Brun expansions of stepped surfaces

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WORDS'07, September 18, 2007

Main result:

Action of *dual maps of free group morphisms* over stepped planes and surfaces (extends substitutions on words).

Applications :

Brun expansions of stepped planes and surfaces.

Recognition of stepped planes among stepped surfaces.

1. Stepped planes and stepped surfaces
2. Dual maps of free group morphisms
3. Brun expansions of stepped planes
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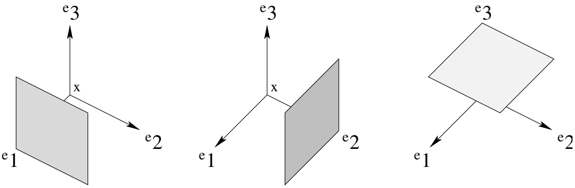
$(\vec{e}_1, \ldots, \vec{e}_d)$ basis of $\mathbb{R}^d$. $\vec{x} \in \mathbb{Z}^d$, $i \in \{1, \ldots, d\} \rightsquigarrow$ *face* $(\vec{x}, i^*)$:

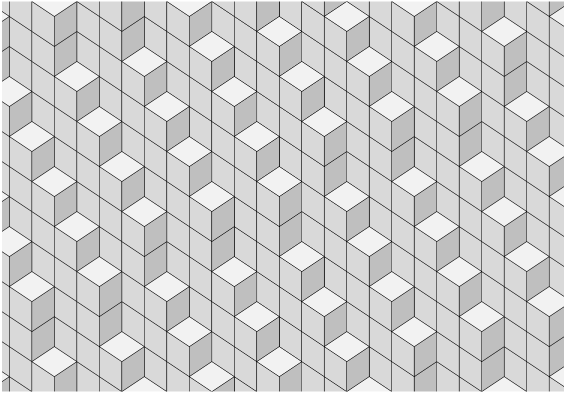


Definition

Stepped plane of normal vector $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \{0\}$:

$$\mathcal{P}_{\vec{\alpha}} = \{(\vec{x}, i^*) \mid \langle \vec{x}, \vec{\alpha} \rangle \leq 0 < \langle \vec{x} + \vec{e}_i, \vec{\alpha} \rangle\}.$$

Stepped plane



A stepped plane.

Let π be the orthogonal projection along $\vec{u} = \vec{e}_1 + \ldots + \vec{e}_d$.

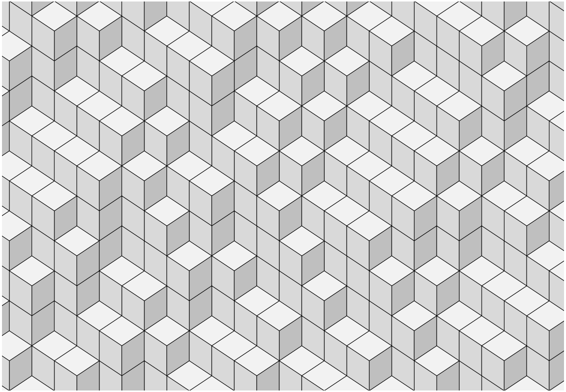
Proposition

Stepped planes are homeomorphic to $\vec{u}^{\perp}$ by π.

By extension:

Definition [Jamet]

Stepped surfaces : any set of faces homeomorphic to $\vec{u}^{\perp}$ by π.



A stepped surface.

1 Stepped planes and stepped surfaces

2 Dual maps of free group morphisms

3 Brun expansions of stepped planes

4 Brun expansions of stepped surfaces

Morphism of the free group over $\{1, \ldots, d\}$ (here, $d = 3$):

$$\sigma \ : \ \begin{cases} 1 & \mapsto & 3 \\ 2 & \mapsto & 3^{-1}1 \\ 3 & \mapsto & 3^{-1}2 \end{cases}$$

For example: $\sigma(1^{-1}312) = \sigma(1)^{-1}\sigma(3)\sigma(1)\sigma(2) = 3^{-2}21$.

Incidence matrix: $(M_\sigma)_{ij} = |\sigma(i)|_j - |\sigma(i)|_{j^{-1}}$. Here:

$$M_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.$$

σ *unimodular* f. g. morph. $\rightsquigarrow$ dual map $E_1^*(\sigma)$ (Arnoux-Ito, Ei).

$E_1^*(\sigma)$: linear map over weighted sums of faces.

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For σ previously defined:

$$E_1^*(\sigma) \ : \ \begin{cases} (\vec{0}, 1^*) & \mapsto & (\vec{e}_1, 2^*) \\ (\vec{0}, 2^*) & \mapsto & (\vec{e}_1, 3^*) \\ (\vec{0}, 3^*) & \mapsto & (\vec{0}, 1^*) - (\vec{e}_1, 2^*) - (\vec{e}_1, 3^*). \end{cases}$$

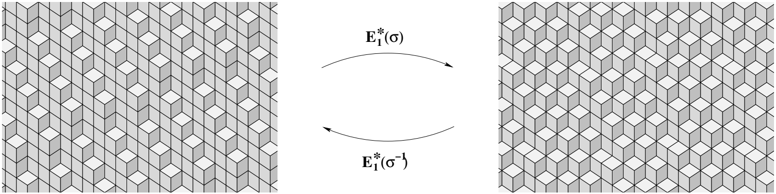
and, for $\lambda \in \mathbb{Z}$, $\vec{x} \in \mathbb{Z}^d$:

$$E_1^*(\sigma)(\lambda.(\vec{x}, i^*)) = M_\sigma^{-1}\vec{x} + \lambda.E_1^*(\sigma)(\vec{0}, i^*).$$

Theorem (B. F. 2007)

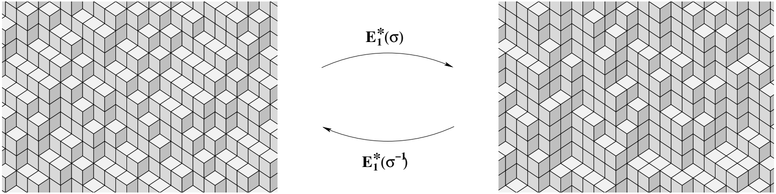
For σ unimodular free group morphism and $\vec{\alpha} \in \mathbb{R}^d_+ \setminus \vec{0}$:

$$M_\sigma^\top \vec{\alpha} \in \mathbb{R}^d_+ \;\Rightarrow\; E_1^*(\sigma)(\mathcal{P}_{\vec{\alpha}}) = \mathcal{P}_{M_\sigma^\top \vec{\alpha}}.$$



Theorem (B. F. 2007)

For σ unimodular free group morphism: if the image by $E_1^*(\sigma)$ of a stepped surface has faces with weights in $\{0,1\}$, then it is a stepped surface. This holds, in particular, when $M_\sigma \geq 0$.



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Brun map T, defined for $\vec{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \backslash \{0\}$:

$$T(\alpha_1, \ldots, \alpha_d) = \left(\frac{\alpha_1}{\alpha_i}, \ldots, \frac{\alpha_{i-1}}{\alpha_i}, \frac{1}{\alpha_i} - a, \frac{\alpha_{i+1}}{\alpha_i}, \ldots, \frac{\alpha_d}{\alpha_i} \right),$$

where $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_\infty\}$ and $a = \lfloor 1/\alpha_i \rfloor$. Matrix viewpoint:

$$(1, \vec{\alpha})^\top \propto B_{a,i}(1, T(\vec{\alpha}))^\top \quad \text{with} \quad B_{a,i} = \begin{pmatrix} a & & 1 & \\ & \mathrm{I}_{i-1} & & \\ 1 & & 0 & \\ & & & \mathrm{I}_{d-i} \end{pmatrix}.$$

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Brun expansion of $\vec{\alpha}$: sequence $(a_n, i_n)_{n \geq 0}$ of $\mathbb{N}^* \times \{1, \ldots, d\}$:

$$a_n = \left\lfloor ||T^n(\vec{\alpha})||_\infty^{-1} \right\rfloor \quad \text{and} \quad i_n = \min\{j \mid \langle T^n(\vec{\alpha}) | \vec{e}_j \rangle = ||T^n(\vec{\alpha})||_\infty\}.$$

Let $\beta_{a,i}$ be an automorphism with incidence matrix $B_{a,i}$ (it exists).

If $i = \min\{j \mid \alpha_j = ||\vec{\alpha}||_\infty\}$ and $a = \lfloor 1/\alpha_i \rfloor$ are known:

$$E_1^*(\beta_{a,i}^{-1})(\mathcal{P}_{(1,\vec{\alpha})}) = \mathcal{P}_{(1,T(\vec{\alpha}))}.$$

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Brun expansions could be computed directly over stepped planes by "reading" (a,i). By abuse: *Brun expansions of stepped planes*.

Note: we do not need to know $\vec{\alpha}$ but just to perform entries comparisons and floor computation.

Definition

An (i,j)-*run* of length a is a set of faces of the form:

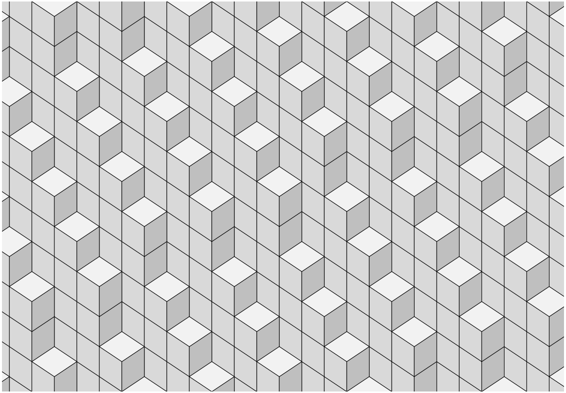
$$\{(\vec{x} + k\vec{e}_j, i^*) \mid 0 \leq k < a\}.$$

entries comparisons:

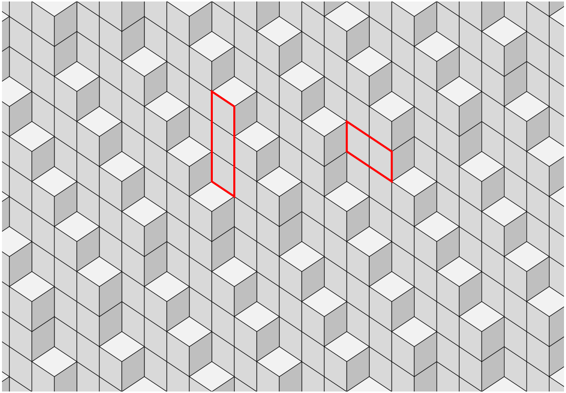
$\mathcal{P}_{\vec{\alpha}}$ admits an (i,j)-run of length at least 2 iff $\alpha_i > \alpha_j$.

floor computation:

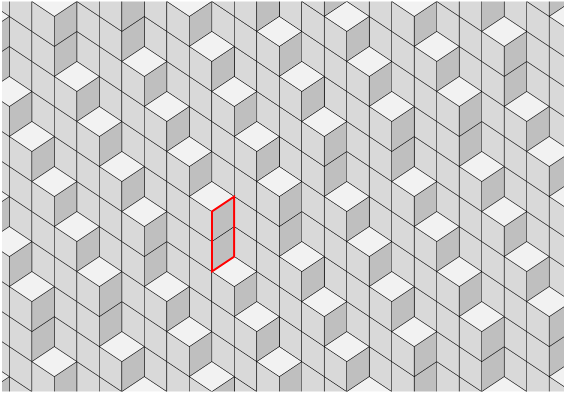
The smallest (i,j)-run of $\mathcal{P}_{\vec{\alpha}}$ has length $\max(\lfloor \alpha_i / \alpha_j \rfloor, 1)$.



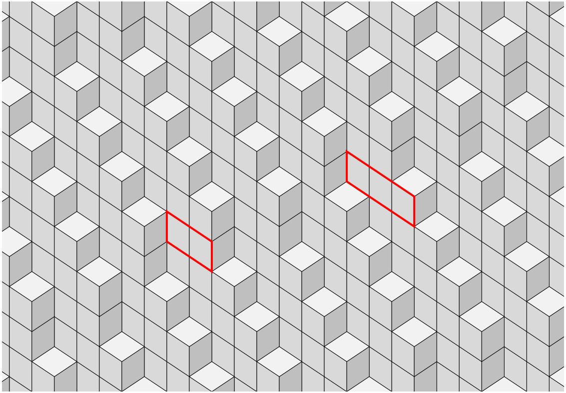
Stepped plane $\mathcal{P}_{(1,\alpha,\beta)}$, with unkown $\alpha, \beta \geq 0$.



$(1,2)$-run and $(1,3)$-run of length at least 2 $\rightsquigarrow$ $(\alpha, \beta) \in [0,1]^2$

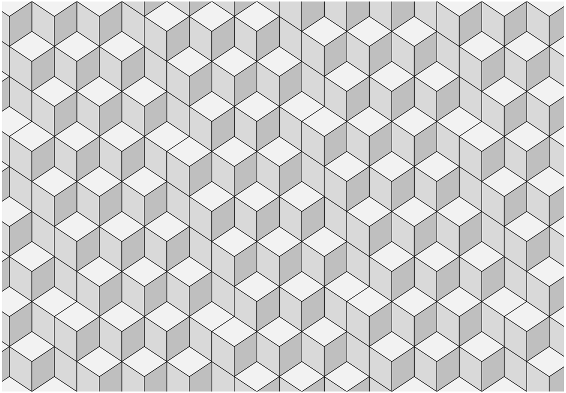


$(2,3)$-run of length $2 \rightsquigarrow \alpha > \beta \rightsquigarrow i = 1$.



Smallest $(1, 2)$-run of length 2 $\rightsquigarrow$ $a = \lfloor 1/\alpha \rfloor = 2$.

From vectors to stepped planes



Finally: $E_1^*(\beta_{2,1}^{-1}(\mathcal{P}_{(1,\alpha,\beta)}) = \mathcal{P}_{(1,T(\alpha,\beta))}$.

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Reading over stepped planes $\rightsquigarrow$ Brun exp. of stepped planes.

By analogy (runs and dual maps are still defined):

Reading over stepped surfaces $\rightsquigarrow$ Brun exp. of stepped surfaces.

Relation with Brun exp. of vectors? (no more normal vectors)

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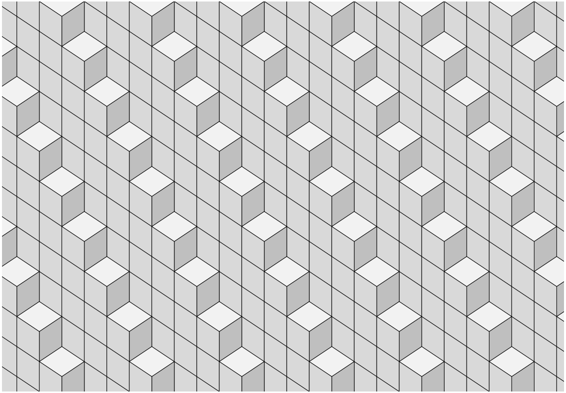
Relation with Brun exp. of vectors? (no more normal vectors)

Theorem (B. F. 2007)

Stepped surfaces having the Brun expansion of $\vec{\alpha} \in \mathbb{R}^d_+ \backslash \{0\}$ are:

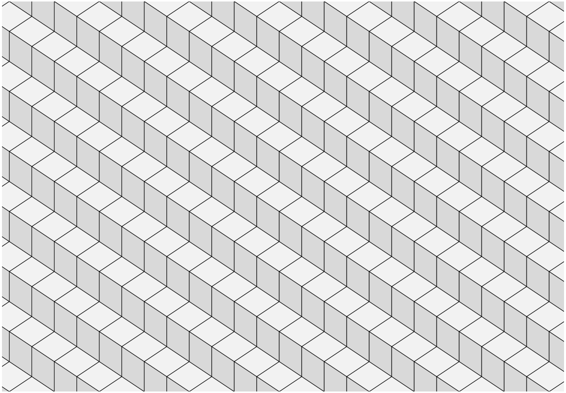
- *the stepped plane $\mathcal{P}_{(1,\vec{\alpha})}$ (finite or infinite expansion);*
- *some stepped surfaces almost equal to $\mathcal{P}_{(1,\vec{\alpha})}$ (idem);*
- *some non-plane stepped surfaces (only finite expansion).*

The stepped plane case (finite or infinite expansion)



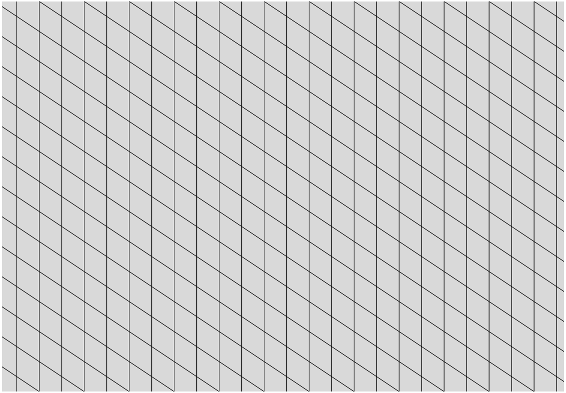
$$(a, i) = (4, 1)$$

The stepped plane case (finite or infinite expansion)



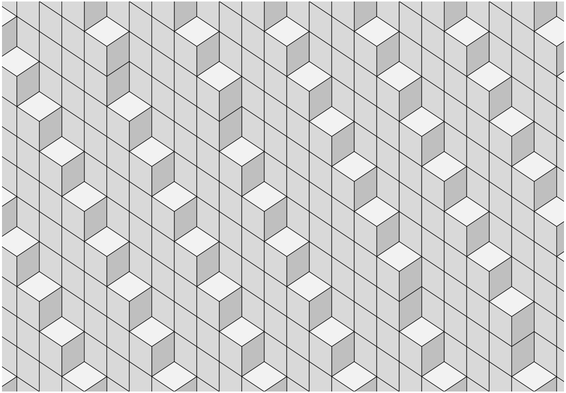
$$(a, i) = (1, 2)$$

The stepped plane case (finite or infinite expansion)



$a = \infty$: (rational) stepped plane recognized.

The stepped quasi-plane case (finite or infinite expansion)



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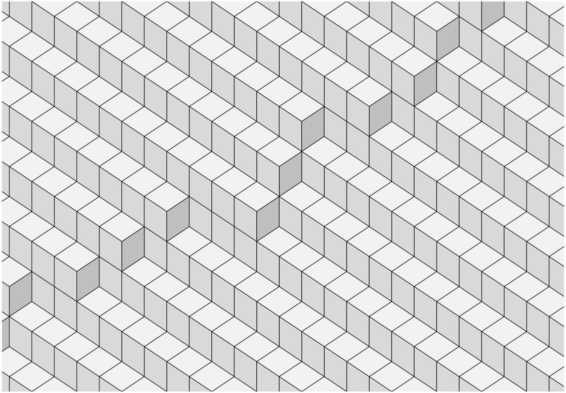

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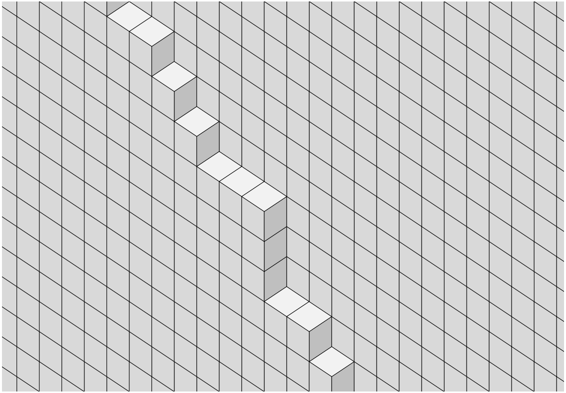
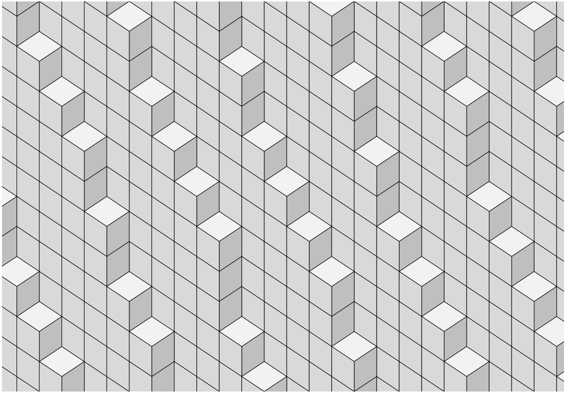


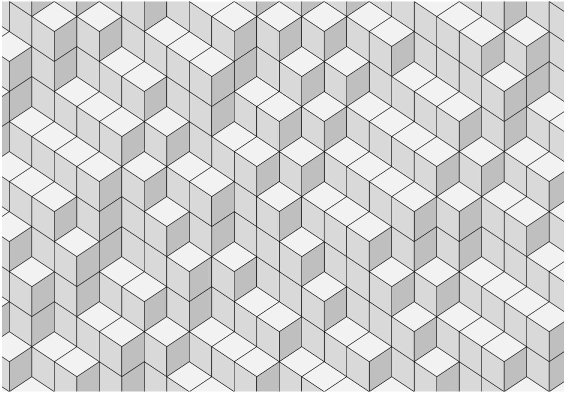
$a = \infty$: not a stepped plane. . . but almost.

The stepped surface case (only finite expansion)



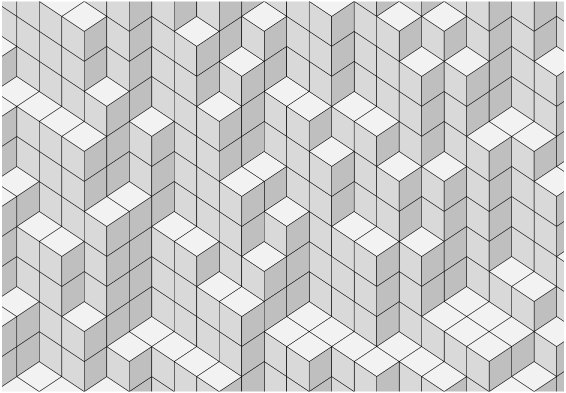
$$(a, i) = (4, 1)$$

The stepped surface case (only finite expansion)



$$(a, i) = (1, 2)$$

The stepped surface case (only finite expansion)



(a, i) undefined: not at all a stepped plane.

Where is "digital plane recognition"?

A stepped surface is a rational stepped plane iff it has a finite Brun expansion, with the last obtained stepped surface being $\mathcal{P}_{(1,\vec{0})}$.

Can be extended for finite subset of stepped surfaces.