# Correctness of Multiplicative Additive Proof Structures is $NL$-Complete

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## Abstract

*We revisit the correctness criterion for the multiplicative additive fragment of linear logic. We prove that deciding the correctness of corresponding proof structures is NL-complete.*

## Introduction

The *proof nets* [5, 3] of Linear logic (LL) are a parallel syntax for logical proofs without all the bureaucracy of sequent calculus. They are a non-sequential graph-theoretic representation of proofs, where the order in which some rules are used in a sequent calculus derivation, when irrelevant, is neglected. The unit-free multiplicative additive proof nets are inductively defined from sequent calculus rules of unit-free Multiplicative Additive Linear Logic (MALL[1]). The MALL *proof structures* are freely built on the same syntax as proof nets, without any reference to a sequent calculus derivation. The same holds for MLL and MELL proof nets and proof structures with respect to MLL and MELL sequent calculus.

In LL we are mainly interested in the following decision problems: Deciding the provability of a given formula, which gives the expressiveness of the logic; deciding if two given proofs reduce to the same normal form, i.e. the cut-elimination problem which corresponds to program equivalence using the Curry-Howard isomorphism; and deciding the correctness of a given proof structure, i.e. whether it comes from a sequent calculus derivation. For this last decision problem, one uses a *correctness criterion* to distinguish proof nets among proof structures. We recall the following main results [13, 16, 15] and as for MLL and MELL [10], we prove that the correctness decision problem for MALL is $NL$-complete:

| fragment | | decision problem | |
|---|---|---|---|
| | units | provability | cut-elimination |
| MLL | no | $NP$-complete | $P$-complete |
| MELL | yes | open | non-elementary |
| MALL | no | $PSPACE$-complete | $coNP$-complete |

One can observe that there is a long story of correctness criteria for MLL: Long-trip [5] based on travels, Acyclic-Connected [3] based on switchings i.e. the choice of one premise for each $\parr$ connective, Contractibility [2] based on graph rewriting rules, Graph Parsing [14] a strategy for Contractibility, etc.... A feature of these criteria is that they successively lower the complexity of sequential, deterministic algorithms deciding correctness for MLL until linear time [7].

For MALL the additives were initially treated with "boxes" and "slices". This allows to work with each additive component (the slices) ignoring the superimposition notion underliyng the connective $\&$ but it is not sufficient to ensure the correctness of the whole proof structure (even without cuts). Better solutions have been proposed in [6] without "boxes" but with "$\&$-jumps" and "boolean weights" allowing to have a correctness criterion, also in [4] with "multi-boxes" that superimpose several $\&$ connectives to manage additive behaviours. Finally D. Hughes and R. van Glabbeek [8] introduce a good representation of proof net for cut-free MALL.

Switching from proof structures to paired graphs, that is undirected graphs with a distinguished set of edges, we give in [10] a new correctness criterion for MLL and we use it here for revisiting the MALL correctness criterion of [8]. This gives us a lower bound for the correctness decision problem for MALL (MALL-CORR). This lower bound yields an exact characterization of the complexity of this problem, and induces naturally efficient parallel algorithms for it.

The paper is organized as follows: we recall preliminary definitions and results in linear logic and complexity theory in Section 1. Section 2 is devoted to the proof of the $NL$-membership of MALL-CORR. This is obtained by the exposition of a new equivalent set of properties that are decidable in $NL$. The $NL$-completeness of MALL-CORR is established in Theorem 2.25.

# 1 Background

## 1.1 MLL, MALL and Proof Nets

Roman capitals $A, B$ stand for MALL formulae, which are given by the following grammar, where the multiplicative connectives $\otimes$ and $\parr$ are duals for the negation $^\perp$, as well as the additive connectives $\oplus$ and $\&$, accordingly to De Morgan laws:

$$F ::= A \mid A^\perp \mid F \otimes F \mid F \parr F \mid F \oplus F \mid F \& F$$

[1] As usual M, A and E denote respectively for Multiplicative, Additive and Exponential fragments of LL

Greek capitals $\Gamma, \Delta$ stand for sequents, which are multiset of formulae, so that exchange is implicit. The MLL sequent calculus is given by the following rules:

$$\frac{}{\vdash A, A^\perp}\ (ax) \qquad \frac{\vdash \Gamma, C \quad \vdash \Delta, C^\perp}{\vdash \Gamma, \Delta}\ (cut)$$

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

The MALL sequent calculus is MLL extended by the following rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_1 \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_2 \quad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \&$$

In the rest of this paper every definition on MALL applies to MLL by restricting the connectives. We recall (and adapt to our formalism) the notion of MALL proof structures and proof nets defined in [8].

**Definition 1.1.** A MALL *skeleton* is a directed acyclic graph (DAG) whose edges are labelled with MALL formulae, and whose nodes are labelled, and defined with an arity and co-arity as follows:

| node label | arity and edges | | coarity and edges | |
|---|---|---|---|---|
| atom | 0 | $\emptyset$ | 1 | $A$ |
| cut | 2 | $A, A^\perp$ | 0 | $\emptyset$ |
| $\otimes$ | 2 | $A, B$ | 1 | $A \otimes B$ |
| $\parr$ | 2 | $A, B$ | 1 | $A \parr B$ |
| $\oplus$ | 2 | $A, B$ | 1 | A$\oplus$B |
| $\&$ | 2 | $A, B$ | 1 | A&B |

We allow edges with a source but no target (i.e pending or dandling edges), they are called the *conclusions* of the skeleton. The set of conclusions of a MALL skeleton is clearly a MALL sequent. We also denote as *premises* of a node the edges incident to it, and *conclusion* of a node its outgoing edge. For a given node $x$ of arity 2, its left (respectively right) parent is denoted $x^l$ (resp. $x^r$).

**Definition 1.2.** Let $\mathcal{S}$ be a MALL skeleton. An *additive resolution* of $\mathcal{S}$ is any result of deleting one argument subtree of each additive ($\oplus$ or $\&$) node in $\mathcal{S}$. A *&-resolution* of $\mathcal{S}$ is any result of deleting one argument subtree of each $\&$-node in $\mathcal{S}$.
An *axiom-link*, or simply *link* on a MALL skeleton $\mathcal{S}$ is a bidirected edge between complementary atoms in $\mathcal{S}$, i.e. atoms labeled with dual literals $P$ and $P^\perp$.
A *linking* on a MALL skeleton $\mathcal{S}$ is a set of distinct links on $\mathcal{S}$ such that its set of vertices is the set of atoms of an additive resolution of $\mathcal{S}$. Note that in the case where $\mathcal{S}$ contains no additive node, a linking on $\mathcal{S}$ is simply a partitioning of the atom nodes of $\mathcal{S}$ into links, i.e. a set of disjoint links whose union contains every atom of $\mathcal{S}$. The additive resolution of $\mathcal{S}$ induced by a linking $\lambda$ is denoted $\mathcal{S}\lfloor\lambda$.
A MALL *proof structure* is $(\mathcal{S}, \Theta)$, where $\mathcal{S}$ is a MALL skeleton and $\Theta$ is a set of linkings on $\mathcal{S}$. In the case of MLL proof structure, $\Theta$ is simply a singleton, so we often omit the set notation.

*Remark* 1.3. The set of conclusions of a MALL proof structure is a MALL sequent.
An additive resolution of $\mathcal{S}$ naturally induces a MLL skeleton, and, for any linking $\lambda$, $(\mathcal{S}\lfloor\lambda, \lambda)$ induces a MLL proof structure.

**Definition 1.4.** A MALL *proof net* is a MALL proof structure inductively defined as follows:

($ax$): $((\{A, A^\perp\}, \emptyset), \{\{(A, A^\perp)\}\})$ is a MALL proof net with conclusions $A, A^\perp$.

$\parr$: if $(\mathcal{S}, \Theta)$ is a MALL proof net with conclusions $\Gamma, A, B$, then $(\mathcal{S}', \Theta)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a $\parr$-node of premises $A$ and $B$ is a MALL proof-net with conclusions $\Gamma, A \parr B$.

$\otimes$: if $(\mathcal{S}_1, \Theta_1)$ with conclusions $\Gamma, A$ and $(\mathcal{S}_2, \Theta_2)$ with conclusions $\Delta, B$ are disjoint MALL proof nets, $(\mathcal{S}, \Theta)$ where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a $\otimes$-link of premises $A$ and $B$ and $\Theta$ is $\{\lambda_1 \uplus \lambda_2, \lambda_1 \in \Theta_1, \lambda_2 \in \Theta_2\}$) is a MALL proof net with conclusions $\Gamma, A \otimes B, \Delta$.

($cut$): if $(\mathcal{S}_1, \Theta_1)$ with conclusions $\Gamma, A$ and $(\mathcal{S}_2, \Theta_2)$ with conclusions $\Delta, A^\perp$ are disjoint MALL proof nets, $(\mathcal{S}, \Theta)$ where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a cut-link of premises $A$ and $A^\perp$ and $\Theta$ is $\{\lambda_1 \uplus \lambda_2, \lambda_1 \in \Theta_1, \lambda_2 \in \Theta_2\}$) is a MALL proof net with conclusions $\Gamma, \Delta$.

$\&$: if $(\mathcal{S} \uplus \mathcal{S}_A, \Theta_A)$, where $\mathcal{S}$ (respectively $\mathcal{S}_A$) has conclusions $\Gamma$ (resp. $A$) and $(\mathcal{S} \uplus \mathcal{S}_B, \Theta_B)$, where $\mathcal{S}_B$ has conclusion $B$ are MALL proof nets, then $(\mathcal{S} \uplus \mathcal{S}', \Theta_A \uplus \Theta_B)$, where $\mathcal{S}'$ is $\mathcal{S}_A \uplus \mathcal{S}_B$ extended with a $\&$-node of premises $A$ and $B$, is a MALL proof net with conclusions $\Gamma, A\&B$.

$\oplus$: for any MALL formula $B$, if $(\mathcal{S}, \Theta)$ is a MALL proof net with conclusions $\Gamma, A$, then $(\mathcal{S}', \Theta)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with the syntactic tree of $B$ and a $\oplus$ node of premises $A$ and $B$ (respectively $B$ and $A$) is a MALL proof net with conclusions $\Gamma, A \oplus B$ (resp. $\Gamma, B \oplus A$).

The inductive definition of MALL proof nets corresponds to a graph theoretic abstraction of the derivation rules of MALL; any proof net is sequentializable, i.e. corresponds to a MALL derivation: given a proof net $P$ of conclusion $\Gamma$, there exists a sequent calculus proof of $\vdash \Gamma$ which infers $P$.

**Definition 1.5.** A *paired graph* is an undirected graph $G = (V, E)$ with a set of *pairs* $C(G) \subseteq E \times E$ which are pairwise disjoint couples of edges with the same target, called a *pair-*

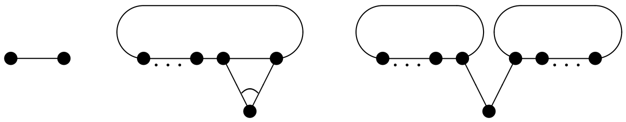

Figure 1: Paired graph constructors associated to MLL proof nets: axiom-link, ⅋-node and $\otimes$ (cut)-node.

*node*, and two (possibly distinct) sources called the *premise-nodes*.
A *switching* $S$ of $G$ is the choice of an edge for every pair of $C(G)$. With each switching $S$ is associated a subgraph $S(G)$ of $G$: for every pair of $C(G)$, erase the edges which are not selected by $S$. When $S$ selects the (abusively speaking) left edge of each pair, $S(G)$ is denoted as $G[\forall \mapsto \cdot\!\setminus\!\cdot]$. Also, $G[\forall \mapsto \cdot\,.\,\cdot]$ stands for $G \setminus \{e, e' \mid (e, e') \in C(G)\}$.

*Remark* 1.6. Without loss of generality we allow tuples of edges, i.e. $C(G) \subseteq \bigcup_{n\in\mathbb{N}} E$. A tuple of edges incident to a node $x$ can be seen as a binary tree rooted at $x$ with all ingoing edges being coupled.

Let $\mathcal{S} = (V, E)$ be a MLL skeleton. To $\mathcal{S}$, we associate the paired graph $G_{\mathcal{S}} = (V, E)$, where $C(G_{\mathcal{S}})$ contains the premises of each ⅋-link of $\mathcal{S}$. To a MLL proof structure $(\mathcal{S}, \lambda)$, we associate the paired graph $G_{(\mathcal{S},\lambda)} = G_{\mathcal{S}} \uplus \lambda$, where $C(G_{(\mathcal{S},\lambda)}) = C(G_{\mathcal{S}})$ (Figure 1).
For a pair of edges $(v, x), (w, x)$, we adopt the representation of Figure 1, where the two edges of the pair are joined by an arc.

**Definition 1.7.** Let $(\mathcal{S}, \Theta)$ be a MALL proof structure.
Let $W$ be a $\&$-resolution of $\mathcal{S}$ and let $\lambda \in \Theta$ be a linking on $\mathcal{S}$. We note $\lambda \sqsubseteq W$ if and only if every vertex of every link in $\lambda$ is a leaf of $W$.
Let $\Lambda \subseteq \Theta$ be a set of linkings on $\mathcal{S}$.
$\Lambda$ is said to *toggle* a $\&$ node $x_{\&}$ (respectively a $\oplus$ node $x_{\oplus}$) of $\mathcal{S}$ if there exists $\lambda_1, \lambda_2 \in \Lambda$ such that $x^l_{\&} \in \mathcal{S}\lfloor\lambda_1$ and $x^r_{\&} \in \mathcal{S}\lfloor\lambda_2$ (resp. $x^l_{\oplus} \in \mathcal{S}\lfloor\lambda_1$ and $x^r_{\oplus} \in \mathcal{S}\lfloor\lambda_2$).
Let $\mathcal{S}\lfloor\Lambda = \bigcup_{\lambda\in\Lambda} \mathcal{S}\lfloor\lambda$, and $G_{\mathcal{S}\lfloor\Lambda} = \bigcup_{\lambda\in\Lambda} G_{(\mathcal{S}\lfloor\lambda,\lambda)}$.[2]
Let $x_{\&}$ be a $\&$ node in $\mathcal{S}$ and $a$ be an atom of $\mathcal{S}$. Let $\{\lambda_1, \lambda_2\} \subseteq \Lambda$. A *jump edge* $(x_{\&}, a)$ is admissible for $\{\lambda_1, \lambda_2\}$ if and only if

1. $x_{\&}$ is the *unique* $\&$ node toggled by $\{\lambda_1, \lambda_2\}$, and,
2. there exists a link $l = (a, b) \in \lambda_1 \setminus \lambda_2$.

Let $H_{\mathcal{S}\lfloor\Lambda}$ be $G_{\mathcal{S}\lfloor\Lambda}$ extended with all admissible jump edges for all $\{\lambda_1, \lambda_2\} \subseteq \Lambda$, and where $C(H_{\mathcal{S}\lfloor\Lambda})$ contains the premise - and jump - edges incident to all ⅋/$\&$ nodes of $\mathcal{S}\lfloor\Lambda$. (the pair edges are actually tuples as in Remark 1.6)

**Definition 1.8.** A MLL proof structure $(\mathcal{S}, \lambda)$ is *DR-correct* if for all switching $S$ of $G_{(\mathcal{S},\lambda)}$, the graph $S(G_{(\mathcal{S},\lambda)})$ is acyclic and connected. Let $G$ be a paired graph. A *switching cycle* $\mathcal{C}$ in $G$ is a cycle in $S(G)$ for some switching $S$ of $G$.

[2] $G_{\mathcal{S}\lfloor\Theta}$ can be defined similarly to the $G_{(\mathcal{S},\lambda)}$ of Figure 1

**Theorem 1.9** (MLL Correctness Criterion, [3])**.** *A* MLL *proof structure* $(\mathcal{S}, \lambda)$ *is a* MLL *proof net iff* $(\mathcal{S}, \lambda)$ *is DR-correct.* □

**Theorem 1.10** (MALL Correctness Criterion, [8])**.** *A* MALL *proof structure* $(\mathcal{S}, \Theta)$ *is a* MALL *proof net iff:*

*(MLL): For every* $\lambda \in \Theta$, $(\mathcal{S}\lfloor\lambda, \lambda)$ *is a* MLL *proof net,*

*(RES): For every* $\&$*-resolution* $W$ *of* $\mathcal{S}$*, there exists a unique* $\lambda \in \Theta$ *such that* $\lambda \sqsubseteq W$*,*

*(TOG): For every* $\Lambda \subseteq \Theta$ *of two or more linkings,* $\Lambda$ *toggles a* $\&$ *node* $x_{\&}$ *such that* $x_{\&}$ *does not belong to any switching cycle of* $H_{\mathcal{S}\lfloor\Lambda}$. □

We define the following decision problem MALL-CORR:
GIVEN: A MALL proof structure $(\mathcal{S}, \Theta)$
PROBLEM: Is $(\mathcal{S}, \Theta)$ a MALL proof net?

### 1.2 Complexity Classes and Related Problems

Let us mention several major complexity classes below $P$, some of which having natural complete problems that we will use in this paper. Let us briefly recall some basic definitions and results:

- $AC^0$ (respectively $AC^1$) is the class of problems solvable by a uniform family of circuits of constant (resp. logarithmic) depth and polynomial size, with NOT gates and AND, OR gates of unbounded fan-in.
- $L$ is the class of problems solvable by a deterministic Turing machine which only uses a logarithmic working space.
- *NL* (respectively *coNL*) is the class of problems solvable by a non-deterministic Turing machine which only uses a logarithmic working space, such that: if the answer is "yes" then at least one (resp. all) computation path accepts, else all (resp. at least one) computation paths reject.

**Theorem 1.11.** *[9, 18] NL = coNL.* □

The following inclusion results are also well known:

$$AC^0 \subseteq L \subseteq NL \subseteq AC^1 \subseteq P,$$

where it remains unknown whether any of these inclusions is strict. It is important to note that our *NL*-completeness result for MALL-CORR is under constant-depth (actually $AC^0$) reductions. From the inclusion above, it should be clear to the reader that the reduction lies indeed in a class small enough for being relevant. For a good exposition of constant-depth reducibility, see [1].
In the sequel, we will often use the notion of a *path* in a directed -or undirected- graph. A path is a sequence of vertices such that there is an edge between any two consecutive

vertices in the path. A path will be called *elementary* when any node occurs at most once in the path. Let us now list some graph-theoretic problems that will be used in this paper.

SOURCE-TARGET CONNECTIVITY (STCONN): Given a directed graph $G = (V, E)$ and two vertices $s$ and $t$, is there a path from $s$ to $t$ in $G$ ?
STCONN is $NL$-complete under constant-depth reductions [12].

UNIVERSAL SOURCE DAG (SDAG): Given a directed graph $G = (V, E)$, is it acyclic and does there exist a source node $s$ such that there is a path from $s$ to each vertex ?

**Theorem 1.12** ([10])**.** SDAG *is NL-complete under constant-depth reductions.* □

## 2 $NL$-completeness of MALL-CORR

For cut-free MLL, it is clear that the size of a proof structure is linear in the size of its skeleton i.e. in the size of its sequent. MLL-CORR for cut-free MLL proof structures is decidable in nondeterministic space logarithmic in the size of its skeleton and its sequent ([10]). The situation for MALL differs quite a lot from the situation for MLL in the sense that the size of a sequent and of a corresponding proof structure - or proof net - may be of different order: while some cut-free MALL proof structures and proof nets have size linear in the size of their skeleton (e.g. pure MLL proof structures) and their sequent, others have size exponential in the size of their skeleton. Define the following correct sequents:

$$\begin{array}{lcl}
\Gamma_1 & = & A_1^\perp \oplus \ldots \oplus A_n^\perp, A_1 \& \ldots \& A_n \\
\Gamma_2 & = & A^\perp \oplus \ldots \oplus A^\perp, A \& \ldots \& A \\
\Sigma_1 & = & A_1^\perp \otimes \ldots \otimes A_n^\perp, A_1 \& A_1, \ldots, A_n \& A_n \\
\Sigma_2 & = & A^\perp \otimes \ldots \otimes A^\perp, A \& A, \ldots, A \& A.
\end{array}$$

For each of these sequents, the size of the corresponding cut-free skeleton is linear in $n$. The following table shows, for a cut-free MALL skeleton for each of these sequents, its number of additive resolutions, &-resolutions and possible links. The last two lines show the number of links in any cut-free MALL proof net, and the number of different cut-free MALL proof nets for each of these sequents.

| sequent | $\Gamma_1$ | $\Gamma_2$ | $\Sigma_1$ | $\Sigma_2$ |
|---|---|---|---|---|
| # add-resolutions | $n^2$ | $n^2$ | $2^n$ | $2^n$ |
| # &-resolutions | $n$ | $n$ | $2^n$ | $2^n$ |
| # links | $n$ | $n^2$ | $2^n$ | $n!2^n$ |
| $\lvert\Theta\rvert$ | $n$ | $n$ | $2^n$ | 2 |
| # $\Theta$ | 1 | $n^2$ | 1 | $n!$ |

This table illustrates how some very simple MALL sequents can yield very large MALL proof nets. These proof-nets are exemplified in Figures 2, 3 and 4 below. Here, the reader should keep in mind that the input to our MALL-CORR problem is actually a MALL proof structure, of size maybe much larger that the size of the corresponding sequent. Recall from Theorem 1.10 that a MALL proof structure is a positive input to MALL-CORR if and only if it satisfies Conditions (MLL), (RES) and (TOG). The $NL$-hardness of MALL-CORR follows directly from the $NL$-hardness of MLL-CORR [10] (since MLL is a sub-system of MALL). The $NL$-membership of Condition (MLL) follows directly from the $NL$-membership of MLL-CORR as established in [10] and recalled here. Therefore, proving the $NL$-membership of MALL-CORR requires to prove the $NL$-membership of (RES) and (TOG). We exhibit in this section algorithms for checking non-deterministically (RES) and (TOG) in space logarithmic in the size of the proof structure, which, in some cases, is actually polynomial in the size of the sequent.

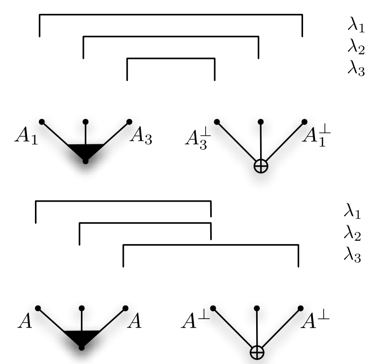


Figure 2: The MALL proof-net on $\Gamma_1$, and an example of proof-net on $\Gamma_2$, with $n = 3$.

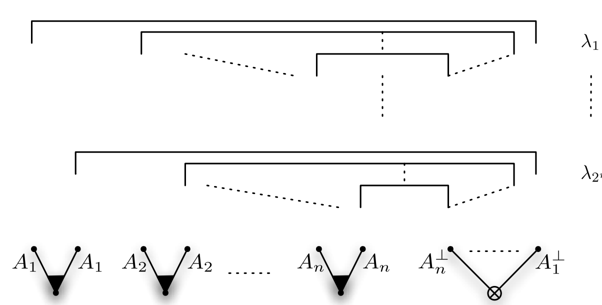


Figure 3: The MALL proof-net $(\Sigma_1, \Theta_1)$ on $\Sigma_1$, with $\Theta_1 = \bigcup_{i=1}^{2^n} \lambda_i$.

### 2.1 Checking (MLL)

We recall here the definitions and the results which are proved in [10]. For a given paired graph, the following no-

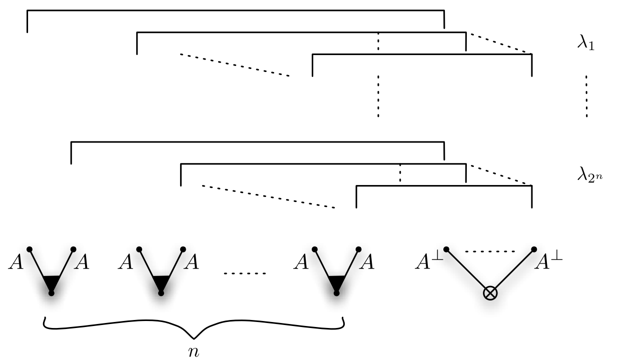


Figure 4: An example of MALL proof-net $(\Sigma_2, \Theta_{n!})$ on $\Sigma_2$, with $\Theta_{n!} = \bigcup_{i=1}^{2^n} \lambda_i$. Note that the set $\Theta_1$ of figure 3 yields another proof-net $(\Sigma_2, \Theta_1)$ on $\Sigma_2$, as well as the $n!$ possible combination of choices among the order in which the premises of the $\otimes$ node are linked to the $\&$ nodes.

tion of dependency graph provides a partial order among its pair-nodes This yields a new correctness criterion for MLL-CORR given by Theorem 2.2.

**Definition 2.1.** Let $G$ be a paired graph. The *dependency graph* $D(G)$ of $G$ is the directed graph $(V_G, E_G)$ defined as follows:

- $V_G = \{v \mid v \text{ is a pair-node in } G\} \cup \{s\}$.
- Let $x$ be a pair-node in $G$, with premise-nodes $x_l$ and $x_r$. The edge $(s \to x)$ is in $E_G$ if and only if:
  1. There exists an elementary path $p_x = x_l, \ldots, x_r$ in $G[\forall \mapsto \backslash\cdot]$,
  2. $x \notin p_x$, and for all pair-node $y$ in $G$, $y \notin p_x$.
- Let $x$ be a pair-node in $G$, with premise-nodes $x_l$ and $x_r$, and let $y \neq x$ be another pair-node in $G$. The edge $(y \to x)$ is in $E_G$ if and only if:
  1. There exists an elementary path $p_x = x_l, \ldots, x_r$ in $G[\forall \mapsto \backslash\cdot]$,
  2. $x \notin p_x$, and for every elementary path $p_x = x_l, \ldots, x_r$ in $G[\forall \mapsto \backslash\cdot]$ with $x \notin p_x$, $y \in p_x$.

For examples of MLL proof structures, corresponding paired graphs and their dependency graphs, see Figure 5.
Define a paired-graph $G$ to be *D-R-connected* if and only if, for any switching $S$ of $G$, the switched graph $S(G)$ is connected.

**Theorem 2.2** (Correctness Criterion, [10])**.** *A* MLL *proof structure* $(\mathcal{S}, \lambda)$ *is a* MLL *proof net if and only if:*

1. $D(G_{(\mathcal{S},\lambda)})$ *satisfies* SDAG*, and*
2. $G_{(\mathcal{S},\lambda)}[\forall \mapsto \backslash\cdot]$ *is a tree.* □

Theorems 1.9 and 2.2 imply the following lemma:

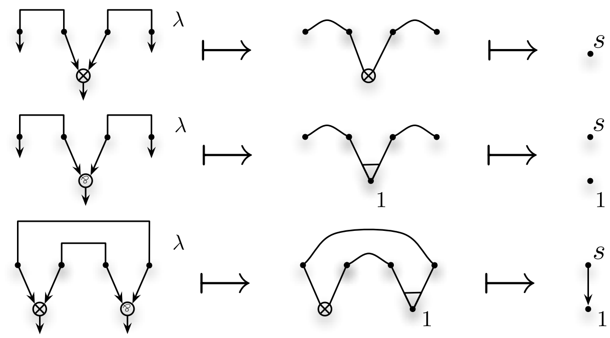


Figure 5: MLL proof structures, corresponding paired graph and dependency graphs, for the sequents $A^\perp, A \otimes B, B^\perp$ (correct), $A^\perp, A \parr B, B^\perp$ (incorrect), $A \otimes B, A^\perp \parr B^\perp$ (correct)

**Lemma 2.3.** *A paired-graph $G$ is D-R-connected if and only if its dependency graph has a node $s$ from which every node is reachable.* □

**Lemma 2.4** ([10])**.** *The function which associates its dependency graph to a paired graph, is in FL.* □

**Theorem 2.5** ([10])**.** *MLL-*CORR *is NL-Complete under constant-depth reductions.* □

Note that the previous best algorithms [14, 7] are not likely to be implemented in logarithmic space, since they require on-line modification of the structure they manipulate. The purpose of our criterion of Theorem 2.2 is precisely that it allows a space-efficient implementation.

### 2.2 Checking (RES)

We recall Condition (RES) of Theorem 1.10: For every $\&$-resolution $W$ of $\mathcal{S}$, there exists a unique $\lambda \in \Theta$ such that $\lambda \sqsubseteq W$.
Let us illustrate the difficulty in checking (RES) on a simple example. Let us consider the proof-structure $(\Sigma_1, \Theta)$, where $\Sigma_1$ is as above $A_1^\perp \otimes \ldots \otimes A_n^\perp, A_1 \& A_1, \ldots, A_n \& A_n$, and $\Theta$ is a subset of $\Theta_1$ of Figure 3 containing $n^{\lceil \log(n) \rceil}$ linkings. The size of $(\Sigma_1, \Theta)$ is therefore $\mathcal{O}(n^{\lceil \log(n) \rceil})$.
We have seen that the number of $\&$-resolutions of $\Sigma_1$ is $2^n$. Enumerating (and explicitly describing) all $\&$-resolutions requires at least $\Omega(n)$ space, and is not feasible in space $\mathcal{O}(\log(n^{\lceil \log(n) \rceil})) = \mathcal{O}(\log(n)^2)$. Therefore a *NL* algorithm for (RES) may not proceed by first plainly enumerating all $\&$-resolutions.
The idea of our algorithm is to define a notion of distance of edition on the $\&$-resolutions such that one can pass from any $\&$-resolution to any other $\&$-resolution with intermediate steps of distance at most one (Condition L1). Lemma 2.11 shows that (RES) fails if there exists a $\&$-resolution $W$ with $\lambda \sqsubseteq W$ at distance 1 to a $\&$-resolution $W'$ with no $\lambda' \sqsubseteq W'$ (Condition L3). Note however that, as on $(\Sigma_1, \Theta)$, the

working space may not be large enough for describing explicitly the &-resolutions: instead, a &-resolution $W$ with $\lambda \sqsubseteq W$ is implicitly described by $\lambda$. The difficulty then is to describe a &-resolution $W'$ with no $\lambda' \sqsubseteq W'$. We establish in Lemma 2.14 that (RES) fails if there exists a &-resolution $W$ with $\lambda \sqsubseteq W$ at distance 1 to a &-resolution $W'$ with no $\lambda' \sqsubseteq W'$, where moreover $W'$ can be implicitly described by $\lambda$ and some &-node (Condition L4). Our algorithm enumerates (in logarithmic space) the $\lambda$'s and the & nodes in search of such a configuration.

**Definition 2.6** (L1)**.** Let $(\mathcal{S}, \Theta)$ be a MALL proof structure. For any &-resolution $W$ of $\mathcal{S}$, let $\mathsf{switch}_W : \{x_\& : \& \text{ node of } \mathcal{S}\} \to \{l, r\}$ be the following function:

$$\mathsf{switch}_W(x_\&) = \begin{cases} l & \text{if } x^l_\& \in W \text{ or } x_\& \notin W \\ r & \text{if } x^r_\& \in W. \end{cases}$$

Let $\mathcal{W}_\mathcal{S}$ be the set of &-resolutions of $\mathcal{S}$.
Let $\mathcal{W}_\Theta = \{W \in \mathcal{W}_\mathcal{S} : \exists \lambda \in \Theta,\ \lambda \sqsubseteq W\}$.
We define the following distance Dist on $\mathcal{W}_\mathcal{S}$ by

$$\mathsf{Dist}(W, W') = |\{x_\& \ \&\text{-node of } \mathcal{S} : \mathsf{switch}_W(x_\&) \neq \mathsf{switch}_{W'}(x_\&)\}|.$$

Let $\mathcal{W} \subseteq \mathcal{W}_\mathcal{S}$. We say that $\mathcal{W}$ satisfies Condition L1 if and only if:

$$\forall W_0, W_k \in \mathcal{W}\ \exists W_1, \ldots, W_{k-1} \in \mathcal{W} \text{ s.t. } \mathsf{Dist}(W_i, W_{i+1})_{0 \leq i < k \leq 1}.$$

**Lemma 2.7.** *$\mathcal{W}_\mathcal{S}$ satisfies condition L1.*

*Proof.* by induction on the skeleton $\mathcal{S}$. □

**Definition 2.8** (L2)**.** Let $(\mathcal{S}, \Theta)$ be a MALL proof structure. $(\mathcal{S}, \Theta)$ is said to satisfy Condition L2 if and only if $\forall y_\oplus$ $\oplus$ node in $\mathcal{S}$, $\forall \lambda_1, \lambda_2 \in \Theta$ that toggle $y_\oplus$, there exists a & node $x_\&$ also toggled by $\{\lambda_1, \lambda_2\}$.

**Lemma 2.9.** *If $(\mathcal{S}, \Theta)$ is a* MALL *proof net, then, it satisfies Condition L2.*

Proof: By induction on $(\mathcal{S}, \Theta)$, along Definition 1.4. The only critical case is that of a & rule:
if $(\mathcal{S} \uplus \mathcal{S}_A, \Theta_A)$, where $\mathcal{S}$ (respectively $\mathcal{S}_A$) has conclusions $\Gamma$ (resp. $A$) and $(\mathcal{S} \uplus \mathcal{S}_B, \Theta_B)$, where $\mathcal{S}_B$ has conclusion $B$ are MALL proof nets, then $(\mathcal{S} \uplus \mathcal{S}', \Theta_A \uplus \Theta_B)$, where $\mathcal{S}'$ is $\mathcal{S}_A \uplus \mathcal{S}_B$ extended with a &-node of premises $A$ and $B$, is a MALL proof net with conclusions $\Gamma, A\&B$.

Two cases arise:

1. Assume there exist a $\oplus$ node $y_\oplus \in \mathcal{S}$, $\lambda \in \Theta_A$, $\lambda' \in \Theta_A$ such that $\lambda, \lambda'$ toggle $y_\oplus$. Then the induction hypothesis on $(\mathcal{S} \uplus \mathcal{S}_A, \Theta_A)$ ensures that there exists a & node $x_\& \in \mathcal{S} \uplus \mathcal{S}_A$ also toggled by $\lambda, \lambda'$. Similarly for $\lambda \in \Theta_B$, $\lambda' \in \Theta_B$.
2. Assume there exist a $\oplus$ node $y_\oplus \in \mathcal{S}$, $\lambda \in \Theta_A$, $\lambda' \in \Theta_B$ such that $\lambda, \lambda'$ toggle $y_\oplus$. Then the & node of premises $A$ and $B$ in $\mathcal{S}'$ is also toggled by $\lambda, \lambda'$. □

**Definition 2.10** (L3)**.** Let $(\mathcal{S}, \Theta)$ be a MALL proof structure.
Let $\lambda \in \Theta$, and define $\mathcal{S}\lfloor_\& \lambda = \{W \in \mathcal{W}_\mathcal{S} : \lambda \sqsubseteq W\}$.
Let $x_\&$ be a & node in $\mathcal{S}$.
$(\lambda, x_\&)$ are said to satisfy Condition L3 in $(\mathcal{S}, \Theta)$ if and only if:
$\exists W^\lambda_+ \in \mathcal{S}\lfloor_\& \lambda, W^\lambda_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$ s.t.
$\mathsf{Dist}(W^\lambda_+, W^\lambda_-) = 1$ and $\mathsf{switch}_{W^\lambda_+}(x_\&) \neq \mathsf{switch}_{W^\lambda_-}(x_\&)$.

**Lemma 2.11.** *Assume $(\mathcal{S}, \Theta)$ is a* MALL *proof structure. Then, $(\mathcal{S}, \Theta)$ satisfies (RES) of Theorem 1.10 if and only if:*

1. *$\forall \lambda, \lambda' \in \Theta,\ \ \lambda \neq \lambda' \Rightarrow \mathcal{S}\lfloor\lambda \neq \mathcal{S}\lfloor\lambda'$, and*
2. *$\forall \lambda \in \Theta,\ \forall x_\&$ & node in $\mathcal{S}$, $(\lambda, x_\&)$ does not satisfy L3 in $(\mathcal{S}, \Theta)$.*

Proof:

1. Let $W \in \mathcal{W}_\Theta$ and $\lambda \in \Theta$ s.t. $\lambda \sqsubseteq W$. By induction on $W$, if there exists $\lambda' \neq \lambda$ s.t. $\lambda' \sqsubseteq W$, then $\mathcal{S}\lfloor\lambda = \mathcal{S}\lfloor\lambda'$. It follows that (1) above is equivalent to the unicity, for any &-resolution $W$ of $\mathcal{S}$, of a $\lambda \in \Theta$ such that $\lambda \sqsubseteq W$.
2. Assume that there exists a &-resolution $W$ of $\mathcal{S}$ s.t. $\forall \lambda \in \Theta$, $\lambda \not\sqsubseteq W$. Then, $\mathcal{W}_\Theta \subsetneq \mathcal{W}_\mathcal{S}$. Assume $\Theta \neq \emptyset$, then, $\mathcal{W}_\Theta \neq \emptyset$. Therefore there exists $W_+ \in \mathcal{W}_\Theta$ and $W_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$. By Lemma 2.9, there exists then $W_1, \ldots, W_k \in \mathcal{W}$ s.t. $\mathsf{Dist}(W_+, W_1) \leq 1$, $\mathsf{Dist}(W_i, W_{i+1})_{0 \leq i < k} \leq 1$, and $\mathsf{Dist}(W_k, W_-) \leq 1$. Since any of the $W_i$ belongs either to $\mathcal{W}_\Theta$ or to $\mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$, there exists $W'_+, W'_- \in \{W_+, W_1, \ldots, W_k, W_-\}$ such that $\mathsf{Dist}(W'_+, W'_-) = 1$, $W'_+ \in \mathcal{W}_\Theta$ and $W'_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$. Let $\lambda \in \Theta$ such that $\lambda \sqsubseteq W'_+$, and $x_\&$ be the & node such that $\mathsf{switch}_{W'_+}(x_\&) \neq \mathsf{switch}_{W'_-}(x_\&)$. Clearly, $(\lambda, x_\&)$ satisfy Condition L3.
Conversely, if there exists $\lambda \in \Theta$ and $x_\&$ a & node in $\mathcal{S}$ such that $(\lambda, x_\&)$ satisfies L3 in $(\mathcal{S}, \Theta)$, then there exists a &-resolution $W$ of $\mathcal{S}$ s.t. $\forall \lambda \in \Theta$, $\lambda \not\sqsubseteq W$. It follows that (2) above is equivalent to the existence, for any &-resolution $W$ of $\mathcal{S}$, of a $\lambda \in \Theta$ such that $\lambda \sqsubseteq W$. □

**Definition 2.12** (L4)**.** Let $(\mathcal{S}, \Theta)$ be a MALL proof structure.
Let $x_\&$ be a & node in $\mathcal{S}$. Define:

$$\mathcal{W}^l_{x_\&} = \{W \in \mathcal{W}_\mathcal{S} \text{ s.t. } \forall x'_\& \text{ s.t. there exists a path } x'_\& \to \cdots \to x^l_\&, \mathsf{switch}_W(x'_\&) = l\}$$

$$\mathcal{W}^r_{x_\&} = \{W \in \mathcal{W}_\mathcal{S} \text{ s.t. } \forall x'_\& \text{ s.t. there exists a path } x'_\& \to \cdots \to x^r_\&, \mathsf{switch}_W(x'_\&) = l\}$$

Let $\lambda \in \Theta$, and define $\mathsf{Mirror}(\lambda, x_\&)$, the set of $W \in \mathcal{W}_\mathcal{S}$ such that

$$\exists W' \in \mathcal{S}\lfloor_\& \lambda \cap \mathcal{W}^l_{x_\&} \cap \mathcal{W}^r_{x_\&} : \\ \mathsf{Dist}(W, W') = 1 \text{ and } \mathsf{switch}_W(x_\&) \neq \mathsf{switch}_{W'}(x_\&).$$

$(\lambda, x_\&)$ are said to satisfy Condition L4 in $(\mathcal{S}, \Theta)$ if and only if:

$$\forall \lambda' \in \Theta, \forall W \in \mathsf{Mirror}(\lambda, x_\&), \lambda' \not\sqsubseteq W.$$

**Lemma 2.13.** *Assume $(\mathcal{S}, \Theta)$ is a* MALL *proof structure satisfying Condition L2. Let $\lambda \in \Theta$ and $x_\&$ be a $\&$ node in $\mathcal{S}$ such that*

1. *$(\lambda, x_\&)$ satisfies Condition L3 in $(\mathcal{S}, \Theta)$, and*
2. *$\forall y_\oplus$ $\oplus$ node in $\mathcal{S}\lfloor\lambda$, $\forall \lambda' \in \Theta$ such that $\lambda, \lambda'$ toggle $y_\oplus$, $x_\&$ is not toggled by $\lambda, \lambda'$.*

*Then, $(\lambda, x_\&)$ satisfies Condition L4 in $(\mathcal{S}, \Theta)$.*

*Proof.* Let $y_\oplus$ be a $\oplus$ node in $\mathcal{S}\lfloor\lambda$. Without loss of generality, let assume that $y^l_\oplus \in \mathcal{S}\lfloor\lambda$ and $x^l_\& \in \mathcal{S}\lfloor\lambda$. Assume $(\lambda, x_\&)$ satisfies Condition L3 in $(\mathcal{S}, \Theta)$:

$$\exists W^\lambda_+ \in \mathcal{S}\lfloor_\& \lambda, W^\lambda_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta \text{ such that} \\ \mathsf{Dist}(W^\lambda_+, W^\lambda_-) = 1 \text{ and } \mathsf{switch}_{W^\lambda_+}(x_\&) \neq \mathsf{switch}_{W^\lambda_-}(x_\&).$$

Let $\theta_\lambda = \{\lambda_i \in \Theta : \ \lambda_i \sqsubseteq W_i \in \mathsf{Mirror}(\lambda, x_\&)\}$.
Assume by contradiction that $\theta_\lambda \neq \emptyset$.
Let us show by contradiction that for all $\lambda' \in \theta_\lambda$, $y^r_\oplus \notin \mathcal{S}\lfloor\lambda'$. Assume $\exists \lambda' \in \theta_\lambda$, $y^r_\oplus \in \mathcal{S}\lfloor\lambda'$. Then $\lambda, \lambda'$ toggle $y_\oplus$. By Condition L2, there exists a $\&$ node $x'_\& \neq x_\&$ also toggled by $\lambda, \lambda'$. Assume without loss of generality that ${x'_\&}^l \in \mathcal{S}\lfloor\lambda$ and ${x'_\&}^r \in \mathcal{S}\lfloor\lambda'$.
Since ${x'_\&}^l \in \mathcal{S}\lfloor\lambda$, for all $W \in \mathsf{Mirror}(\lambda, x_\&)$, $\mathsf{switch}_W(x'_\&) = l$. Since ${x'_\&}^r \in \mathcal{S}\lfloor\lambda'$, for any $W' \in \mathsf{Mirror}(\lambda, x_\&)$ s.t. $\lambda' \sqsubseteq W'$, $\mathsf{switch}_{W'}(x'_\&) = r$: contradiction.
Therefore, for all $\lambda' \in \theta_\lambda$, $y^r_\oplus \notin \mathcal{S}\lfloor\lambda'$.
Let $\lambda' \in \theta_\lambda$, and let $x'_\&$ (respectively $y'_\oplus$) be any $\&$ node (resp. $\oplus$ node) such that there exists no path $x'_\& \to \cdot\cdot \to x_\&$ (resp. $y'_\oplus \to \cdot\cdot \to x_\&$). Then, by induction on $\mathcal{S}$,

$$\begin{array}{ll} x'_\& \in \mathcal{S}\lfloor\lambda \Rightarrow x'_\& \in \mathcal{S}\lfloor\lambda', & y'_\oplus \in \mathcal{S}\lfloor\lambda \Rightarrow y'_\oplus \in \mathcal{S}\lfloor\lambda', \\ {x'_\&}^l \in \mathcal{S}\lfloor\lambda \Rightarrow {x'_\&}^l \in \mathcal{S}\lfloor\lambda', & {y'_\oplus}^l \in \mathcal{S}\lfloor\lambda \Rightarrow {y'_\oplus}^l \in \mathcal{S}\lfloor\lambda', \\ {x'_\&}^r \in \mathcal{S}\lfloor\lambda \Rightarrow {x'_\&}^r \in \mathcal{S}\lfloor\lambda', & {y'_\oplus}^r \in \mathcal{S}\lfloor\lambda \Rightarrow {y'_\oplus}^r \in \mathcal{S}\lfloor\lambda'. \end{array}$$

It follows that $\lambda' \sqsubseteq W^\lambda_-$: contradiction. □

**Lemma 2.14.** *Assume $(\mathcal{S}, \Theta)$ is a* MALL *proof structure satisfying L2. Let $\lambda \in \Theta$ and $x_\&$ be a $\&$ node in $\mathcal{S}$ such that*

1. *$(\lambda, x_\&)$ satisfy Condition L3 in $(\mathcal{S}, \Theta)$, and*
2. *$\exists y_\oplus$ $\oplus$ node in $\mathcal{S}\lfloor\lambda$, and $\lambda' \in \Theta$ such that $\lambda, \lambda'$ toggle both $y_\oplus$ and $x_\&$.*

*Then, there exists $x'_\&$ $\&$ node in $\mathcal{S}$ such that $(\lambda', x'_\&)$ satisfies Condition L4 in $(\mathcal{S}, \Theta)$.*

*Proof:* By induction on the maximal number of $\&$ and $\oplus$ nodes traversed along a path $x \to \cdot\cdot \to x_\&$ or $x \to \cdot\cdot \to y_\oplus$ in $\mathcal{S}$. Since $\mathcal{S}$ is acyclic, this number is well defined. Assume $(\lambda, x_\&)$ satisfies Condition L3 in $(\mathcal{S}, \Theta)$:

$$\exists W^\lambda_+ \in \mathcal{S}\lfloor_\& \lambda, W^\lambda_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta \text{such that} \\ \mathsf{Dist}(W^\lambda_+, W^\lambda_-) = 1 \text{ and } \mathsf{switch}_{W^\lambda_+}(x_\&) \neq \mathsf{switch}_{W^\lambda_-}(x_\&).$$

Without loss of generality, assume $y^l_\oplus \in \mathcal{S}\lfloor\lambda$ and $x^l_\& \in \mathcal{S}\lfloor\lambda$.
Let $\theta_\lambda = \{\lambda_i \in \Theta : \ \lambda_i \sqsubseteq W_i \in \mathsf{Mirror}(\lambda, x_\&)\}$. If there is no $\&$ or $\oplus$ node along any path $x \to \cdot\cdot \to x_\&$ or $x \to \cdot\cdot \to y_\oplus$, $\theta_\lambda = \emptyset$. If $\theta_\lambda = \emptyset$, $(\lambda, x_\&)$ satisfies Condition L4 in $(\mathcal{S}, \Theta)$. Assume in the following that $\theta_\lambda \neq \emptyset$.

1. Let $y'_\oplus$ be a $\oplus$ node in $\mathcal{S}\lfloor\lambda$ such that there exists no path $y'_\oplus \to \cdot\cdot \to y_\oplus$ and no path $y'_\oplus \to \cdot\cdot \to x_\&$. Let us show by contradiction that $y'_\oplus$ is toggled by no $(\lambda, \lambda_i)$, $\lambda_i \in \theta_\lambda$.

   Assume $y'_\oplus$ is toggled by $(\lambda, \lambda_i)$, $\lambda_i \in \theta_\lambda$, and, without loss of generality, ${y'_\oplus}^l \in \mathcal{S}\lfloor\lambda$, ${y'_\oplus}^r \in \mathcal{S}\lfloor\lambda_i$. Then, by Condition L2, there exists a $\&$ node $x'_\& \in \mathcal{S}\lfloor\lambda \cap \mathcal{S}\lfloor\lambda_i$ toggled by $(\lambda, \lambda_i)$, and, without loss of generality, ${x'_\&}^l \in \mathcal{S}\lfloor\lambda$ and ${x'_\&}^r \in \mathcal{S}\lfloor\lambda_i$. Let $W'_i$ be any $\&$-resolution such that $\lambda_i \sqsubseteq W'_i$: $\forall W \in \mathcal{S}\lfloor_\& \lambda \cap \mathcal{W}^l_{x_\&} \cap \mathcal{W}^r_{x_\&}$ $x^l_\& \in W$, ${x'_\&}^l \in W$, $x^r_\& \in W'_i$, ${x'_\&}^r \in W'_i$, and $\mathsf{Dist}(W, W') \geq 1$. Therefore, $W'_i$ cannot possibly be in $\mathsf{Mirror}(\lambda, x_\&)$, which contradicts the hypothesis that $y'_\oplus$ is toggled by $(\lambda, \lambda_i)$, $\lambda_i \in \theta_\lambda$.
2. By Condition L3, $\forall \lambda_i \in \theta_\lambda$, $\exists (x_i, y_i) \in \lambda_i : x_i \notin W^\lambda_-$. Let us show that $\forall (x_i, y_i) \in \lambda_i \in \theta_\lambda$, $x_i \notin W^\lambda_-$, there exists a path $x_i \to \cdot\cdot \to y^r_\oplus$ or a path $x_i \to \cdot\cdot \to x^r_\&$.

   Assume there exists no such path. For any $\oplus$ node $y'_\oplus$ such that there exists a path $x_i \to \cdot\cdot \to y'_\oplus$, there exists no path $y'_\oplus \to \cdot\cdot \to y_\oplus$ and no path $y'_\oplus \to \cdot\cdot \to x_\&$. By (1) above, $y'_\oplus$ is toggled by no $(\lambda, \lambda_i)$, $\lambda_i \in \theta_\lambda$. Moreover, for any $\&$ node $x'_\&$ such that there exists a path $x_i \to \cdot\cdot \to x'_\&$, there exists no path $x'_\& \to \cdot\cdot \to x_\&$. By definition of $\theta_\lambda$, $x'_\&$ is then toggled by no $(\lambda, \lambda_i)$, $\lambda_i \in \theta_\lambda$, and $x_i \in \mathcal{S}\lfloor\lambda$. Therefore, $\forall W' \in \mathcal{S}\lfloor_\& \lambda$, $x_i \in W'$. By Condition L3, there exists $W^\lambda_+ \in \mathcal{S}\lfloor_\& \lambda$ s.t. $\mathsf{Dist}(W^\lambda_+, W^\lambda_-) = 1$ and $\mathsf{switch}_{W^\lambda_+}(x_\&) \neq \mathsf{switch}_{W^\lambda_-}(x_\&)$. Since $x_i \in W^\lambda_+$ and since there exists no path $x_i \to \cdot\cdot \to x_\&$, it follows that $x_i \in W^\lambda_-$: contradiction.
3. By hypothesis, $W^\lambda_+ \in \mathcal{S}\lfloor_\& \lambda$, and $\mathsf{switch}_{W^\lambda_+}(x_\&) = l$. Since $\mathsf{Dist}(W^\lambda_+, W^\lambda_-) = 1$ and $\mathsf{switch}_{W^\lambda_-}(x_\&) = r$, it follows that $W^\lambda_+ \in \mathcal{W}^l_{x_\&} \cap \mathcal{W}^r_{x_\&}$, and therefore $W^\lambda_- \in \mathsf{Mirror}(\lambda, x_\&)$.

4. It is clear that $\mathcal{S}\lfloor_{\&}\lambda$, $\mathcal{W}^l_{x_\&}$ and $\mathcal{W}^r_{x_\&}$ satisfy condition L1. Therefore, so does $\mathsf{Mirror}(\lambda, x_\&)$. Since $W^\lambda_- \in \mathsf{Mirror}(\lambda, x_\&)$ and $\theta_\lambda \neq \emptyset$, there exist $W^{\lambda_i}_+, W^{\lambda_i}_- \in \mathsf{Mirror}(\lambda, x_\&)$, $\lambda_i \in \theta_\lambda$ such that $\lambda_i \sqsubseteq W^{\lambda_i}_+$, $W^{\lambda_i}_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$ and $\mathsf{Dist}(W^{\lambda_i}_+, W^{\lambda_i}_-) = 1$. Let $x'_\&$ be the unique $\&$ node in $\mathcal{S}$ such that $\mathsf{switch}_{W^{\lambda_i}_+}(x'_\&) \neq \mathsf{switch}_{W^{\lambda_i}_-}(x'_\&)$. By (2) above, there exists a path $x'_\& \rightarrow \cdot\cdot \rightarrow y_\oplus$. If there exists a $\oplus$ node $y'_\oplus$ in $\mathcal{S}\lfloor\lambda_i$ and $\lambda_j \in \theta_\lambda$ such that $\lambda_i, \lambda_j$ toggle both $x'_\&$ and $y'_\oplus$, by (1) above, there exists a path $y'_\oplus \rightarrow \cdot\cdot \rightarrow y_\oplus$ or a path $y'_\oplus \rightarrow \cdot\cdot \rightarrow x_\&$. Therefore we can apply the induction hypothesis to conclude that $(\lambda', x'_\&)$ satisfies Condition L4 in $(\mathcal{S}, \Theta)$. □

**Proposition 2.15.** *Assume $(\mathcal{S}, \Theta)$ is a* MALL *proof structure. Then, $(\mathcal{S}, \Theta)$ satisfies (RES) of Theorem 1.10 if and only if:*

1. $\forall\lambda, \lambda' \in \Theta,\ \ \lambda \neq \lambda' \Leftrightarrow \mathcal{S}\lfloor\lambda \neq \mathcal{S}\lfloor\lambda'$,
2. $(\mathcal{S}, \Theta)$ *satisfies Condition L2, and*
3. $\forall\lambda \in \Theta$, $\forall x_\&$ $\&$ *node in* $\mathcal{S}$, $(\lambda, x_\&)$ *does not satisfy L4 in* $(\mathcal{S}, \Theta)$.

*Proof.* Apply Lemmas 2.11, 2.13 and 2.14. □

A consequence of proposition 2.15 is a *NL* algorithm deciding whether a given MALL proof structure satisfies (RES). Indeed (1), Conditions L2 and L4 can easily be checked in *NL* by parsing the set of linkings and the skeleton.

### 2.3 Checking (TOG)

We recall Condition (TOG) of Theorem1.10:
For every $\Lambda \subseteq \Theta$ of two or more linkings, $\Lambda$ toggles a $\&$ node $x_\&$ such that $x_\&$ does not belong to any switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$.
Checking Condition (TOG) in non-deterministic logarithmic space involves two difficulties, which we address in this section:

1. The number of sets $\Lambda \subseteq \Theta$ of two or more linkings is exponential in the size of $\Theta$, i.e. exponential in the size of the input in the worst case. Consider for instance the sequent $\Gamma = A\&\ldots\&A, A^\perp$ of figure 6 below: a proof-net $(\Gamma, \Theta)$ contains $n$ linkings, each linking containing a single link. The number of sets $\Lambda \subseteq \Theta$ of two or more linkings is then $2^n - n - 1$. Clearly, there is no possibility to enumerate all the sets $\Lambda \subseteq \Theta$ of two or more linkings in logarithmic space[3]. Lemma 2.17 below shows that it is actually enough to consider only a quadratic number of well chosen such sets of linkings.

[3]It is mentioned in [8] that it suffices to check (TOG) merely for *saturated* sets $\Lambda$ of linkings only, namely, such that any strictly larger subset of $\Theta$ toggles more $\&$ nodes than $\Lambda$. Note however that the saturated sets of linkings are also exponentially many, and cannot be enumerated in logpsace.

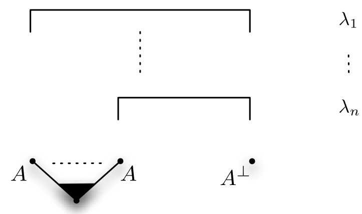


Figure 6: A proof-net $(\Gamma, \Theta)$, with $\Theta = \bigcup_{i=1}^n \lambda_i$.

2. Given a set $\Lambda \subseteq \Theta$ of two or more linkings and a $\&$ node $x_\&$ toggled by $\Lambda$, it remains to be checked whether $x_\&$ belongs to a switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$. In the worst case, the number of switched graphs of $H_{\mathcal{S}\lfloor\Lambda}$ to be investigated may be also exponential in the size of the input. Moreover, it is unclear whether $H_{\mathcal{S}\lfloor\Lambda}$ enjoys properties such as D-R correctness that allow space-efficient algorithms. Lemma 2.22 below shows that the switching cycles of $H_{\mathcal{S}\lfloor\Lambda}$ are actually the switching cycles of a graph $I_{\mathcal{S}\lfloor\Lambda}$ which, in turns, enjoys the property of being D-R connected.

The two points above are necessary step-stones towards an *NL* algorithm for condition (TOG) exhibited in Proposition 2.23.

**Definition 2.16.** Let $\{\lambda_1, \lambda_2\} \subseteq \Theta$, we define $\Theta_{\lambda_1,\lambda_2} = \{\lambda \in \Theta :\ \mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2 \subseteq \mathcal{S}\lfloor\lambda\}$.

**Lemma 2.17.** *Let $(\mathcal{S}, \Theta)$ be a* MALL *proof structure satisfying (RES).*
*$(\mathcal{S}, \Theta)$ satisfies (TOG) if and only if, for all $\{\lambda_1, \lambda_2\} \subseteq \Theta$, there exists a $\&$ node $x_\&$ toggled by $\lambda_1, \lambda_2$ such that $x_\&$ does not belong to any switching cycle of $H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$.*

*Proof.* Only if direction is trivial. We prove the if direction. In a first step, we show by induction on $\mathcal{S} \setminus (\mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2)$ that, for all $\Lambda \subseteq \Theta_{\lambda_1,\lambda_2}$ with at least two linkings, $\Lambda$ toggles a $\&$ node $x'_\&$ such that $x'_\&$ does not belong to any switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$.
Let $\lambda_1, \lambda_2 \in \Theta$, $x_\&$ a $\&$ node toggled by $\{\lambda_1, \lambda_2\}$ and $\Lambda \subseteq \Theta_{\lambda_1,\lambda_2}$. Then, $H_{\mathcal{S}\lfloor\Lambda} \subseteq H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$, and the switching cycles of $H_{\mathcal{S}\lfloor\Lambda}$ are switching cycles of $H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$.

1. If $\Lambda$ toggles $x_\&$, then $x_\&$ belongs to no switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$ (otherwise it would belong to a switching cycle of $H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$)
2. Assume $\Lambda$ does not toggle $x_\&$. Then, $(\mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2) \subsetneq \bigcap_{\lambda\in\Lambda} \mathcal{S}\lfloor\lambda$. Let $W^l_\Lambda$ be the $\&$-resolution of $\mathcal{S}$ defined as follows:

$$\bigcap_{\lambda\in\Lambda} \mathcal{S}\lfloor\lambda \quad \subseteq \quad W^l_1, \text{ and } \forall\& \text{ node } x'_\& \in \mathcal{S},$$

$$x'_\& \notin \bigcap_{\lambda\in\Lambda} \mathcal{S}\lfloor\lambda \quad \Rightarrow \quad x'^r_\& \text{ is erased in } W^l_1,$$

and $W^r_\Lambda$ as follows:

$$\bigcap_{\lambda\in\Lambda} \mathcal{S}\lfloor\lambda \quad\subseteq\quad W^r_1, \text{ and } \forall \& \text{ node } x'_\& \in \mathcal{S},$$
$$x'_\& \notin \bigcap_{\lambda\in\Lambda} \mathcal{S}\lfloor\lambda \quad\Rightarrow\quad x'^l_\& \text{ is erased in } W^r_1.$$

By Condition (RES), there exist $\lambda^l, \lambda^r \in \Theta$ s.t. $\lambda^l \sqsubseteq W^l_\Lambda$ and $\lambda^r \sqsubseteq W^r_\Lambda$. Then, clearly, $\Lambda \subseteq \Theta_{\lambda^l,\lambda^r} \subsetneq \Theta_{\lambda_1,\lambda_2}$. Since $|\Theta_{\lambda^l,\lambda^r}| > 2$, by Condition (RES), $\Theta_{\lambda^l,\lambda^r}$ toggles a $\&$ node $x'_\& \neq x_\&$. By construction, $x'_\&$ is also toggled by $\Lambda$. The induction hypothesis on $\Theta_{\lambda^l,\lambda^r}$, and the arguments of (1) above yield that $x'_\&$ belongs to no switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$.

The second step is to show that there exist $\lambda_1, \lambda_2 \in \Theta$ s.t. $\Theta = \Theta_{\lambda_1,\lambda_2}$. Consider $W_l$ the $\&$-resolution of $\mathcal{S}$ where all right premises of $\&$ nodes are erased, and $W_r$ the one where all left premises of $\&$ nodes are erased. By Condition (RES), there exists $\lambda_1, \lambda_2 \in \Theta$ such that $\lambda_1 \sqsubseteq W_l$ and $\lambda_2 \sqsubseteq W_r$. It is clear that, for all $\lambda \in \Theta$, $\mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2 \subseteq \mathcal{S}\lfloor\lambda$. Therefore, $\Theta \subseteq \Theta_{\lambda_1,\lambda_2}$. ☐

**Definition 2.18.** Let $(\mathcal{S}, \Theta)$ be a MALL proof structure. Let $x_\&$ be a $\&$ node in $\mathcal{S}$. $x_\&$ is said to be *environment-free* if, for all $\lambda \in \Theta$, for all link $(a, b) \in \lambda$, there exists a path $a \to \cdots \to x_\&$ if and only if there exists a path $b \to \cdots \to x_\&$. If $x_\&$ is not environment-free, it is said to be environment linked.

**Lemma 2.19.** *If $(\mathcal{S}, \Theta)$ is a* MALL *proof net then, for all $\&$ node $x_\&$, $x_\&$ is environment-free if and only if, for any sequentialization of $(\mathcal{S}, \Theta)$, any $\&$-rule applied on $x_\&$ has an empty environment $\Gamma$.*

*Proof.* Straightforward proof by induction. ☐

**Definition 2.20.** Let $(\mathcal{S}, \Theta)$ be a MALL proof structure. Let $I_{\mathcal{S}\lfloor\Lambda}$ be $G_{\mathcal{S}\lfloor\Lambda}$ extended with all admissible jump edges for all $\{\lambda_1, \lambda_2\} \subseteq \Lambda$ and where $C(I_{\mathcal{S}\lfloor\Lambda})$ contains the premise - and jump - edges incident to all $⅋$ nodes and environment-linked $\&$ nodes of $\mathcal{S}\lfloor\Lambda$, and the jump edges only incident to all environment-free $\&$ nodes of $\mathcal{S}\lfloor\Lambda$.

**Lemma 2.21.** *If $(\mathcal{S}, \Theta)$ is a* MALL *proof net then, for all $\{\lambda_1, \lambda_2\} \subseteq \Theta$, $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$, is D-R-connected.*

*Proof.* We actually prove the lemma for the graph $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ without jumps. An easy graph-theoretic proof by induction shows that adding the jumps does not D-R-Disconnect the paired graph.

The proof is by induction on $(\mathcal{S}, \Theta)$, along Definition 1.4. The only critical case is that of a $\&$ rule on $\Gamma, A\&B$, where the $\&$ node $x_\&$ introduced by the rule is environment-linked and is toggled by $\lambda_1, \lambda_2$. Assume without loss of generality that $x^l_\& \in \mathcal{S}\lfloor\lambda_1$ and $x^r_\& \in \mathcal{S}\lfloor\lambda_2$.

By Definition 1.4, $\Theta = \Theta_A \uplus \Theta_B$, and $\mathcal{S}$ is $\mathcal{S}_\Gamma \uplus \mathcal{S}_A \uplus \mathcal{S}_B$ (with respective conclusions $\Gamma$, $A$ and $B$) extended with $x_\&$, and $(\mathcal{S}_\Gamma \uplus \mathcal{S}_A, \Theta_A)$, $(\mathcal{S}_\Gamma \uplus \mathcal{S}_B, \Theta_B)$ are both MALL proof nets, and by Lemma 2.19, $\mathcal{S}_\Gamma \neq \emptyset$.

Let $\Lambda_A = \{\lambda \in \Theta_A : \mathcal{S}_\Gamma\lfloor\lambda_1 \cap \mathcal{S}_\Gamma\lfloor\lambda_2 \subseteq \mathcal{S}_\Gamma\lfloor\lambda\}$ and $\Lambda_B = \{\lambda \in \Theta_B : \mathcal{S}_\Gamma\lfloor\lambda_1 \cap \mathcal{S}_\Gamma\lfloor\lambda_2 \subseteq \mathcal{S}_\Gamma\lfloor\lambda\}$. Then, clearly, $\Theta_{\lambda_1,\lambda_2} = \Lambda_A \uplus \Lambda_B$, $\lambda_1 \in \Lambda_A$ and $\lambda_2 \in \Lambda_B$.

Let $W^l_1$ be the $\&$-resolution of $\mathcal{S}$ defined as follows:

$$\mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2 \quad\subseteq\quad W^l_1, \text{ and } \forall \& \text{ node } x'_\& \in \mathcal{S},$$
$$x'_\& \notin \mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2 \quad\Rightarrow\quad x'^r_\& \text{ is erased in } W^l_1, \text{ and}$$
$$x^r_\& \text{ is erased in } W^l_1,$$

and $W^r_1$ as follows:

$$\mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2 \quad\subseteq\quad W^r_1, \text{ and } \forall \& \text{ node } x'_\& \in \mathcal{S},$$
$$x'_\& \notin \mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2 \quad\Rightarrow\quad x'^l_\& \text{ is erased in } W^r_1, \text{ and}$$
$$x^r_\& \text{ is erased in } W^r_1.$$

Then, by Condition (RES), there exists $\lambda^l_1, \lambda^r_1 \in \Theta$ s.t. $\lambda^l_1 \sqsubseteq W^l_1$ and $\lambda^r_1 \sqsubseteq W^r_1$. Moreover, $\lambda^l_1 \in \Theta_A$, $\lambda^r_1 \in \Theta_A$ and $\mathcal{S}\lfloor\lambda^l_1 \cap \mathcal{S}\lfloor\lambda^r_1 = \mathcal{S}\lfloor\lambda_1 \cap \mathcal{S}\lfloor\lambda_2$. Therefore, $\Lambda_A = \Theta_{\lambda^l_1,\lambda^r_1}$.

Similarly, there exists $\lambda^l_2, \lambda^r_2 \in \Theta$ s.t. $\Lambda_B = \Theta_{\lambda^l_2,\lambda^r_2}$.

By induction hypothesis, $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}} = I_{\mathcal{S}\lfloor\Theta_{\lambda^l_1,\lambda^r_1}} \cup I_{\mathcal{S}\lfloor\Theta_{\lambda^l_2,\lambda^r_2}}$ where $I_{\mathcal{S}\lfloor\Theta_{\lambda^l_1,\lambda^r_1}}$ and $I_{\mathcal{S}\lfloor\Theta_{\lambda^l_2,\lambda^r_2}}$ are both D-R-connected.

Moreover, by Condition (RES), neither $I_{\mathcal{S}\lfloor\Theta_{\lambda^l_1,\lambda^r_1}}$ nor $I_{\mathcal{S}\lfloor\Theta_{\lambda^l_2,\lambda^r_2}}$ contains a unary couple of edges except for $x_\&$. Therefore, for any switching $S$ of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$, $x^l_\&$ is connected through $S(I_{\mathcal{S}\lfloor\Theta_{\lambda^l_1,\lambda^r_1}})$ to some vertex $y \in I_{\mathcal{S}\lfloor\Theta_{\lambda^l_1,\lambda^r_1}} \cap I_{\mathcal{S}\lfloor\Theta_{\lambda^l_2,\lambda^r_2}} \neq \emptyset$, and back to $x^r_\&$ through $S(I_{\mathcal{S}\lfloor\Theta_{\lambda^l_2,\lambda^r_2}})$. ☐

**Lemma 2.22.** *Let $(\mathcal{S}, \Theta)$ be a* MALL *proof structure satisfying (RES) and let $\Lambda \subseteq \Theta$ with at least two linkings.*
*$\Lambda$ toggle a $\&$ node $x_\&$ such that $x_\&$ belongs to a switching cycle of $I_{\mathcal{S}\lfloor\Lambda}$ if and only if it belongs to a switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$.*

*Proof.* Condition (RES) implies that no premise edge of any environment-free $\&$ node belongs to any switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$. Therefore, the switching cycles of $H_{\mathcal{S}\lfloor\Lambda}$ are switching cycles of $I_{\mathcal{S}\lfloor\Lambda}$, hence the "if" direction. The "only if" direction proceeds from the fact that the switching cycles of $I_{\mathcal{S}\lfloor\Lambda}$ are switching cycles of $H_{\mathcal{S}\lfloor\Lambda}$. ☐

Lemmas 2.17 and 2.22 yield the following proposition:

**Proposition 2.23.** *Let $(\mathcal{S}, \Theta)$ be a* MALL *proof structure satisfying (RES). $(\mathcal{S}, \Theta)$ satisfies (TOG) iff, for all $\{\lambda_1, \lambda_2\} \subseteq \Theta$, $\Theta_{\lambda_1,\lambda_2}$ toggles a $\&$ node $x_\&$ such that $x_\&$ does not belong to any switching cycle of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$.* ☐

**Proposition 2.24.** *Let* $(\mathcal{S}, \Theta)$ *be a* MALL *proof structure satisfying (RES) and (MLL). The following algorithm decides whether* $(\mathcal{S}, \Theta)$ *satisfies (TOG) in non-deterministic logarithmic space:*

FOR ALL $\lambda_1, \lambda_2 \in \Theta$
  COMPUTE $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$,
  COMPUTE $D(I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}})$ the dependency graph of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$,
  IF $\forall s \in D(I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}})$, $\exists x \in D(I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}})$
    such that $\neg$STCONN$(s, x)$ THEN REJECT
  ELSE
    LET `tog`= false
    FOR ALL $\&$ node $x_\&$ in $\mathcal{S}$
      LET $I_{x_\&}$ be $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}[\forall \mapsto \backslash\!\cdot]$ whithout any premise
        -or jump- edge to $x_\&$,
      IF no premise-argument or jump-argument of $x_\&$ is
        connected to $x_\&$ in $I_{x_\&}$ THEN `tog`=true
    END FOR ALL
  END IF
  IF `tog`=false THEN REJECT
END FOR ALL
ACCEPT

*Proof.* By Proposition 2.24, $(\mathcal{S}, \Theta)$ satisfies (TOG) if and only if, for all $\{\lambda_1, \lambda_2\} \subseteq \Theta$, $\Theta_{\lambda_1,\lambda_2}$ toggles a $\&$ node $x_\&$ such that $x_\&$ does not belong to any switching cycle of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$. By Lemma 2.21, if $(\mathcal{S}, \Theta)$ satisfies (TOG), then $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ is D-R-connected, and, by Lemma 2.3, its dependency graph has a node $s$ from which every node is reachable. Now, if $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ is D-R-connected, a $\&$ node $x_\&$ belongs to a switching cycle of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ if and only if it belongs to a cycle of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}[\forall \mapsto \backslash\!\cdot]$, therefore the algorithm above decides whether $(\mathcal{S}, \Theta)$ satisfies (TOG).
It is clear that the enumeration of the $\lambda_1, \lambda_2 \in \Theta$, and the computation of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ and $D(I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}})$ can be performed in logarithmic space. Since STCONN $\in NL$, the whole algorithm works in $NL$. □

Theorem 2.5 and propositions 2.15 and 2.24 yield the following result:

**Theorem 2.25.** *MALL*-CORR *is NL-complete under constant-depth reductions.* □

Since the size of a MALL proof structure is at most exponential in the size of its skeleton and *PSPACE=NPSPACE*, a consequence of Theorem 2.25 is that MALL-CORR can be decided in (deterministic) polynomial space in the size of the skeleton.
For other presentations of additive proof structures, as with boxes [5], weights [6] or multiboxes [4], it seems reasonable to expect the same result.

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