcup_{\substack{j,p'\neq\varepsilon,s\neq\varepsilon \\ |\sigma(j)=p'\cdot s}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s)), j^*) \cup \underbrace{\bigcup_{j} (M_\sigma^{-1}\boldsymbol{x}, j^*)}_{\check{c}_{M_\sigma^{-1}\boldsymbol{x}}} .
\end{aligned}$$

On the other hand, using that $\boldsymbol{e_i} = \boldsymbol{f}(i)$ and $\boldsymbol{f}(\sigma(j)) = M_\sigma \boldsymbol{e_j}$, one computes:

$$\begin{aligned}
\Theta^*_\sigma(\hat{c}_{\boldsymbol{x}}) &= \bigcup_{i} \Theta^*_\sigma(\{(\boldsymbol{x} + \boldsymbol{e_i}, i^*)\}) \\
&= \bigcup_{\substack{j,p,i,s \\ |\sigma(j)=p\cdot i\cdot s}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{e_i} + \boldsymbol{f}(s)), j^*) \\
&= \bigcup_{\substack{j,p,i,s \\ |\sigma(j)=p\cdot i\cdot s}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(i\cdot s)), j^*) \\
&= \bigcup_{\substack{j,p,s'\neq\varepsilon \\ |\sigma(j)=p\cdot s'}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s')), j^*) \\
&= \bigcup_{\substack{j,p\neq\varepsilon,s'\neq\varepsilon \\ |\sigma(j)=p\cdot s'}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s')), j^*) \cup \bigcup_{j} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(\sigma(j))), j^*) \\
&= \bigcup_{\substack{j,p\neq\varepsilon,s'\neq\varepsilon \\ |\sigma(j)=p\cdot s'}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s')), j^*) \cup \bigcup_{j} (M_\sigma^{-1}(\boldsymbol{x} + M_\sigma \boldsymbol{e_j}), j^*) \\
&= \bigcup_{\substack{j,p\neq\varepsilon,s'\neq\varepsilon \\ |\sigma(j)=p\cdot s'}} (M_\sigma^{-1}(\boldsymbol{x} + \boldsymbol{f}(s')), j^*) \cup \underbrace{\bigcup_{j} (M_\sigma^{-1}\boldsymbol{x} + \boldsymbol{e_j}, j^*)}_{\hat{c}_{M_\sigma^{-1}\boldsymbol{x}}}
\end{aligned}$$

The result easily follows. □