Coverability in a NonFunctional Extension of BVASS

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Abstract

We define Vector Addition with Sates and Split/Join Transitions, a new model that extends VASS and BVASS. We define a suitable notion of covering graph for the model, and prove its finiteness and effective constructibility, and prove a coverability theorem.

Introduction

Petri nets, and equivalent models of computation such as Vector Addition Systems (VAS) and Vector Addition Systems with States (VASS) are a natural parallel and resource sensitive model of computation, for which reachability, among other key properties, has been proven decidable [Kos82, May84, May81, Lam92, Reu88]. After the introduction of Linear Logic by Girard [Gir87], several authors focused on the syntactical and semantical relations between Linear Logic and Petri Nets [MOM89, Bro90, AFG90, Far99].
Results more relevant to our current considerations has been obtained by Kanovich, who established several equivalence results between decision problems for various kinds of Petri Nets and different fragments of Linear Logic [Kan92, Kan94, Kan96, Kan95]. In particular, he proved in [Kan95] the equivalence between the !-Horn fragment of Mutiplicative Exponential Linear Logic (MELL) and the reachability problem for Petri Nets, which is known to be decidable. His result, however, does not yield the decidability of MELL, a problem still open today.
A further step connecting MELL decidability and reachability properties for Petri Nets has been achieved by de Groote, Guillaume and Salvati [dGGS04], who established the equivalence between MELL and the reachability problem for Vector Addition Tree Automata (VATA), a generalization of Vector Addition Systems. Independently, Verma and Goubault-Larrecq introduced a branching extension of VASS, called BVASS, for which they proposed a notion of Karp and Miller Tree that allows to establish the decidability of properties such as finiteness, boundedness and emptiness of their model [VGL05]. Yet, they do not obtain the decidability or undecidability of the reachability problem. It turns out that the BVASS model is actually equivalent to the VATA model of de Groote, Guillaume and Salvati: therefore, the work of Verma and Goubault-Larrecq is indeed an interesting step towards the decidability of MELL. Further work on the complexity of decision problems for the BVASS model has been done by Demri [DJLL09] and by Finkel and Goubault-Larrecq [FG12], who present an extensive study of the coverability problem for complete well-strucured transition systems, extending the Karp and Miller procedure to a larger family of functional transition systems.

A careful reading of the reachability algorithm of Kobayashi and Mayr [Kos82, May84, May81] for Petri Nets or for VASS reachability reveals a feature of the model that is central in the construction of the algorithm: Petri Nets, as well as its equivalent models, VAS and VASS, are symmetric, in the sense that the model is stable by an inversion of the arrows. This feature is not present in the VATA or in the BVASS model, hence it seems unlikely that one can adapt the algorithm of Kobayashi and Mayr to the setting of VATA or BVASS.
Our contribution to this line of research consists in the symmetrization of the BVASS model, that we call Vector Addition Systems with States and Split/Join transitions (VASS-SJ). A consequence of this symmetrization is that, in contrast with the VASS or BVASS model from which it originates, the VASS-SJ model is no longer functional, and is not a well-structured transition system in the sense of Finkel and Goubault-Larrecq. Yet, a careful study of the execution semantics of the VASS-SJ model allows us nonetheless to introduce a suitable extension of the classical notion of Karp and Miller Tree for VASS to this new model of VASS-SJ. Karp and Miller Tree for VASS-SJ are given by a pair $(G, \mathcal{B})$, where G is a finite, directed acyclic graph and $\mathcal{B}$ a partitioning of its vertices, such that the quotient graph $G/\mathcal{B}$ is a tree. When restricted to the setting of VASSs, this quotient graph $G/\mathcal{B}$ is then isomorphic to the classical Karp and Miller tree. Finally, this construction allows us to prove a coverability theorem for VASS-SJ.

1 Vector Addition Systems with States and Split/Join Transitions

We introduce in this section the model of Vector Addition system with States and Split/Join Transitions, a symmetrization of the BVASS model. We give its execution semantics, and introduce its reachability problem.

1.1 Definitions

Definition 1.1. Vector Addition Systems with States and Split/Join Transitions.

A Vector Addition Systems with States and Split/Join transitions (VASS-SJ) is a 4-tuple $S = (G, T, m)$, where:

- $G = (Q, A)$ is a finite directed graph, where Q is the set of *states*,
- $m \geq 1$ is a natural number, the *dimension* of the system, and
- $T \subseteq (Q \times \mathbb{Z}^n \times Q) \cup (Q \times \{s\} \times Q \times Q) \cup (Q \times Q \times \{j\} \times Q)$ is a (finite) set of *transitions*, where
  1. $T \cap Q \times \mathbb{Z}^n \times Q$ is the set of *regular* transitions,
  2. $T \cap Q \times \{s\} \times Q \times Q$ is the set of *split* transitions, and
  3. $T \cap Q \times Q \times \{j\} \times Q$ is the set of *join* transitions.

A VASS-SJ with only regular and split transitions is called a *Vector Addition Systems with States and Split Transitions* (VASS-S), and a VASS-SJ with only regular and join transitions is a BVASS of [VGL05], and is called in the sequel a *Vector Addition Systems with States and Join Transitions* (VASS-J).

Definition 1.2. Configuration of a VASS-SJ.

Let $S = (G, m, v)$ be a VASS-SJ. A *single configuration* of G is a 2-tuple $c = (q, x)$, where $q \in Q$ is a *state* and $x \in \mathbb{N}^m$ is a *value*. A single configuration c is *non-negative* if and only if, for all $0 \leq i \leq m$, $c_i \geq 0$. A *configuration* of S is a finite multiset $\mathcal{C}$ of single configurations of S. For each $c \in \mathcal{C}$, the multiplicity of c in $\mathcal{C}$ is noted $M_{\mathcal{C}}(c)$. When using (multi-)set notations, we indicate the multiplicity of a element with a power notation, and in Example 1.4 below. A configuration is *non-negative* if and only if all its single configurations are non-negative.

For the sake of simplicity and readability, we will thoroughly use graphical representations for (single) configurations in the sequel, as introduced in the following definition.

Definition 1.3. Base of a configuration

Let $S = (G, T, m)$ be a VASS-SJ, and $\mathcal{C}$ be a configuration of S. A *Base* of $\mathcal{C}$ is a finite set $B_{\mathcal{C}}$ of vertices, each vertex being labelled with some c, where $c \in \mathcal{C}$, such that, for each $c \in \mathcal{C}$ with multiplicity $M_{\mathcal{C}}(c)$, there are exactly $M_{\mathcal{C}}(c)$ corresponding vertices labeled with c in $B_{\mathcal{C}}$.

Example 1.4. Consider a configuration $\mathcal{C} = \{(q_1, 1, 1)^3, (q_2, 0, 0)\}$. The corresponding base $B_{\mathcal{C}}$ is given by Figure 1.

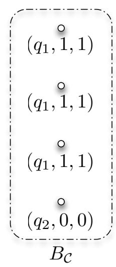


Figure 1: The Base $B_{\mathcal{C}}$ for $\mathcal{C} = \{(q_1, 1, 1)^3, (q_2, 0, 0)\}$.

1.2 Execution Semantics of a VASS-SJ

In this section, we provide a graphical exposition of the execution semantics for VASS-SJ, based on the notion of bases above. In order to do so, we enrich the set of possible transitions kinds with a new one, *idle* transitions. The set of idle transitions is $\{(q, q),\ q \in Q\}$.

Definition 1.5. Firing of a Transition.

Let $S = (G, T, m)$ be a VASS-SJ, and $B_{\mathcal{C}}$ and $B_{\mathcal{C}'}$ be two bases of S. Let t be a transition of S. The *firing* of t from $B_{\mathcal{C}}$ to $B_{\mathcal{C}'}$ is the relation $B_{\mathcal{C}} \hookrightarrow_t B_{\mathcal{C}'}$ given by Figure 2.

In order to express in a graph theoretic way the execution semantics of a VASS-SJ, we recall these classical notions of graph theory.

Definition 1.6. Quotient graph, quotient tree

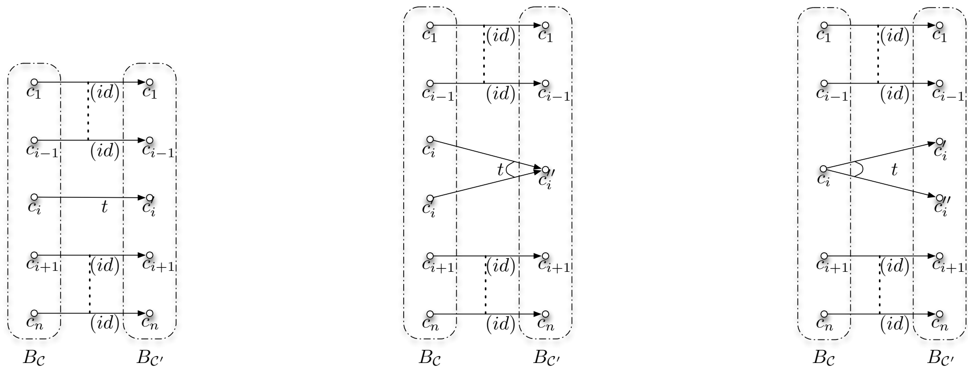
Let $G = (V, E)$ be a finite directed graph. A partitioning of G is a division of its vertices into disjoint subsets $\mathcal{B} = \{B_1, \cdots, B_k\}$. The *quotient graph* induced by the partitioning, written, $G/\mathcal{B}$, is the graph $G' = (B, E')$ where $(B_i, B_j) \in E'$ if and only if there exists $v_i \in B_i$ and $v_j \in B_j$ such that $(v_i, v_j) \in E$. If $G/\mathcal{B}$ is a tree we call it the *quotient tree* of G induced by $\mathcal{B}$.

Now, we are able to express in terms of graphs an execution of a VASS-SJ:

Definition 1.7. Transition Sequence.

Let S be a VASS-SJ. A *transition sequence* on S is a pair $(G, \mathcal{B})$, where $G = (V, E)$ is a labelled graph and $\mathcal{B} = (B_i)_{i \in \mathbb{N}}$ is a partition of V, such that:

1. For any index i, B_i is a base of S, and



t is regular:
$t = (q, a_1, \cdots, a_m, q')$,
$c_i = (q, x_1, \cdots, x_m)$, and
$c'_i = (q', x_1 + a_1, \cdots, x_m + a_m)$.

t is join:
$t = (q, q', j, q'')$,
$c_i = (q, x_1, \cdots, x_m)$,
$c'_i = (q', x'_1, \cdots, x'_m)$, and
$c''_i = (q''_i, x_1 + x'_1, \cdots, x_m + x'_m)$.

t is split:
$t = (q, s, q', q'')$,
$c_i = (q, x'_1 + x''_1, \cdots, x'_m + x''_m)$,
$c'_i = (q', x'_1, \cdots, x'_m)$, and
$c''_i = (q'', x''1, \cdots, x''_m)$.

Figure 2: Firing $B_{\mathcal{C}} \hookrightarrow_t B_{\mathcal{C}'}$ of a transition t from a base $B_{\mathcal{C}}$ to a base $B_{\mathcal{C}'}$.

2. For two consecutive indexes i and $i+1$, the restriction of G to $B_i \uplus B_{i+1}$ is a transition firing $B_i \hookrightarrow_{t_i} B_{i+1}$ for some transition t_i of S.

An example of a transition sequence is given in Figure 3.

Clearly, for a transition sequence $(G, \mathcal{B})$, the quotient graph $G/\mathcal{B}$ is a sequence. Without loss of generality, we assume that the edges (B_i, B_{i+1}) in this sequence are labelled with the corresponding transition fired.
It is clear, from the definition above, the VASS-S and VASS-SJ are not functional transition systems: for a given VASS-SJ S, the split transition rule of Figure 2 induces an unbounded number of transition functions.

Definition 1.8. $\mathbb{Z}$ and $\mathbb{N}$ Reachability Problem.

Let S be a VASS-SJ, B_0, B_1 be two bases of S. The $\mathbb{Z}$ *reachability problem* for S, B_0, B_1 is the following:
Does there exists a finite transition sequence in S from B_0 to B_1.
The $\mathbb{N}$ *reachability problem* for S, B_0, B_1 is the following:
Does there exists a finite transition sequence in S from B_0 to B_1, with only non-negative single configurations.

Theorem 1.9. *The $\mathbb{Z}$ reachability problem for VASS-SJ is decidable.*

Proof. Let $S = (G, T, m)$ be a VASS-SJ with $G = (Q, A)$, and let B_0 and B_n be two bases of configurations of S. We define the VASS $V = (G', T', m')$, where:

- G' has only one state q,
- $m' = m + |Q|$, where the m first coordinates encode the sum, over all vertices of a base of S, of the corresponding coordinate, and the $|Q|$ last ones encode the states of the vertices of bases of S,
- for every regular transition $a = (q_1, v, q_2) \in T$, there exists a regular transition $t_a = (q, v', q) \in T'$, where
  - for $i = 1 \cdots m$, $v'_i = v_i$, and
  - for $i = m+1 \cdots m+|Q|$, $v'_i = -1$ if the state q_{i-m} is the origin of a in S, $v'_i = +1$ if the state q_{i-m} is the destination of a in S, and $v'(a)_i = 0$ otherwise,

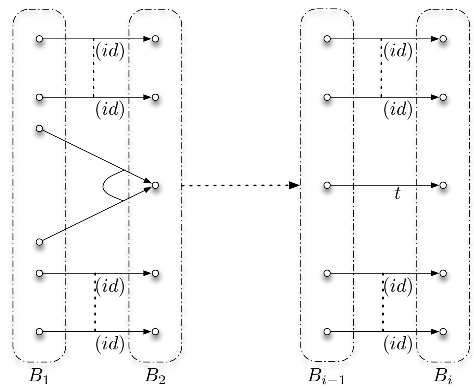


Figure 3: An example of a transition sequence.

- for every split transition $a = (q_1, s, q_1^2, s, q_2^2) \in T$, there exists a regular transition $t_a = (q, v', q) \in T'$, where
  - for $i = 1 \cdots m$, $v'_i = 0$, and
  - for $i = m + 1 \cdots m + |Q|$, $v'_i = -1$ if the state q_{i-m} is the origin of a in S, $v'_i = +1$ if the state q_{i-m} is a destination of a in S, and $v'(a)_i = 0$ otherwise, and
- for every join transition $a = (q_1^1, q_2^1, j, q_2) \in T$, there exists a regular transition $t_a = (q, v', q) \in T'$, where
  - for $i = 1 \cdots m$, $v'_i = 0$, and
  - for $i = m + 1 \cdots m + |Q|$, $v'_i = -1$ if the state q_{i-m} is an origin of a in S, $v'_i = +1$ the state if q_{i-m} is the destination of a in S, and $v'(a)_i = 0$ otherwise.

For a base B on S, we define the configuration $f(B) = (q, x)$ on V as follows:

- for $i = 1 \cdots m$, x_i is the sum, over all vertices v of B labelled with $(q, y_1, \cdots, y_m)$ for any state q, of the corresponding coordinate values y_i, and
- for $i = m + 1 \cdots m + |Q|$, x_i is the number of vertices v of B labelled with $(q_{i-m}, y_1, \cdots, y_m)$ for any $(y_1, \cdots, y_m) \in \mathbb{Z}^m$.

Then, there exists a transition sequence $f(B_0) \hookrightarrow \cdots \hookrightarrow f(B_n)$ of V, positive with respect to the $|Q|$ last coordinates, if and only if there exists a transition sequence $B'_0 \hookrightarrow \cdots \hookrightarrow B'_n$ on S, such that $B'_0 = B_0$ and, for all $i = 0, \cdots, n$, $f(B'_i) = f(B_i)$. If, moreover, the graph of the transition sequence $B'_0 \hookrightarrow \cdots \hookrightarrow B'_n$ is simply connected, one can derive a transition sequence $B_0 \hookrightarrow \cdots \hookrightarrow B_n$ on S that its graph is also simply connected.
Indeed, it is easy to see that, if there exists a firing $B_i \hookrightarrow_t B_{i+1}$ in S, then there exists a firing $f(B_i) \hookrightarrow_{f(t)} f(B_{i+1})$ in V, where $f(t)$ is the regular transition of V associated to the transition t of S. Moreover, the last $|Q|$ coordinates of $f(B_i)$ and $f(B_{i+1})$ are by construction, non-negative. Therefore, if there exists a transition sequence $B_0 \hookrightarrow \cdots \hookrightarrow B_n$ in S, an easy proof by induction on the length of the sequence shows that there exists a transition sequence $f(B_0) \hookrightarrow \cdots \hookrightarrow f(B_n)$ in V, positive with respect to the $|Q|$ last coordinates.
For the converse, consider a transition firing $F_i \hookrightarrow_{f(t)} F_{i+1}$ in V, positive with respect to the $|Q|$ last coordinates. Assume that at least one of the $|Q|$ last coordinates of F_i is non-zero: then, there exists a base B'_i of S, such that $F_i = f(B'_i)$. By construction of V, there exists then a transition firing $B'_i \hookrightarrow_t B'_{i+1}$ in S,

where $F_{i+1} = f(B'_{i+1})$. Therefore, if there exists a transition sequence $F_0 \hookrightarrow \cdots \hookrightarrow F_n$ in V, non-negative with respect to the last $|Q|$ coordinates, such that $F_0 = f(B_0)$, an easy induction on the length of the sequence shows that there exists a transition sequence $B'_0 \hookrightarrow \cdots \hookrightarrow B'_n$ on S, such that $B'_0 = B_0$ and, for all $i = 0, \cdots, n$, $f(B'_i) = F_i$.
Assume now moreover that the graph of the transition sequence $B'_0 \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-1}} B'_n$ is simply connected. We prove by induction on the length of the transition sequence that there exists a transition sequence $B_0 \hookrightarrow \cdots \hookrightarrow B_n$, with the same graph.

- If $n = 0$, the case is trivial
- Assume it holds for $n-1$. Two case arise:
  1. The graph of the transition sequence $B'_0, \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-2}} B'_{n-1}$ is simply connected. By construction of V, there exists a base B_{n-1} and a transition firing $B_{n-1} \hookrightarrow_{t_{n-1}} B_n$ in S, with $F(B_{n-1}) = F(B'_{n-1})$. We apply the induction hypothesis on $B'_0, \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-2}} B'_{n-1}$ to obtain a transition sequence $B_0, \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-2}} B_{n-1} \hookrightarrow_{t_{n-1}} B_n$.
  2. The transition sequence $B'_0, \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-2}} B'_{n-1}$ has two simply connected components $B'^1_0, \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-2}} B'^1_{n-1}$ and $B'^2_0, \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-2}} B'^2_{n-1}$. Then, t_{n-1} is necessarily a join transition. Let $v'^1_{n-1} \in B'^1_{n-1}$ and $v'^2_{\in} B'^2_{n-1}$ be its two source vertices, one in each simply connected component. Let v'_n be the destination vertex of this join transition in $B'_0, \hookrightarrow_{t_0} \cdots \hookrightarrow_{t_{n-1}} B'_n$, and v_n be the corresponding vertex in B_n. Then, there exist B^1_{n-1} and B^2_{n-1} such that:
     - $F(B^1_{n-1}) = F(B'^1_{n-1})$ and $F(B^2_{n-1}) = F(B'^2_{n-1})$, and
     - for $i = 1, \cdots, m$, the sum of the i^{th} coordinates of the vertices $v^1_{n-1} \in B^1_{n-1}$ and $v^2_{n-1} \in B^2_{n-1}$, corresponding to $v'^1_{n-1} \in B'^1_{n-1}$ and $v'^2_{\in} B'^2_{n-1}$ respectively, equals the i^{th} coordinate of v_n.

     Then, it suffices to apply the induction hypothesis on both simply connected components, with final bases B^1_{n-1} and B^2_{n-1} respectively, to conclude.

Now, given such a transition sequence $f(B_0) \hookrightarrow \cdots \hookrightarrow f(B_n)$ of V, it is not clear how to decide whether one of the corresponding transition sequence $B'_0 \hookrightarrow \cdots \hookrightarrow B'_n$ on S has a simply connected graph. It may be the case that some graphs are simply connected while some are not. We do so by enriching V as follows.
Assume the vertices of B_0 are numbered $v^0_0, \cdots, v^0_{s-1}$, and let $P = \{p_1, \cdots, p_n\}$ be the set of partitions of B_0. Note that any transition sequence on S starting on B_0 has at most s simply connected components, and that the set of these simply connected components corresponds to an element in P (a partition of B_0). Let now $V = (G', T', m', v')$, where:

- G' has n states $q_1, \cdots, q_n$. The semantics of V being in state q_i is that V simulates a transition sequence on S starting on B_0 whose simply connected components correspond to the partition p_i of B_0.
- $m' = m.s + |Q|.s$. The value space keeps track of the states of all vertices for any simply connected component of the transition sequence on S (the last $|Q|.s$ components of the space), as well as the sum of the values for any simply connected component of the transition sequence (the first $m.s$ components).
- for any regular transition $a = (q_1, v(a), q_2) \in T$, there exist $n.s$ regular transitions $a^i_j = (q_i, v'(a^i_j), q_i)$, for $i = 1, \cdots, n$, $j = 0, \cdots, s-1$, where
  - for $t = m.j + 1, \cdots, m.(j+1)$, $v'(a^i_j)_t = v(a)_{t-m.j}$,
  - for $t = m.s + |Q|.j + 1, \cdots, m.s + |Q|.(j+1)$, $v'(a^i_j)_t = -1$ if the state $q_{t-m.s-|Q|.j}$ is the origin of a in S, $v'(a)_i = +1$ if the state $q_{t-m.s-|Q|.j}$ is the destination of a in S,
  - and $v'(a^i_j)_t = 0$ otherwise.
- for any split transition $a \in T$, there exist $n.s$ regular transitions $a^i_j = (q_i, v'(a^i_j), q_i)$, for $i = 1, \cdots, n$, $j = 0, \cdots, s-1$, where

- for $t = m.s + |Q|.j + 1, \cdots, m.s + |Q|.(j+1)$, $v'(a^i_j)_t = -1$ if the state $q_{t-m.s-|Q|.j}$ is the origin of a in S, $v'(a)_i = +1$ if the state $q_{t-m.s-|Q|.j}$ is a destination of a in S,
- and $v'(a^i_j)_t = 0$ otherwise.

- for any join transition $a \in T$, there exist $\frac{n.(n-1)}{2}.\frac{s.(s-1)}{2}$ regular transitions $a^{i,k}_{j,l} = (q_i, v'(a^{i,k}_{j,l}), q_k)$, for $i \leq k = 1, \cdots, n$, $j \leq l = 0, \cdots, s-1$, where
  - for $t = m.s + |Q|.j + 1, \cdots, m.s + |Q|.(j+1)$, $v'(a^{i,k}_{j,l})_t = -1$ if the state $q_{t-m.s-|Q|.j}$ is an origin of a in S, $v'(a^{i,k}_{j,l})_t = +1$ if the state $q_{t-m.s-|Q|.j}$ is the destination of a in S,
  - for $t = m.s + |Q|.l + 1, \cdots, m.s + |Q|.(l+1)$, $v'(a^{i,k}_{j,l})_t = -1$ if the state $q_{t-m.s-|Q|.l}$ is the other origin of a in S,
  - $v'(a^{i,k}_{j,l})_t = 0$ otherwise,
  - and $p_k \in P$ is the partition of $\mathcal{C}_0$ obtained from p_i by merging the simply connected components corresponding to c_j and c_l.

Denote by $p_1 \in P$ the partition of B_0 in exactly s components, and by p_n that of B_0 in exactly 1 component. B_0 can be encoded by a configuration (q_1, x_1) of V, where x_1 is the concatenation of values and states of vertices in B_0. Then, as above, there exists a transition sequence $(q_1, x_1) \to \cdots \to (q_n, x_n)$ on V positive with respect to the $|Q|.s$ last coordinates, for some x_n encoding the set of states of vertices in B_n, and the sum of their values, if and only if there exists a transition sequence $B_0 \hookrightarrow \cdots \hookrightarrow B'_n$ on S , such that $F(B'_n) = F(B_n)$. Moreover, by construction, the transition sequence $B_0 \hookrightarrow \cdots \hookrightarrow B'_n$ is now simply connected, and there exists therefore a simply connected transition sequence $B_0 \hookrightarrow \cdots \hookrightarrow B_n$ on S.
By the decidability of the $\mathbb{N}$ reachability for VASS, it follows that it is decidable whether there exists a simply connected transition sequence $B_0 \hookrightarrow \cdots \hookrightarrow B_n$ on S. The decidability of the $\mathbb{Z}$ reachability problem for VASS-SJ follows by performing the reduction above for all possible partitions of B_0 and B_n, and checking whether there exists such a partition that allows for simply connected transition sequences. □

2 Increasing Transition Sequences, their Repetitions, and an Abstraction: Generalized Transition Sequences

2.1 Repeating non-negative Transition Sequences

As a first step towards a good notion of coverability for VASS-SJs, let us exhibit the conditions needed for being able to indefinitely repeat a given (non-negative) transition sequence of a given VASS-SJ. Of course, our aim is to ensure that the corresponding transition sequence remains non-negative. This question, in the setting of well structured transition systems of [FG12] or, more generally, functional transition systems, is simple: since the number of transition functions is finite, a repeatable transition sequence does not induce any increase in the number of admissible transition firings, and notion such as composition, or repetition of transition sequences, are straightforward.
In our setting, this is no longer the case. As an example, let us consider the transition sequence from base $B_{\mathcal{C}}$ to base $B_{\mathcal{C}'}$ given in Figure 4, that we assume non-negative. This transition sequence induce an increase in the number of vertices from its initial to its final base: it contains therefore some split transition firings. This transition sequence can be indefinitely repeated only if there exists a way to "plug" the destination base $B_{\mathcal{C}'}$ into the source base $B_{\mathcal{C}}$, in order to be able to fire the very same sequence of transitions. This "plugging" is described by a graph, that associates injectively every vertex v of $B_{\mathcal{C}}$ to a vertex v' of $B_{\mathcal{C}'}$, the two single configurations labeling the two associated vertices sharing the same state. An additional requirement is that, for each vertex v in $B_{\mathcal{C}}$, the single configuration of its associated vertex v' in $B_{\mathcal{C}'}$, is greater than its own single configuration. Clearly, if some single configuration of a vertex in v' $B_{\mathcal{C}'}$, source of a split transition firing, is strictly greater than the single configuration of its associated vertex v in $B_{\mathcal{C}}$, the number of possible firings for this split transition from v', is also strictly greater than the number of possible firings for this

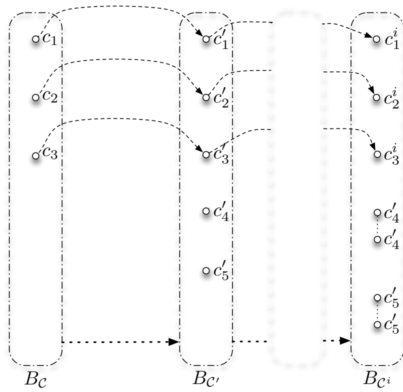


Figure 4: The i^{th} repetition of a transition sequence.

split transition from v. From this simple observation, it follows that, the fact that our VASS-SJ model is nonfunctional induces that composition or repetition of non-negative transition sequence in our setting is an ambiguous notion, which needs to be detailed further.
An example of a graph, associating vertices in $B_{\mathcal{C}}$ with vertices in $B_{\mathcal{C}'}$, that we call *welding graph*, together with a corresponding repetition of the transition sequence, is given in Figure 4, where edges of this welding graph are given by the dotted arrows.
In order to ensure that there exists a non-negative i^{th} repetition of this transition sequence, as in Figure 4, we require that, for $n = 1, 2$ or 3, no coordinate of c'_n is strictly smaller than the corresponding coordinate of c_n. For the coordinates of c'_n strictly greater than the corresponding coordinate of c_n, the i^{th} repetition of the transition sequence may induce an increase of the corresponding coordinate in c^i_n. Moreover, the i^{th} repetition of the transition sequence may also induce the creation of i copies of the isolated points c'_4 and c'_5 of the welding graph. As it turns out, allowing an unbounded number of repetition of this transition sequence induces two levels of unboundedness in the resulting base: unbounded coordinates in some single configurations, and unbounded number of occurrences of some single configurations. This leads us to the idea of introducing a generalization of our notions of configurations and bases, allowing ∞ values for coordinates in single configurations, as in the classical case, but also ∞ multiplicities of single configurations. Of course, when dealing with ∞ values and multiplicities, one needs to extend the notions of bases, transition firings, transitions sequences, welding graphs and so on. These ideas are detailed in the definitions below.

2.2 Generalized Configurations

Definition 2.1. Generalized configurations.
For $m \geq 1$, we denote by $\leq$ the usual product order on $(\mathbb{N} \cup \infty)^m$.
Let $S = (G, T, m)$ ba a VASS-SJ.

1. A *generalized single configuration* of S is a 2-tuple $g = (q, x)$, where $q \in Q$ is a *state* and $x \in (\mathbb{Z} \cup \infty)^m$ is a *generalized value*. g is *non-negative* if and only if $x \geq (0, \cdots, 0)$.

2. A *generalized configuration* of S is a finite set $\mathcal{G}$ of generalized single configurations of S, together with a multiplicity function $M_{\mathcal{G}} : \mathcal{G} \rightarrow \mathbb{N} \cup \infty$, i.e. a finite multiset whose elements may have finite or infinite multiplicity. $\mathcal{G}$ is *non-negative* if and only if all its single generalized configurations are non-negative.

We extend to multiset with possible ∞ multiplicity the usual set operations. Moreover, for the sake of readability, we will usually write a generalized configuration $\mathcal{G}$ with a set notation as follows:

$$\mathcal{G} = \{c_1^{k_1}, \cdots, c_n^{k_n}\},$$

where each of the single generalized configurations c_i, $i = 1, \cdots, n$ has multiplicity k_i in $\mathcal{G}$.
For a set $\mathcal{D}$ of generalized single configurations, we write also $\mathcal{D}$ for the generalized configuration with underlying set $\mathcal{D}$, where all elements have multiplicity 1.

As above we introduce *bases* of generalized configurations, as their graphical representations.

Definition 2.2. Base of a generalized configuration

Let $S = (G, T, m)$ be a VASS-SJ, and $\mathcal{G}$ be a generalized configuration of S. A *Base* of $\mathcal{G}$ is a finite set $B_\mathcal{G}$ of vertices, each vertex being labelled with some g^k, where $g \in \mathcal{G}$ and $k \in \{1, \infty\}$, such that:

1. For each $g \in \mathcal{G}$, $M_\mathcal{G}(g)$ is the sum, over all vertices labelled with g^k for some k, of these k.

2. For each $g \in \mathcal{G}$ with multiplicity ∞, exactly one vertex with labelled with g^∞.

Moreover, $B_\mathcal{G}$ is in *normal form* if, for any $g \in \mathcal{G}$ with ∞ multiplicity, no vertex is labelled with g^1. Note that, for every generalized configuration, the corresponding base in normal form is unique. The *normalization* of a base consists in merging all vertices labelled with g^1 with the vertex labelled with g^∞, when it exists, for all single generalized configurations g. The merging of a vertex labelled with g^1 with a vertex labelled with g^∞ is a *normalization step.*

Example 2.3. Consider a generalized configuration $\mathcal{G} = \{(q_1, 1, \infty)^\infty, (q_2, 0, 0)^2\}$. A corresponding base $B_\mathcal{G}$ is given by Figure 5.

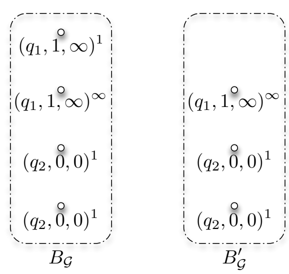


Figure 5: A Base $B_\mathcal{G}$ for $\mathcal{G} = \{(q_1, 1, \infty)^\infty, (q_2, 0, 0)^2\}$, and its normal form $B'_\mathcal{G}$.

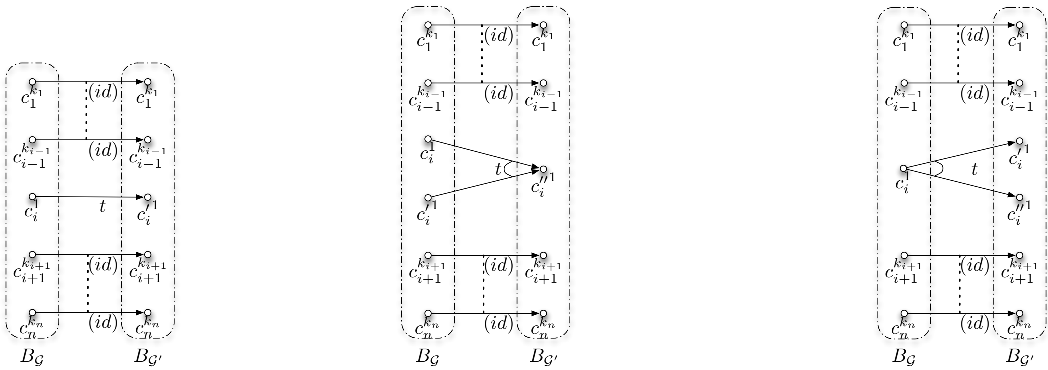
Let us now extend the firing of a transition over bases of generalized configurations.

Definition 2.4. Firing of a Transition

Let $S = (G, T, m)$ be a VASS-SJ, and $B_\mathcal{G}$ and $B_{\mathcal{G}'}$ be two bases of generalized configurations of S. Let t be a transition of S. Assume the vertices that are source or target of the edge(s) labelled with t are labelled with c_i^1 and $c_i'^1$ (case of a regular transition), or by c_i^1, $c_i'^1$ and $c_i''^1$ (case of a join or a split transition) respectively. Then, the *firing* of t from $B_\mathcal{G}$ to $B_{\mathcal{G}'}$ is the relation $B_\mathcal{G} \hookrightarrow_t B_{\mathcal{G}'}$ given by Figure 6. Note that the firing of a split on a ∞ coordinate produces a corresponding ∞ coordinate in both destination vertices. The general case of a transition firing (∞ multiplicity of the source or destination of the transition) is obtained from Figure 6 by one or several normalization steps on the vertices labelled with c_i^1, $c_i'^1$ or $c_i''^1$.

2.3 Welding Graphs

Let us now build upon the idea exposed in Figure 4: under which circumstances can one safely repeat a given non-negative transition firing sequence, from a base $B_\mathcal{G}$ of a generalized configuration to a base $B_{\mathcal{G}'}$ of another generalized configuration sequence, without risking integer coordinates going into the negatives? In order to do so, one needs again a proper way to "plug" $B_{\mathcal{G}'}$ into $B_\mathcal{G}$, so that:



t is regular:
$t = (q, a_1, \cdots, a_m, q')$,
$c_i = (q, x_1, \cdots, x_m)$, and
$c'_i = (q', x_1 + a_1, \cdots, x_m + a_m)$.

t is join:
$t = (q, q', j, q'')$,
$c_i = (q, x_1, \cdots, x_m)$,
$c'_i = (q', x'_1, \cdots, x'_m)$, and
$c''_i = (q''_i, x_1 + x'_1, \cdots, x_m + x'_m)$.

t is split:
$t = (q, s, q', q'')$,
$c_i = (q, x'_1 + x''_1, \cdots, x'_m + x''_m)$,
$c'_i = (q', x'_1, \cdots, x'_m)$,
$c''_i = (q'', x''1, \cdots, x''_m)$, and
$x'_1 = \infty \Leftrightarrow x''_1 = \infty, \cdots, x'_m = \infty \Leftrightarrow x''_m = \infty$.

Figure 6: Firing $B_{\mathcal{G}} \hookrightarrow_t B_{\mathcal{G}'}$ of a transition t from a base $B_{\mathcal{G}}$ to a base $B_{\mathcal{G}'}$.

1. the sequence of transitions can be fired again and,
2. no integer coordinate decreases in the process.

These properties are ensured by the following notion of *welding graph*, which describes how vertices of $B_{\mathcal{G}}$ are associated to vertices of $B_{\mathcal{G}'}$, and the corresponding $\preceq$ order relation, which ensures that integer coordinates do not decrease in the process.

Definition 2.5. $\preceq$ relation, welding graphs

Let $S = (G, T, m)$ be a VASS-SJ, and $B_{\mathcal{G}}$ and $B_{\mathcal{G}'}$ be two bases of generalized configurations $\mathcal{G}$ and $\mathcal{G}'$ respectively. Then, $B_{\mathcal{G}} \preceq B_{\mathcal{G}'}$ if and only if there exists a directed graph $G_{B_{\mathcal{G}} \preceq B_{\mathcal{G}'}} = (V, E)$ such that:

- $V = B_{\mathcal{G}} \uplus B_{\mathcal{G}'}$,
- $E \subseteq B_{\mathcal{G}} \times B_{\mathcal{G}'}$,
- any $v \in B_{\mathcal{G}}$ labelled with g^1 is source of exactly one edge $(v, v') \in E$, where v' is labelled with ${g'}^1$, and $g \leq g'$,
- any $v \in B_{\mathcal{G}}$ labelled with g^∞ is source of at least one edge $(v, v') \in E$, where v' is labelled with ${g'}^\infty$. Moreover, for any edge $(v, v'') \in E$, v'' is labelled with some ${g''}^{k''}$, and $g \leq g''$. Finally,
- any $v' \in B_{\mathcal{G}'}$ labelled with ${g'}^1$ is destination of at most one edge $(v, v') \in E$.

The graph $G_{B_{\mathcal{G}} \preceq B_{\mathcal{G}'}}$ is called a *welding* graph for the relation $B_{\mathcal{G}} \preceq B_{\mathcal{G}'}$. Note that a given relation $B_{\mathcal{G}} \preceq B_{\mathcal{G}'}$ may have different welding graphs.
If, moreover, for any edge $(v_0, v_1) \in G_{B_{\mathcal{G}} \preceq B_{\mathcal{G}'}}$, with v_1 labelled with $g_1^{k_1}$ and v_0 labelled with $g_0^{k_0}$, each coordinate in $(g_1 - g_0)$ is either 0 or ∞, we write $g_0 \leq_\infty g_1$ and $B_{\mathcal{G}} \preceq_\infty B_{\mathcal{G}'}$ respectively.

Clearly, the $\preceq$ relation is an order relation on bases.

Remark 2.6. For a given pair of bases $B_{\mathcal{G}}$ and $B_{\mathcal{G}'}$, and a given transition firing sequence from $B_{\mathcal{G}}$ to $B_{\mathcal{G}'}$, there may exist several different welding graphs $G_{B_{\mathcal{G}} \preceq B_{\mathcal{G}'}}$ corresponding to the relation $B_{\mathcal{G}} \preceq B_{\mathcal{G}'}$. For an example, see Figure 7 below.

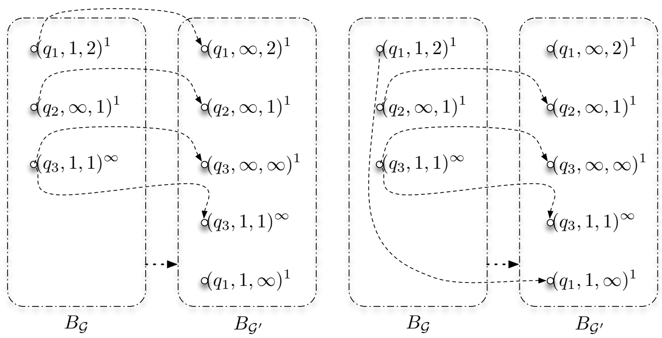


Figure 7: Two different welding graphs for a given pair of bases $B_{\mathcal{G}}$ and $B_{\mathcal{G}'}$.

We are now in a position to introduce the tools we will use to express synthetically the repetition of transition sequences that increase coordinate and multiplicities. A first such tool is the notion of lifting, which extends the classical one presented in [Reu88], and allows to transform a transition sequence with only finite coordinates and multiplicities, to a generalized version of it with infinite coordinates and multiplicities.

Definition 2.7. Liftings

Let S ba a VASS-SJ, and B be the base of a generalized configuration. A *single lifting* on B consists in one of the two following operations:

1. Making a copy v' labelled with $(q, y_1, \cdots, y_m)^\infty$, of a vertex v labelled with $(q, x_1, \cdots, x_m)^1$, with $(x_1, \cdots, x_m) \leq (y_1, \cdots, y_m)$ and $(x_1, \cdots, x_m) \not\leq_\infty (y_1, \cdots, y_m)$.
2. Replacing some finite coordinate of a vertex label of multiplicity 1, with ∞.

A *lifting* is a finite sequence of single liftings.

Note that we do not allow any coordinate increase of a vertex label of infinite multiplicity. The reason behind is the need to be consistent with the fact that a transition firing on a base of a generalized configuration transfers all labels with infinite multiplicity through idle transitions, without modifying them.
Note also that liftings are monotonous with respect to the $\preceq$ order relation.
As in the classical case for VASS, liftings are a key tool in the construction of a Karp and Miller Tree, which is a synthetic representation of all possible sequences of transition firings with unbounded values. In order to allow a similar construction in our setting of VASS-SJ, we need to extend or notions of transition sequences to the case of infinite values and multiplicities. This is the purpose of the following definition.

Definition 2.8. Generalized Transition Tree

Let S be a VASS-SJ. A *Generalized transition tree* $(G, \mathcal{B})$ on S is a labelled acyclic directed finite graph $G = (V, E)$, and a partitioning $\mathcal{B} = \{B_1, \cdots, B_k)$ of V such that:

- Every B_i is a base of a generalized configuration of S,
- $G/\mathcal{B}$ is a quotient tree,
- for any edge (B_i, B_j) of $G/\mathcal{B}$, there exists a transition firing $B_i \hookrightarrow_t B'_j$ such that:

1. the edges of the restriction of G to $B_i \uplus B_j$ are exactly those of $B_i \hookrightarrow_t B'_j$, and
2. B_j can be obtained from B'_j with one or several liftings.

Moreover, $(G, \mathcal{B})$ is in *normal form* if all bases in $\mathcal{B}$ are in normal form.

Note that, in a generalized transition tree $(G, \mathcal{B})$, where B_1 is the root of $G/\mathcal{B}$, the only vertices of in-degree 0 are the vertices of B_1. When $G/\mathcal{B}$ is a sequence, G will be refereed as a *Generalized transition sequence*. When moreover all bases B_i are actually bases of configurations, G will be refereed as a *transition sequence*. Finally, a (generalized) transition sequence, from B_i to B_j, is *increasing* when $B_i \preceq B_j$.
Clearly, generalized transition sequences extend the notion of transition sequences. Our purpose is to use generalized transition sequence to represent in some succinct way all possible non-negative transition sequence starting from a given base. In order to do so we need to relate the two concepts, with a notion of *compatibility*, as follows.

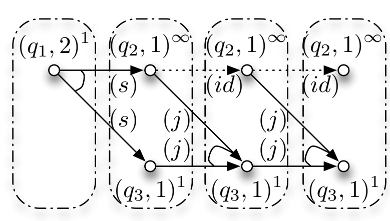
Definition 2.9. Transition sequence compatible with a generalized transition sequence

Let S be a VASS-SJ. Let $(G, \mathcal{B})$, with $\mathcal{B} = B_1, \cdots, B_f$, be a generalized transition sequence on S, with initial base B_1 and final base B_f.
Let D_1 and D_f be two bases of configurations, and $(P, \mathcal{D})$ be a non-negative transition sequence on S, with initial base D_1 and final base D_f, where $\mathcal{D} = D_1, \cdots, D_f$.
$D_1, \cdots, D_f$ are *compatible* with $B_1, \cdots, B_f$ if and only if there exist functions $f_{D_i \to B_i} : D_i \to B_i$, for $i = 1, \cdots, f$, called *compatibility* functions, and non-negative integer constants $c_1, \cdots, c_f$, called *compatibility caps*, such that, for all $i = 1, \cdots, f$

1. $\forall v \in V_i$ labelled with $(q, x_1, \cdots, x_m)$, $f_{D_i \to B_i}(v)$ is labelled with h^l with $h = (q, y_1, \cdots, y_m)$ and, for $j = 1, \cdots, m$:
   - $y_j = \infty \Rightarrow x_j \geq c_i$, and
   - $y_j \neq \infty \Rightarrow x_j = y_j$, and
2. for all $w \in B_i$ labelled with h^1, $|f^{-1}_{D_i \to B_i}(w)| = 1$,
3. for all $w \in B_i$ labelled with h^∞, $|f^{-1}_{D_i \to B_i}(w)| \geq c_i$,

$(P, \mathcal{D})$ is *compatible* with $(G, \mathcal{B})$ if and only if $D_1, \cdots, D_f$ are compatible with $B_1, \cdots, B_f$, and the normal form of $(G, \mathcal{B})$ can be obtained from $(P, \mathcal{D})$ by performing some liftings, and normalizing.

Remark 2.10. All generalized transition sequences do not necessarily admit compatible (non-negative) transition sequences. See the following example.

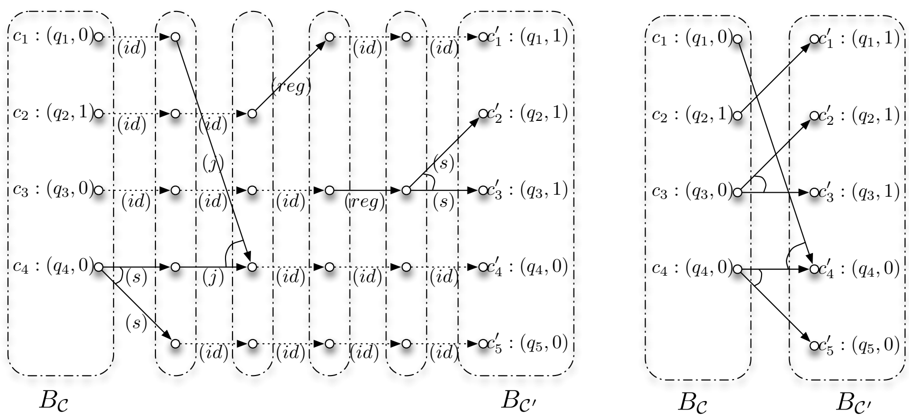


2.4 Increasing Transition Sequences

As for the classical case of Karp and Miller trees for VASS, our purpose here is to abstract increasing transition sequences by generalized transition sequences, by performing one or several liftings onto the sequence. Theses liftings need to take into account all, and only all possible ways of increasing values and

multiplicities arbitrarily high when repeating the increasing transition sequence. As seen before, there may exist several ways of repeating a single increasing transition sequence, each way of repeating it inducing different increases in coordinate values and in multiplicities of single configurations. All these different possibilities need to be taken into account when performing the said lifting.
For the sake of simplicity, let us begin with an example, in order to illustrate how to associate a lifting to an increasing transition sequence.

Example 2.11. Consider the following increasing sequence of dimension 1, from a base base $B_{\mathcal{C}}$ to a base $B_{\mathcal{C}'}$, with $B_{\mathcal{C}} \preceq B_{\mathcal{C}'}$.



Here, there is only one possible welding graph, it is therefore omitted in the figure. For the sake of readability, the transition sequence is informally presented in Figure 8 as one single step, where only non-(id) transitions are represented (note that such a presentation is not always possible).
In this example, we have a pair of vertices (c_1, c'_1) such that:

1. (c_1, c'_1) is an edge of the welding graph, and
2. $c_1 < c'_1$.

It would seem at first sight that, when repeating i times this increasing sequence, the (only) coordinate of the resulting vertex c_1^i would increase linearly to i. Yet, the i^{th} repetition of the transition sequence given in Figure 8 shows the opposite: the highest coordinate value reached by one of the vertices c_1^i in this repetition is 1. Conversely, we also have a pair of vertices (c_4, c'_4) such that:

1. (c_4, c'_4) is an edge of the welding graph, and
2. $c_4 = c'_4$.

It would seem then that, when repeating i times this increasing sequence, the (only) coordinate of the resulting vertex c_4^i would remain 0. Yet, again, Figure 8 shows the opposite: $c_4^i = (q_4, i-1)$.
There is more: the transition sequence contains the firing of a split transition from c_3 to (c'_2, c'_3). When repeating the sequence, the coordinate of the source vertex of the split increases, from 0 in the first occurrence of the sequence, to at least 1 in the following occurrences. As a result, there exist several (non-deterministic) ways of repeating the increasing transition sequence: the only limiting factor being to ensure that the repetition is indeed non-decreasing. Figure 8 shows one choice, where every firing of the split transition is constrained to produce a pair of single configurations (c_2^i, c_3^i) where $c_2^i = (q_2, 1)$. Another choice, where every firing of the split transition is constrained to produce a pair of single configurations (c_2^i, c_3^i) where $c_3^i = (q_3, 1)$, gives the repetition of Figure 9. Clearly, the resulting base $B_{\mathcal{C}^i}$ differs in both cases.

There is even more: take an arbitrary $N \in \mathbb{N}$. Then, from the i^{th} repetition of Figure 8 with $i \geq 3N$, followed by two other occurrences of the same transition sequence with a well chosen firing of the split transition (c_3, s, c'_2, c'_3), on can reach a configuration

$$B_{\mathcal{C}^N} = \quad \{c_1^N,\ c_2^N, c_3^N, c_4^N, \quad \underbrace{c_5^N, \cdots, c_5^N}_{n_5}\}$$

where

1. $c_1^N \geq (q_1, N)$,
2. $c_2^N \geq (q_2, N)$,
3. $c_3^N \geq (q_3, N)$,
4. $c_4^N \geq (q_4, N)$, and
5. $n_5 \geq N$.

Therefore, the correct lifting for abstracting this increasing transition sequence needs to be the one of Figure 11.

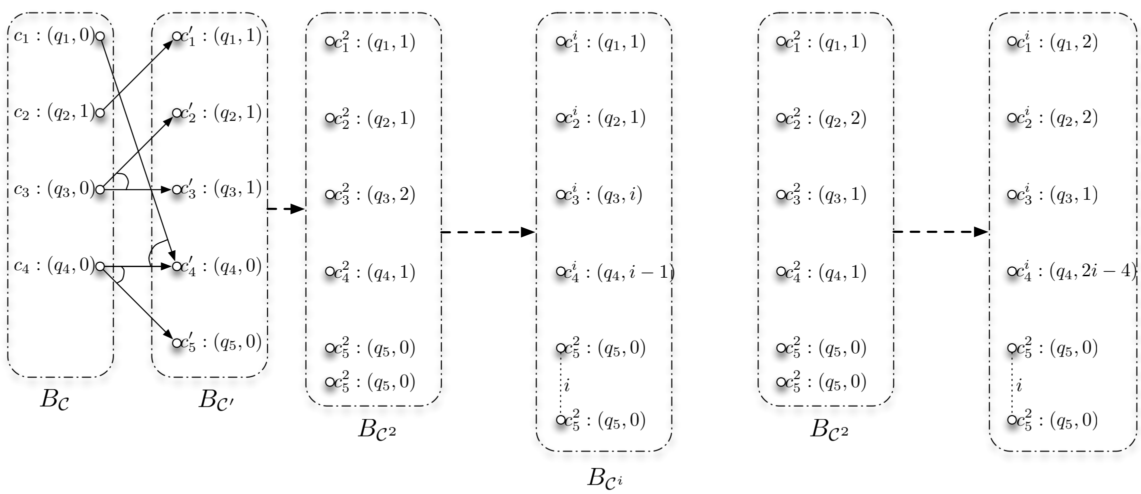


Figure 8: An increasing transition from $B_{\mathcal{C}}$ to $B_{\mathcal{C}'}$, and one of its i^{th} repetition to $B_{\mathcal{C}^i}$.

Let us now formalize these ideas.

Definition 2.12. Switchings

Let $(G, \mathcal{B})$ be a transition sequence from B_1 to B_n. Let (v_i, s, v_j, v_k) be the firing of a split transition in $(G, \mathcal{B})$. A *switching* of (v_i, s, v_j, v_k) is a choice of one of the destination vertices v_j and v_k. Let $\mathcal{S}$ be a set of switchings for all firings of spit transitions of $(G, \mathcal{B})$.
$\mathcal{S}$ induces the following *switched* transition sequence $(G_{\mathcal{S}}, \mathcal{B})$, obtained from $(G, \mathcal{B})$ be erasing every edge (v_i, v_j), where:

1. (v_i, v_j) is an edge of a split transition firing in $(G, \mathcal{B})$, and
2. v_j is not the destination of this split transition firing selected by the corresponding switching in $\mathcal{S}$.

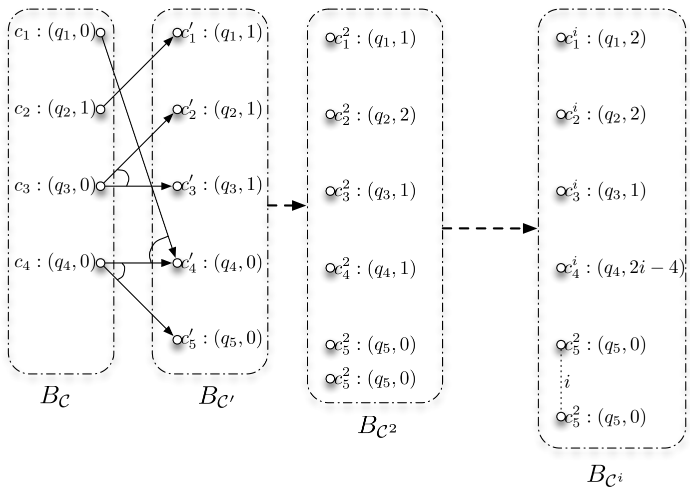


Figure 9: Another i^{th} repetition of the same transition sequence to $B_{\mathcal{C}^i}$.

Let $\mathcal{S}$ be a switching of $(G, \mathcal{B})$, and B'_1 be a base, such that $B_1 \preceq B'_1$ with welding graph $G_{B_1 \preceq B'_1}$. $\mathcal{S}$ and $(G, \mathcal{B})$ induce a transition sequence $(G', \mathcal{B}')$ from B'_1 to B'_n such that:

1. The vertices of B'_1 isolated in the welding graph $G_{B_1 \preceq B'_1}$ are transferred identically from B'_1 to B'_n through (id) edges. Let $G'_{isolated}$ be the corresponding sub-graph of G'.
2. The graph $G' \setminus G'_{isolated}$ is isomorphic to G. Abusing notations, denote by $(G' \setminus G'_{isolated}, \mathcal{B}')$ the restriction of $(G', \mathcal{B}')$ to its sub-graph $G' \setminus G'_{isolated}$.
3. For any firing (c'_i, s, c'_j, c'_k) of a split transition in $(G' \setminus G'_{isolated}, \mathcal{B}')$, isomorphic to (c_i, s, c_j, c_k) in $(G, \mathcal{B})$,
   (a) if c_j does not belong to the switching $\mathcal{S}$, c'_j is labelled with the same single configuration as c_j, and otherwise
   (b) c_k does not belong to the switching $\mathcal{S}$, and c'_k s labelled with the same single configuration as c_k.

It is clear that, for a given increasing non-negative transition sequence and a given welding graph, a switching induces a unique way of repeating the transition sequence. For a given increasing non-negative transition sequence $(G, \mathcal{B})$, a given welding graph $G_{B_1 \preceq B_n}$ and a given switching $\mathcal{S}$ of $(G, \mathcal{B})$, we denote by $(G^i_{\mathcal{S}}, \mathcal{B}^i_{\mathcal{S}})$ the corresponding i^{th} repetition of the sequence. Two examples of such repetitions of an increasing transition sequence are given in Figures 8 and 9.
The following Lemma sorts out which coordinates actually increase when repeating an increasing transition sequence accordingly to a given switching and a given welding graph. As actually shown by Figures 8 and 9, this is not as straightforward as one could expect.

Lemma 2.13. *Let $(G, \mathcal{B})$ be an increasing transition sequence from B_1 to B_n, with corresponding welding graph $G_{B_1 \preceq B_n}$. Let $\mathcal{S}$ be a switching of $(G, \mathcal{B})$, and, for $i \in \mathbb{N}$, let $(G^i_{\mathcal{S}}, \mathcal{B}^i_{\mathcal{S}})$ the corresponding i^{th} repetition of $(G, \mathcal{B})$, where $\mathcal{B}^i_{\mathcal{S}} = (B_1, \cdots, B^i_n)$. Denote by $G_{B_1 \preceq B^i_n}$ the corresponding welding graph.*
Let $G^{-1}_{B_1 \preceq B_n}$ be the mirror graph of $G_{B_1 \preceq B_n}$, where all edges are reversed, and $(G_{\mathcal{S}}, \mathcal{B})$ be the switched transition sequence induced by $\mathcal{S}$, and consider the graph $G' = G_{\mathcal{S}} \uplus G^{-1}_{B_1 \preceq B_n}$.
Then, for any vertex t of B^i_n labelled with $(q_t, x^i_{t,1}, \cdots, x^i_{t,n})$, for any coordinate $l = 1, \cdots, m$, $x^i_{t,l} = \Theta(i)$ if and only if:

1. *there exists* $v \in B_1$ *labelled with* $(q_v, x_{v,1}, \cdots, x_{v,n})$, $w \in B_n$ *labelled with* $(q_w, x_{w,1}, \cdots, x_{w,n})$, *and an edge* $(v, w) \in G_{B_1 \preceq B_n}$, *with* $x_{w,l} > x_{v,l}$,

2. *there exists* $z \in B_1$, $t \in B_n$, *and an edge* $(z, t) \in G_{B_1 \preceq B_n}$,

3. *there exists a path from* v *to* t *in* G', *and*

4. *there exists a path from* z *to* t *in* G'.

Additionally, there exists a constant $c_{(G,\mathcal{B})}$ *such that, for any vertex* t *of* ${B'}^i_n$ *labelled with* $(q_t, x^i_{t,1}, \cdots, x^i_{t,n})$, *there exists* $s \in B_1$ *labelled with* $(q_t, x^i_{s,1}, \cdots, x^i_{s,n})$ *and an edge* (s, t) *in* $G_{B_1 \preceq B^i_n}$ *such that, for any coordinate* $l = 1, \cdots, m$,

$$x^i_{t,l} \neq \Theta(i) \Rightarrow x^i_{t,l} \leq x^i_{s,l} + c_{(G,\mathcal{B})}.$$

For a given increasing transition sequence, with a given corresponding welding graph, and a given switching, the vertices and coordinates satisfying the conditions of Lemma 2.13 are called *augmentation coordinates.* In the previous example (Figure 9): $v = c_1$, $w = c'_1$, $z = c_4$, $t = c'_4$, and $c_{(G,\mathcal{B})} = 2$.

Proof. For the "if" direction, assume t satisfies the conditions of the Lemma.
In a first step, assume $v = z$ and $w = t$. Let k_v be the number of vertices of B_n in the path from v to t in G'. Without loss of generality, we can assume that $k_v = 1$. If it is not the case, it suffices to replace $(G, \mathcal{B})$ by $(G^{k_v}_{\mathcal{S}}, \mathcal{B}^{k_v}_{\mathcal{S}})$ in the proof.
For a vertex $x \in (G, \mathcal{B}) \setminus B_n$, we denote x^1 its occurrence in $(G, \mathcal{B})$, and x^i its last occurrence in $(G^i_{\mathcal{S}}, \mathcal{B}^i_{\mathcal{S}})$, modulo the identification induced by the welding graph. Following this notation, $v = v^1$ and $w = v^2$. Let $v^1_1, \cdots, v^1_k, v^2_1$ be the path from v to w in $(G, \mathcal{B})$, where $v^1 = v^1_1$. Then, an easy double induction on j and i shows that, for all i and j, $v^i_j > v^{i-1}_j$. This ensures that the increase in the coordinate is at least linear in i. The upper bound follows from the fact that the firing of any transition increase the sum of all coordinates of all vertices of the transition sequence by at most an additive constant.
In a second step, assume $v \neq z$. Let k_v be the number of vertices of B_n in the path from v to t in G', and k_z be the the number of vertices of B_n in the path from z to t in G'. Without loss of generality assume $k_v \leq k_z$: if it is not the case, it suffices to repeat the path from z to t a large enough number of times.
Denote by $v_1, \ldots, v_i$ the vertices in $B_1, \cdots, B^i_n$ corresponding to v in $(G^i_{\mathcal{S}}, \mathcal{B}^i_{\mathcal{S}})$, where $v = v_1$ and $w = v_2$. Similarly, denote by $z_1, \ldots, z_i$ the vertices in $B_1, \cdots, B^i_n$ corresponding to z in $(G^i_{\mathcal{S}}, \mathcal{B}^i_{\mathcal{S}})$, where $z = z_1$ and $t = z_2$.
Since $w > v$, and since the sequence is increasing, following the path from v to z, starting from w, yields that $z_{k_v} > z_1$, and $z_{k_z} \geq z_{k_v} > z_1$. Then, it suffices to replace $(G, \mathcal{B})$ by $(G^{k_z}_{\mathcal{S}}, \mathcal{B}^{k_z}_{\mathcal{S}})$, and to apply the first case above.
For the "only if" direction, let V_t be the set of vertices v in B_1 such that there exists a path from v to t in G'. Assume that, for any $v \in V_t$, any $(v, w) \in G_{B_1 \preceq B_n}$, if there exists a path from v to w in G', then v and w have the same label. Then, in any (possibly infinite) path leading to t in G', the only vertices having an infinite number of occurrences, if there is any, are the vertices v such that v and w have the same label, where $(v, w) \in G_{B_1 \preceq B_n}$. It follows that the increase in any coordinate in t is uniquely determined by the portions of these paths with no loop in G', and is therefore bounded by a constant. □

Lemma 2.14. *Let* $(G, \mathcal{B})$ *be an increasing transition sequence from* B_1 *to* B_n, *with corresponding welding graph* $G_{B_1 \preceq B_n}$. *Let* $\mathcal{S}$ *be a switching of* $(G, \mathcal{B})$, *and, for* $i \in \mathbb{N}$, *let* $(G^i_{\mathcal{S}}, \mathcal{B}^i_{\mathcal{S}})$ *the corresponding* i^{th} *repetition of* $(G, \mathcal{B})$, *where* $\mathcal{B}^i_{\mathcal{S}} = (B_1, \cdots, B^i_n)$. *Denote by* $G_{B_1 \preceq B^i_n}$ *the corresponding welding graph.*
Then, There exist $\Theta(i)$ *vertices in* B^i_n *labelled with the same* $(q_t, x_{t,1}, \cdots, x_{t,n})$ *if and only if there exists a vertex* t *in* B_n *labelled with some* $(q_t, x'_{t,1}, \cdots, x'_{t,n})$, *isolated in* $G_{B_1 \preceq B_n}$, *with* $x'_{t,1}, \cdots, x'_{t,n} \leq x_{t,1}, \cdots, x_{t,n}$. *Moreover, the values* $x_{t,1}, \cdots, x_{t,n}$ *are uniquely determined by* $(G, \mathcal{B})$ *and* $\mathcal{S}$, *and do not depend on* i.

For a given increasing transition sequence, with a given corresponding welding graph, and a given switching, we say that the vertex t has the associated label $(q_t, x_{t,1}, \cdots, x_{t,n})$ with augmentation multiplicity.

Proof. For he "if" direction, assume t satisfies the conditions of the Lemma. Since every vertex in G' has by construction at most one outgoing edge, t does not satisfy the conditions of Lemma 2.13, and has no augmentation coordinate. Since t is isolated in the welding graph $G_{B_1 \preceq B_n}$, every vertex corresponding to t produced by a $i-1$ repetition $(G_{\mathcal{S}}^{i-1}, \mathcal{B}_{\mathcal{S}}^{i-1})$ of $(G, \mathcal{B})$, is transferred to the final base of $(G_{\mathcal{S}}^{i}, \mathcal{B}_{\mathcal{S}}^{i})$ through (id) transitions. Since t has no augmentation coordinate, the coordinates of these vertices are bounded: let $x_{t,1}, \cdots, x_{t,n}$ be their maximal value. By monotonicity, if one such vertex, corresponding to t and labelled with $(q_t, x_{t,1}, \cdots, x_{t,n})$, appears in the last base of a j^{th} repetition of $(G, \mathcal{B})$, every vertex corresponding to t produced in the last base of a i^{th} repetition of $(G, \mathcal{B})$, with $i > j$, is also labelled with $(q_t, x_{t,1}, \cdots, x_{t,n})$.
for the "only if" direction, assume by contradiction that every isolated point in $G_{B_1 \preceq B_n}$ is either labelled with a state different from t, or with coordinates strictly greater than $x'_{t,1}, \cdots, x'_{t,n}$. Since, by construction, the only vertices in a i^{th} repetition of $(G, \mathcal{B})$ created in a non-constant number are those corresponding to isolated vertices in the welding graph $G_{B_1 \preceq B_n}$, and, since, by monotonicity, their configuration is greater or equal to the configuration of the corresponding vertex in B_n, a contradiction follows immediately.
Let now $G^{-1}_{B_1 \preceq B_n}$ be the mirror graph of $G_{B_1 \preceq B_n}$, where all edges are reversed, and $(G_{\mathcal{S}}, \mathcal{B})$ be the switched transition sequence induced by $\mathcal{S}$, and consider the graph $G' = G_{\mathcal{S}} \uplus G^{-1}_{B_1 \preceq B_n}$. The coordinates $x_{t,1}, \cdots, x_{t,n}$ depend only on the longest path leading to t in G': let v be the vertex in B_1 source of this path, and $w \in B_n$ be its associated through the welding graph. The coordinates $x_{t,1}, \cdots, x_{t,n}$ are computed by firing all transitions along this path, starting with the initial configuration of vertex w instead of v. An example is shown in Figure 10. $\square$

Remark 2.15. As shown by Figure 10, the $\Theta(i)$ vertices in ${B'}^{i}_{n}$, induced by the vertex t of B_n with augmentation multiplicity, do not necessarily have the same coordinates as t.

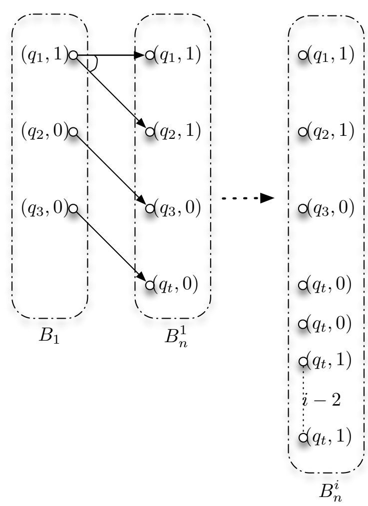


Figure 10: A i^{th} repetition of an increasing transition sequence, with a vertex with augmentation multiplicity.

2.5 Increasing generalized transition sequences

Our purpose here is to extend the approach of the previous subsection to the case of generalized transition sequence. In order to do so, we first need to explicit how to repeat generalized transition sequences, or, more generally, how to compose them.

Definition 2.16. Expansion of a generalized transition sequence

Let S be a VASS-SJ. Let $(G, \mathcal{B})$ with $\mathcal{B} = B_1, \cdots, B_f$ be a generalized transition sequence with initial base B_1 and final base B_n. Let D_1 be a base of a generalized configuration of S, such that $B_1 \preceq D_1$ with welding graph $G_{B_1 \preceq D_1}$. An *expansion* of $(G, \mathcal{B})$ with respect to the welding graph $G_{B_1 \preceq_\infty D_1}$ is a generalized transition sequence $(H, \mathcal{D})$, with $\mathcal{D} = D_1, \cdots, D_n$, and for all $i = 1, \cdots, f$, $B_i \preceq D_i$ with welding graph $G_{B_i \preceq D_i}$, inductively obtained as follows.
For all $i = 1, \cdots, n-1$, B_{i+1} is by hypothesis obtained from B_i by a transition firing, and a lifting. Then,

1. For every edge (c, c') from B_i to B_{i+1} labelled with (id), assume c is labelled with $(q, x_1, \cdots, x_m)$ and c' with $(q, y_1, \cdots, y_m)$. Let (c, d) be an edge in $G_{B_i \preceq D_i}$, where d is labelled with $(q, s_1, \cdots, s_m)$. Define a vertex $d' = (q, t_1, \cdots, t_m) \in D_{i+1}$, an edge (d, d') labelled with (id) in $\mathcal{H}$, and an edge (c', d') in $G_{B_{i+1} \preceq D_{i+1}}$, such that, for all $j = 1, \cdots, m$, $t_j = y_j + s_j - x_j$.

2. For any vertex $d \in D_i$ isolated in $G_{B_i \preceq D_i}$, make d' a copy of d in D_{i+1}, isolated in $G_{B_{i+1} \preceq D_{i+1}}$, and an (id) edge (d, d') in H.

3. Finally, Let t be the transition fired:

   - If $t = (q, a_1, \cdots, a_m, q')$ is regular: there exists one edge (c, c') in G labelled with (t), where $c = (q, x_1, \cdots, x_m)$ belongs to B_i, $c' = (q', y_1, \cdots, y_m)$ belongs to B_{i+1}, and, for all $j = 1, \cdots, m$, $y_j \in \{x_j + a_j, \infty\}$. Since $B_i \preceq D_i$ with welding graph $G_{B_i \preceq D_i}$, there exists $d = (q, s_1, \cdots, s_m) \in D_i$, where (c, d) is an edge of $G_{B_i \preceq D_i}$ and, for all $j = 1, \cdots, m$, $s_j \geq x_j$.

     Then, define a vertex $d' = (q', t_1, \cdots, t_m) \in D_{i+1}$, an edge (d, d') labelled with (t) in $\mathcal{H}$, and an edge (c', d') in $G_{B_{i+1} \preceq D_{i+1}}$, such that, for all $j = 1, \cdots, m$, $t_j = y_j + s_j - x_j$.
   - If $t = (q, q', j, q'')$ is join: there exists two edges (c, c'') and (c', c'') in G labelled with $(join)$, where $c = (q, x_1, \cdots, x_m)$ belongs to B_i, $c' = (q', x'_1, \cdots, x'_m)$ belongs to B_i, $c'' = (q'', y_1, \cdots, y_m)$ belongs to B_{i+1}, and , for all $j = 1, \cdots, m$, $y_j \in \{x_j + x'_j, \infty\}$. Since $B_i \preceq D_i$ with welding graph $G_{B_i \preceq D_i}$, there exists $d = (q, s_1, \cdots, s_m)$ and $d' = (q', s'_1, \cdots, s'_m)$ in D_i, where (c, d) and (c', d') are edges of $G_{B_i \preceq D_i}$ and, for all $j = 1, \cdots, m$, $s_j \geq x_j$ and $s'_j \geq x'_j$.

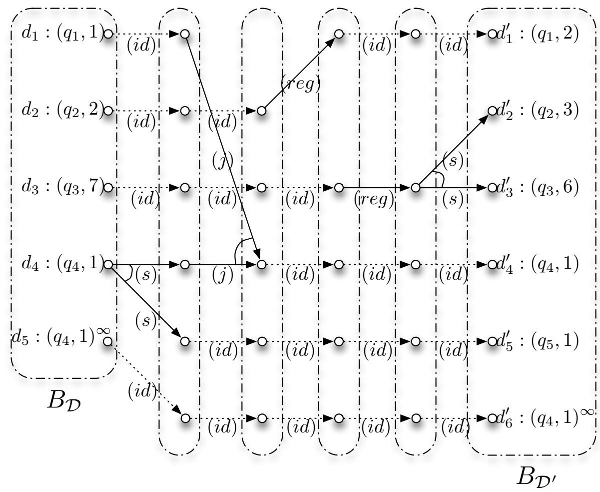
     Then, define a vertex $d'' = (q'', t_1, \cdots, t_m) \in D_{i+1}$, two edges (d, d'') and (d', d'') labelled with $(join)$ in $\mathcal{H}$, and an edge (c'', d'') in $G_{B_{i+1} \preceq D_{i+1}}$, such that, for all $j = 1, \cdots, m$, $t_j = y_j + s_j + s'_j - x_j - x'_j$.
   - If $t = (q, q', j, q'')$ is split: there exists two edges (c, c') and (c, c'') in G labelled with $(split)$, where $c = (q, x_1, \cdots, x_m)$ belongs to B_i, $c' = (q', y'_1, \cdots, y'_m)$ belongs to B_{i+1}, $c'' = (q'', y''_1, \cdots, y''_m)$ belongs to B_{i+1}, and , for all $j = 1, \cdots, m$, $y'_j + y''_j \in \{x_j, \infty\}$, $y'_j \in [0, \cdots, x_j] \uplus \infty$ and $y''_j \in [0, \cdots, x_j] \uplus \infty$. Since $B_i \preceq D_i$ with welding graph $G_{B_i \preceq D_i}$, there exists $d = (q, s_1, \cdots, s_m)$ in D_i, where (c, d) is an edge of $G_{B_i \preceq D_i}$ and, for all $j = 1, \cdots, m$, $s_j \geq x_j$.

     Then, for all $j = 1, \cdots, m$, choose t'_j and t''_j such that $t'_j + t''_j \in \{s_j, \infty\}$, $t'_j \in [0, \cdots, s_j] \uplus \infty$, $t''_j \in [0, \cdots, s_j] \uplus \infty$, $t'_j \geq y'_j$ and $t''_j \geq y''_j$. Define two vertices $d' = (q', t'_1, \cdots, t'_m)$ and $d'' = (q'', t''_1, \cdots, t''_m)$ in D_{i+1}, two edges (d, d') and (d, d'') labelled with $(split)$ in $\mathcal{H}$, and two edges (c', d') and (c'', d'') in $G_{B_{i+1} \preceq D_{i+1}}$.

It is clear from the definition above that the expansion $(H, \mathcal{D})$ of a generalized transition sequence $(G, \mathcal{B})$ is a generalized transition sequence. Also, if $(G, \mathcal{B})$ contains no split transition, $(H, \mathcal{D})$ is uniquely defined. Moreover, if $(G, \mathcal{B})$ contains some split transitions, and if $B_1 \preceq_\infty D_1$, then $(H, \mathcal{D})$ is uniquely defined, and, for all $i = 1, \cdots, n$, $B_i \preceq_\infty D_i$. If $(G, \mathcal{B})$ contains some split transitions, and if $B_1 \not\preceq_\infty D_1$, choosing a switching $\mathcal{S}$ of $(G, \mathcal{B})$ induces a unique expansion $(H, \mathcal{D})$ of $(G, \mathcal{B})$, by enforcing every firing of a split on a finite coordinate to fire accordingly to the switching: the corresponding coordinate of the destination vertex in $(H, \mathcal{D})$ not selected by the switching needs to be equal to the corresponding coordinate of the vertex in $(G, \mathcal{B})$, destination of the same split and not selected by the switching. In that case, we say that the expansion $(H, \mathcal{D})$ is *induced* by the welding graph $G_{B_1 \preceq D_1}$ and by the switching $\mathcal{S}$.

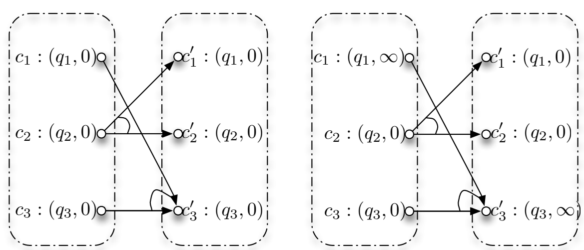
Example 2.17. Consider the increasing transition sequence of Example 2.11, from $B_{\mathcal{C}}$ to $B_{\mathcal{C}'}$. Let $\mathcal{D} = \{(q_1, 0)^1, (q_2, 2)^1, (q_3, 7)^1, (q_4, 1)^\infty\}$ be a generalized configuration, with corresponding base $B_{\mathcal{D}}$, with $B_{\mathcal{C}} \preceq$

$B_{\mathcal{D}}$ with welding graph $G_{B_{\mathcal{C}} \preceq \mathcal{D}}$. An expansion of the transition sequence , with respect to the welding graph $G_{B_{\mathcal{C}} \preceq \mathcal{D}}$, is the following:



Note that this expansion is not unique: there exists different switchings for the two split transitions, leading to other labels for the vertices d'_2, d'_3 , d'_4 and d'_5.

Remark 2.18. The expansion of an increasing (generalized) transition sequence is not necessarily increasing. See the following generalized increasing transition, and an expansion of it.



Definition 2.19. Composition of generalized transition sequences

Let $S = (G, T, m)$ be a VASS-SJ. Let $(G_1, \mathcal{B}_1)$, with $G_1 = (V_1, E_1)$ be a generalized transition sequence with initial base B_1^1 and final base B_1^f. Let $(G_2, \mathcal{B}_2)$, with $G_2 = (V_2, E_2)$ be a generalized transition sequence with initial base B_2^1 and final base B_2^l. Assume $B_2^1 \preceq B_1^f$ with corresponding welding graph $G_{B_2^1 \preceq B_1^f}$.
The *composition* of $(G_1, \mathcal{B}_1)$ and $(G_2, \mathcal{B}_2)$ with respect to the welding graph $G_{B_2^1 \preceq B_1^f}$, denoted as

$$(G_1, \mathcal{B}_1).{}_{G_{B_2^1 \preceq B_1^f}}(G_2, \mathcal{B}_2),$$

or simply as

$$(G_1, \mathcal{B}_1).(G_2, \mathcal{B}_2)$$

when the welding graph $G_{B_2^1 \preceq B_1^f}$ is clear from context, is obtained by extending $(G_1, \mathcal{B}_1)$ with the normal form of an expansion of $(G_2, \mathcal{B}_2)$ with respect to the welding graph $G_{B_2^1 \preceq B_1^f}$.
If, moreover, we choose a switching $\mathcal{S}_2$ of $(G_2, \mathcal{B}_2)$, and denote by $(H_2, \mathcal{D}_2)$ the unique expansion of $(G_2, \mathcal{B}_2)$ induced by the welding graph $G_{B_2^1 \preceq B_1^f}$ and the switching $\mathcal{S}_2$, the composition of $(G_1, \mathcal{B}_1)$ and $(G_2, \mathcal{B}_2)$ with respect to the welding graph $G_{B_2^1 \preceq B_1^f}$ and the switching $\mathcal{S}_2$, denoted as

$$(G_1, \mathcal{B}_1).{}^{\mathcal{S}_2}_{G_{B_2^1 \preceq B_1^f}}(G_2, \mathcal{B}_2),$$

is obtained by extending $(G_1, \mathcal{B}_1)$ with the normal form of $(H_2, \mathcal{D}_2)$.

This definition provides a proper notion of repetition of an increasing generalized transition induced by a welding graph and a switching. This allows us to state the following:

Remark 2.20. Lemma 2.13 transfers to increasing generalized transitions. Under the hypotheses and notations of Lemma 2.13, for any *finite* coordinate $l = 1, \cdots, m$,

- $x^i_{t,l} = \Theta(i)$ if and only if Conditions 1), 2), 3) and 4) of Lemma 2.13 hold, and,
- $x^i_{t,l} \neq \Theta(i) \Rightarrow x^i_{t,l} \leq x^i_{s,l} + c_{(G,\mathcal{B})}$.

The proof is the same as for Lemma 2.13.

2.6 Sub-increasing Generalized Transition Sequences

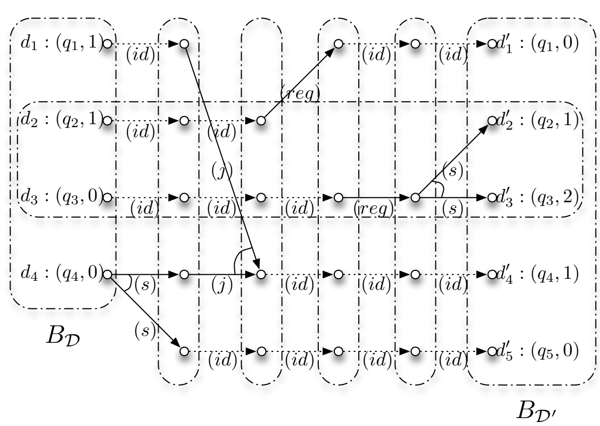
In the previous section, we have seen under which conditions one can safely repeat a generalized transition sequence, how to do it and how to abstract it with the lifting operation. We now need to go one step deeper, as there exists non-increasing generalized transition sequences where one can nonetheless safely repeat one subset of its transition firings. As a simple example, consider a non-increasing generalized transition sequence $(G, \mathcal{B})$ such that G is not simply connected, and such that one of the simply connected components of G induces an increasing generalized transition sequence: clearly, one can repeat only this simple connected part of $(G, \mathcal{B})$, and define the corresponding notion of lifting. But there is even more: a transition sequence, even when simply connected, corresponds to one choice of sequentialization of the transition firings. It may very well happen that a different choice in the order in which the transition are fired reveals an increasing part of the said generalized transition sequence. This is captured by the following definition.

Definition 2.21. Sub-increasing Generalized Transition Sequence
Let $(G, \mathcal{B})$ be a generalized transition sequence from B_1 to B_n. Assume there exist non-empty bases $B'_1, \cdots, B'_n$ such that:

1. for $i = 1, \cdots, n$, $B'_i \subseteq B_i$,
2. for $i = 1, \cdots, n-1$, every edge in G with destination vertex in B'_{i+1} has its source in B'_i, and
3. $B'_1 \preceq B'_n$

Then, $(G, \mathcal{B})$ is said to be *sub-increasing* with respect to $\mathcal{B}' = B'_1, \cdots, B'_n$.

Example 2.22. Consider the following non-increasing transition sequence. It is sub-increasing: the corresponding bases are boxed.



As said before, sub-increasing generalized transition sequences correspond to one choice of sequentialization of the transition firings, where only a subsequence of the sequence of firings yields an increasing generalized transitions. For a given sub-increasing generalized transition sequence $(G, \mathcal{B})$, it is always possible to permute some of the transition firings, in order to turn $(G, \mathcal{B})$ into the composition of an increasing generalized transition sequence, which corresponds to the sequence of transition firings in the B_i''s, and of non sub-increasing generalized transition sequence, which corresponds to the sequence of transition firings in the $B_i \setminus B_i'$'s. This is detailed in the following lemma.

Lemma 2.23. *Let $(G, \mathcal{B})$ be a generalized transition sequence, where $\mathcal{B} = B_1, \cdots, B_n$, and assume $(G, \mathcal{B})$ is sub-increasing with respect to $\mathcal{B}' = B_1', \cdots, B_n'$. Assume exactly k non-(id) transitions $t_1', \cdots, t_k'$ are fired in $(G, \mathcal{B})$ on single generalized configurations in the bases B_i', and $n-(k+1)$ non-(id) transitions $t_1, \cdots, t_{n-(k+1)}$ are fired in $(G, \mathcal{B})$ on single generalized configurations in the bases $B_i \setminus B_i'$. Denote by $B_1'', \cdots, B_{k+1}''$ the sequence of* distinct *bases of $\mathcal{B}'$, and $D_1', \cdots, D_{n-k}'$ the sequence of* distinct *bases of $(B_i \setminus B_i')$ for all $i = 1, \cdots, n$.*
Then, there exists a generalized transition sequence $(H, \mathcal{D})$, with $\mathcal{C} = D_1, \cdots, D_n$, such that

1. *$D_1 = B_1$, $D_n = B_n$,*
2. *for all $i = 1, \cdots, k$, $D_i \hookrightarrow_{t_i'} D_{i+1}$, where $B_i'' \subseteq D_i$ and t_i' is fired on single generalized configurations in B_i'', and*
3. *for all $i = k+1, \cdots, n-1$, $D_i \hookrightarrow_{t_{i-k}} D_{i+1}$, where t_{k+1-i} is fired on single generalized configurations in D_i'.*

Proof. The result is actually valid even when only Conditions 1) and 2) of Definition 2.21 hold. Let $(G, \mathcal{B})$ be a generalized transition sequence, where $\mathcal{B} = B_1, \cdots, B_n$. Assume there exist non-empty bases $B_1', \cdots, B_n'$ such that:

1. for $i = 1, \cdots, n$, $B_i' \subseteq B_i$, and
2. for $i = 1, \cdots, n-1$, every edge in G with destination vertex in B_{i+1}' has its source in B_i'.

Let k be the number of non-(id) transitions fired in $(G, \mathcal{B})$ on single generalized configurations in the bases B_i'. We now prove the result by induction on k.
The case $k = 0$ is trivial.
Assume $k = 1$, and denote by t_1' the only non-(id) transition fired in $(G, \mathcal{B})$ on single generalized configurations in the bases B_i'. Let $1 \leq j < n$ such that $B_j \hookrightarrow_{t_1'} B_{j+1}$ in $(G, \mathcal{B})$, with $B_j' \hookrightarrow_{t_1'} B_{j+1}'$. Denote by s_j, or s_j^1 and s_j^2 the source (or the two sources in case of a join) vertex of the transition firing in B_j', and by t_{j+1}, or t_{j+1}^1 and t_{j+1}^2, its destination, (or two destinations in case of a split) . For all $i \neq j$, denote by $B_i \hookrightarrow_{t_i} B_{i+1}$ the transition firing on single generalized configurations in the bases $B_i \setminus B_i'$.
Then, for all $i \leq j$, all incoming edges to vertices in B_i' are (id) edges, and their source vertex are in B_{i-1}'. It follows that there exists one vertex in B_i' with the same label as s_j (or two vertices in B_i' with the same label as s_j^1 and s_j^2 respectively). Denote by B_i'' the base obtained from B_i' by removing this copy of s_j (or these copies of s_j^1 and s_j^2), and adding a copy of t_j (or a copy of t_{j+1}^1 and of t_{j+1}^2). Then, the generalized transition sequence

$$\begin{gathered} B_1 \hookrightarrow_{t_1'} (B_1 \setminus B_2' \cup B_2'') \hookrightarrow_{t_1} (B_2 \setminus B_3' \cup B_3'') \hookrightarrow_{t_2} (B_3 \setminus B_4' \cup B_4'') \hookrightarrow_{t_3} \cdots \\ \hookrightarrow_{t_{j-2}} B_{j-1} \setminus B_j' \uplus B_j'' \hookrightarrow_{t_{j-1}} B_j \setminus B_j' \uplus B_j'' \hookrightarrow_{t_j} B_{j+1} \hookrightarrow_{t_{j+1}} B_{j+2} \hookrightarrow_{t_{j+2}} \cdots \hookrightarrow_{t_{n-1}} B_n, \end{gathered}$$

where t_1' is fired on the copy of s_j in B_1' (or the two copies of s_j^1 and s_j^2 in B_1'), and the t_i for $i \neq j$ are fired on $B_i \setminus B_i'$, satisfies the lemma.
Assume now that the result holds for $k-1 \geq 0$, and that there exists k non-(id) transitions $t_1', \cdots, t_k'$ fired in $(G, \mathcal{B})$ on single generalized configurations in the bases B_i'. Let $1 \leq j < n$ such that $B_j \hookrightarrow_{t_k'} B_{j+1}$ is the last occurrence of such a non-(id) transition firing. Let $(G_{1,j}, \mathcal{B}_{1,j})$, with $\mathcal{B}_{1,j} = B_1, \cdots, B_j$, be the restriction of $(G, \mathcal{B})$ to the bases $B_1, \cdots, B_j$, and $(G_{j+1,n}, \mathcal{B}_{j+1,n})$, with $\mathcal{B}_{j+1,n} = B_{j+1}, \cdots, B_n$, be the restriction of

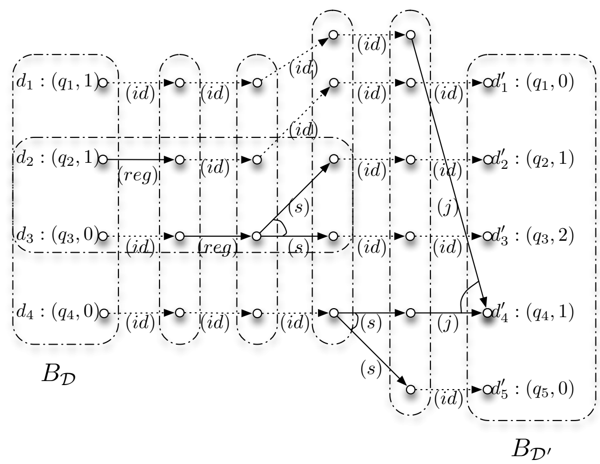
$(G, \mathcal{B})$ to the bases $B_{j+1}, \cdots, B_n$. We can apply the induction hypothesis on $(G_{1,j}, \mathcal{B}_{1,j})$: Let $(H_{1,j}, \mathcal{D}_{1,j})$ be the corresponding generalized transition sequence, from B_1 to B_j. $(H_{1,j}, \mathcal{D}_{1,j})$ is the composition of two generalized transition sequences:

1. $(H_{1,k}, \mathcal{D}_{1,k})$, from $D_1 = B_1$ to D_k, where the transitions are fired only on single generalized configurations in the B_i''s, and
2. $(H_{k+1,j}, \mathcal{D}_{k+1,j})$, from D_{k+1} to $D_j = B_j$, where the transitions are fired only on single generalized configurations in the $B_i \setminus B_i'$'s.

Apply again the induction hypothesis on the composition of $(H_{k+1,j}, \mathcal{D}_{k+1,j})$ and $(G_{j+1,n}, \mathcal{B}_{j+1,n})$, with only one non-(id) transition fired on single generalized configurations in the bases B_i', and denote by $(I_{k+1,n}, \mathcal{E}_{k+1,n})$ the corresponding generalized transition sequence, from D_{k+1} to B_n. Then, the composition of $(H_{1,k}, \mathcal{D}_{1,k})$ and $(I_{k+1,n}, \mathcal{E}_{k+1,n})$ satisfy Conditions 1), 2) and 3) of the Lemma.

□

Example 2.24. Consider the sub-increasing generalized transition sequence of Example 2.22. Firing firstly the transitions corresponding to the sub-increasing part of the sequence, and secondly the remaining transitions, yields the following generalized transition sequence.



3 The Karp and Miller Tree for VASS-SJ

In the previous section, we have seen under which conditions a transition sequence can be safely repeated, and the effects of this repetition on the coordinate values and multiplicities of the labels of its vertices. We have also extended these conditions and effects to our abstraction of transition sequences, the generalized transition sequences. We are now in a good position to use this abstraction to embed all non-negative transition sequences starting from an initial base into an abstract, finite data structure, and to study how to relate this data structure to these non-negative transition sequences. This data structure is a generalized transition tree, and, when restricting our study to the particular case of VASS, is isomorphic to the Karp and Miller tree for VASS: hence, we call it a Karp and Miller tree for VASS-SJ.

3.1 Liftings associated to increasing generalized transition sequences

As for the classical case, our starting point is to use our notion of generalized transition sequence to represent in an abstract way the coordinate and multiplicity increase induced by the repetition of an increasing transition sequence. This abstract representation is performed by an ad-hoc lifting associated to these increasing transition sequences.

Definition 3.1. Single liftings associated to an increasing (generalized) transition sequence and a welding graph.

Let S be a VASS-SJ. Let $(G, \mathcal{B})$ be an increasing (generalized) transition sequence from B_1 to B_n, with corresponding welding graph $G_{B_1 \preceq B_n}$. Let $\mathcal{S}_1, \cdots, \mathcal{S}_l$ be the different possible switchings of the increasing transition sequence. The single lifting associated to $(G, \mathcal{B})$ and $G_{B_1 \preceq B_n}$ is obtained by

1. for each isolated point v in $G_{B_1 \preceq B_n}$ labelled with g^1, with associated label h^1 with augmentation multiplicity, adding an isolated vertex w labelled with h^∞.
2. for any switching $\mathcal{S}_1, \cdots, \mathcal{S}_l$, replace any finite augmentation coordinate for this switching with ∞, and
3. Repeating the following procedure until no new ∞ coordinate is added:
   for any vertex w_l in B_n, *not* isolated in the welding graph $G_{B_1 \preceq B_n}$, such that:
   (a) there exists an edge (v_k, w_k) in the welding graph $G_{B_1 \preceq B_n}$,
   (b) there exists a path from v_k to w_l in $(G, \mathcal{B})$,
   for any ∞ coordinate of w_k, replace the corresponding coordinate in w_l with ∞.

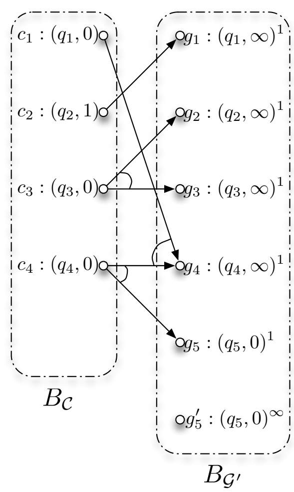


Figure 11: The single lifting corresponding to all possible repetitions of the same transition sequence to $B_{\mathcal{C}^i}$.

An example of such a single lifting, corresponding to the increasing transition sequence of Figures 8 and 9, is given by Figure 11. Following the definition above, performing the lifting consists in:

1. adding a new isolated vertex g_5', labeled with $(q_5, 0)^\infty$,
2. labeling g_3 with $(q_3, \infty)^1$ (following the switching of Figure 8), and g_4 with $(q_4, \infty)^1$ (following the switchings of Figures 8 and 9), and
3. firstly, labeling g_2 with $(q_2, \infty)^1$, and, secondly, labeling g_1 with $(q_1, \infty)^1$.

Step (3) above ensures that the vertices g_1 and g_2 of Figure 11 are labelled with $(q_1, \infty)^1$ and $(q_2, \infty)^1$ respectively, instead of $(q_1, 1)$ and $(q_2, 1)$ (case of the switching Figure 8) or $(q_1, 2)$ and $(q_2, 2)$ (case of the switching Figure 9). Their ∞ coordinate is "inherited" from g_3: indeed, after steps (1) and (2) of the

lifting, two additional repetitions of the increasing transition sequence ensure that the ∞ coordinate of g_3 is transferred to g_2 and g_1.
Note that different choices choices for the welding graph $G_{B_i \preceq B_n}$ yield actually different liftings. Yet, the number of possible choices is clearly exponentially bounded in the size of the bases.

Remark 3.2. Let S be a VASS-SJ. Let $(G, \mathcal{B})$ be an increasing transition sequence from B_1 to B_n, with corresponding welding graph $G_{B_1 \preceq B_n}$. Let $(v, w) \in G_{B_1 \preceq B_n}$, and assume $v < w$. Then, in the single lifting associated to $(G, \mathcal{B})$ and $G_{B_1 \preceq B_n}$, it is not necessarily the case that every coordinate of w strictly greater than the corresponding coordinate of v, is replaced by an ∞. For an example see Figure 12.

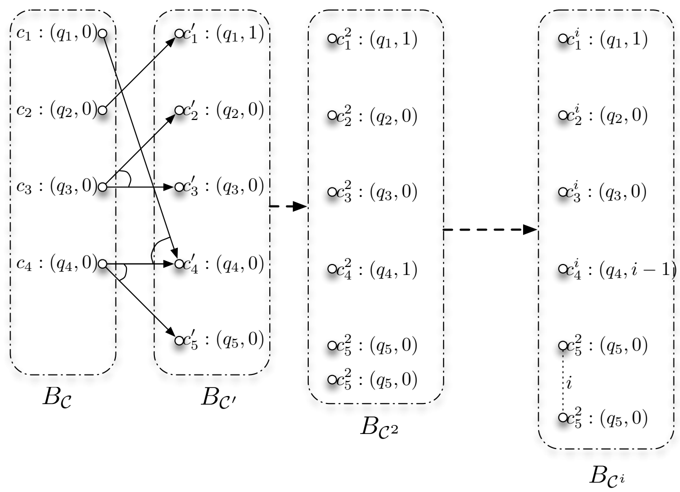


Figure 12: Another example the the i^{th} repetition of an increasing transition sequence.

The following Lemma shows that, when the increase in coordinate values is strict, there exists some augmentation coordinates or some augmentation multiplicity.

Lemma 3.3. *Let S be a VASS-SJ. Let $(G, \mathcal{B})$ be an increasing (generalized) transition sequence from B_1 to B_n, with corresponding welding graph $G_{B_1 \preceq B_n}$. Let $(v, w) \in G_{B_1 \preceq B_n}$, and assume $v < w$. Let $\mathcal{S}$ be a switching of $(G, \mathcal{B})$, $G^{-1}_{B_1 \preceq B_n}$ be the mirror graph of $G_{B_1 \preceq B_n}$, where all edges are reversed, and $(G_{\mathcal{S}}, \mathcal{B})$ be the switched transition sequence induced by $\mathcal{S}$, and consider the graph $G' = G_{\mathcal{S}} \uplus G^{-1}_{B_1 \preceq B_n}$.*
Then, one of the following holds:

1. *there exists a vertex t with a label with augmentation multiplicity, and a path from v to t G', or*

2. *there exists vertices $z \in B_1$ and $t \in B_n$ such that v, w, z and t satisfy the conditions of Lemma 2.13.*

Proof. By construction of G', every vertex in G' has at most one outgoing edge. Therefore, there is exactly one path starting from v in G'. To cases arise:

1. this path is finite. Then, its final vertex is in B_n, and is isolated in the welding graph $G_{B_1 \preceq B_n}$. Then, it has a label with augmentation multiplicity, or

2. this path is infinite. Then, it passes through infinitely many occurrences of a vertex (t, z) in the graph $G^{-1}_{B_1 \preceq B_n}$, and v has an augmentation coordinate.

□

The following Lemma shows the non-commutativity, in the general case, of the single liftings.

Lemma 3.4. *Let S be a VASS-SJ. Let $(G, \mathcal{B})$ be a (generalized) transition sequence from B_1 to B_n. Assume there exists $B_i \in \mathcal{B}$, such that $B_i \preceq B_n$ with welding graph G_i, and $B_j \in \mathcal{B}$, such that $B_j \preceq B_n$ with welding graph G_j. Then, the single liftings associated to the restriction of $(G, \mathcal{B})$ to $(B_i, \cdots, B_n)$ and G_i on one hand, and to the restriction of $(G, \mathcal{B})$ to $(B_j, \cdots, B_n)$ and G_j on the other hand, do not commute. However, steps (1) of the two single liftings commute, as do steps (2) and (3).*

Proof. Consider any vertex $v \in B_n$. The critical case is the non commutation of steps (1) and (2) of two different single liftings. Assume a vertex v is isolated in G_i and not in G_j. Performing the single lifting associated to G_i first consists in making a new copy of v, with ∞ multiplicity. Then, the single lifting associated to G_j does not touch this copy. Performing the single lifting associated to G_j first consist in modifying the coordinates of v. Then, the single lifting associated to G_i makes a new copy of it, with ∞ multiplicity. It is clear that the coordinates of the copy of v with ∞ multiplicity may differ in both cases. The second part of the Lemma is clear. $\square$

This Lemma allows us to extend, in a unambiguous way, the classical notion of lifting used in the construction of the Karp and Miller tree for VASS.

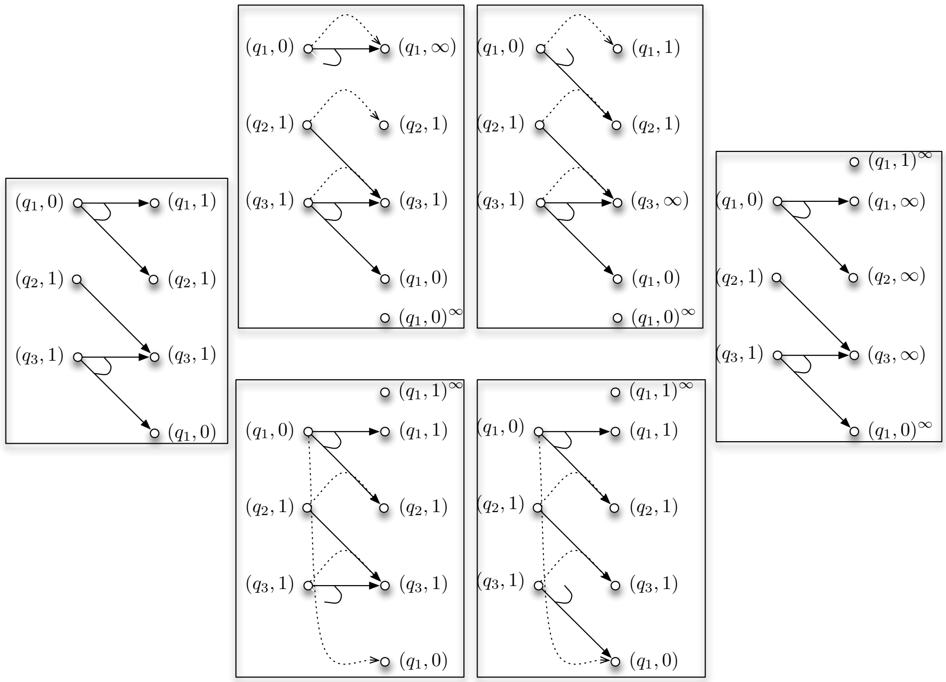
Definition 3.5. Lifting of a Generalized Transition Sequence.

Let $S = (G, T, m)$, with $G = (Q, A)$ be a VASS-SJ. Let $(G, \mathcal{B})$ be a generalized transition sequence on S, from B_1 to B_n. Let $\mathcal{B}' \subset \mathcal{B}$ be the set of bases B such that $B \preceq B_n$. The lifting of $(G, \mathcal{B})$ consists in:

1. performing Step (1) of Definition 3.1 for all bases $B \in \mathcal{B}'$, all corresponding increasing generalized transition sequence from B to B_n, and all corresponding welding graphs, in any order,
2. performing Step (2) of Definition 3.1 for all bases $B \in \mathcal{B}'$, all corresponding increasing generalized transition sequence from B to B_n, and all corresponding welding graphs, in any order,
3. performing Step (3) of Definition 3.1 for all bases $B \in \mathcal{B}'$, all corresponding increasing generalized transition sequence from B to B_n, and all corresponding welding graphs, in any order, and finally
4. normalizing the resulting generalized transition sequence.

Lemma 3.4 ensures that the definition above is unambiguous: commutativity of the steps (1) of the different single liftings, and of the steps (2) and (3), allows to perform each of the corresponding steps in any order. The only necessary condition is sequentiality between Steps (1) and the other two.

Example 3.6. Consider the following increasing generalized transition sequence, from $\{(q_1, 0), (q_2, 1), (q_3, 1)\}$ to $\{(q_1, 1), (q_2, 1), (q_3, 1), (q_1, 0)\}$, given in the leftmost frame. This sequence admits two welding graphs, one in the top two frames, and one in the bottom two frames. The two welding graphs are drawn with dotted arrows. For each of these welding graphs, the transition sequence admits two switchings: each of them is represented by erasing the split edges not selected by the switching. The top two frames represent steps 1) and 2) of the lifting, for the first welding graph. The augmentation coordinates for each switching are replaced with ∞ in the corresponding frame. The bottom two frames represent steps 1) and 2) of the lifting, for the second welding graph. The generalized transition sequence resulting from the lifting (after performing steps 3) and normalizing the bases) is given in the rightmost frame.



3.2 The Karp and Miller Tree

Now, we introduce our construction of a Karp and Miller Tree.

Definition 3.7. Karp and Miller Tree.
Let S be a VASS-SJ and B be a base of S. The *Karp and Miller tree* $\mathfrak{T}$ on (S, B) is a generalized transition tree $(G, \mathcal{B})$, constructed inductively as follows.

- The root of $G/\mathcal{B}$ is B
- Let $B_j \in \mathcal{B}$ be a generalized base in $\mathfrak{T}$. Denote by $(G_j, \mathcal{B}_j)$ the generalized transition sequence corresponding to the path from B to B_j in $G/\mathcal{B}$. Then,

  1. If there exists $B_i \in (G_j, \mathcal{B}_j)$, isomorphic to B_j, the vertices of B_j have out-degree 0 in G (hence B_j has out-degree 0 in $G/\mathcal{B}$).
  2. If there exists B_i in $(G_j, \mathcal{B}_j)$, such that the restriction of $(G_j, \mathcal{B}_j)$ to the bases $B_i \cdots, B_j$ is sub-increasing, the vertices of B_j have out-degree 0 in G (hence B_j has out-degree 0 in $G/\mathcal{B}$). Otherwise,
  3. assume there exists a transition firing $B_j \hookrightarrow_t B_k$. Then, extend $\mathfrak{T}$ with $B_j \hookrightarrow_t B_k$. Denote by $(G', \mathcal{B}')$ the restriction of $\mathfrak{T}$ to the path from B to B_k in $G/\mathcal{B}$, and perform the lifting of $(G', \mathcal{B}')$.

The inductive construction above halts when no new base can be added with these rules.

Remark 3.8. Lemma 2.23 ensures that Case 2) of the definition above does not reduce the expressivity of the tree: indeed, if a branch of the tree admits a sub-increasing generalized transition sequence, Lemma 2.23 ensures that there exists a different choice of sequentialization of the transition firings, which leads to a similar base, where the increasing part of the sequence is fired first. Hence, there exists another branch of the tree, which takes into account the same set of transition firings, in a different order, and which does not admits a sub-increasing subsequence. Note also that case 2) of the definition may seem unnecessary at first sight; it is, however, a necessary condition for proving the finiteness and constructibility of the Karp and

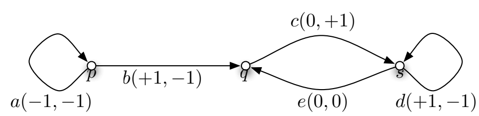
Miller tree: it is indeed possible to build infinite generalized transition trees with only cases 1) and 3) of the definition.

Example 3.9. For the sake of simplicity, we provide here an example of a Karp and Miller tree for a VASS (only regular transitions). Karp and Miller trees for VASS are very similar to Karp and Miller tree for VASS-SJ, with the two following simplifications:

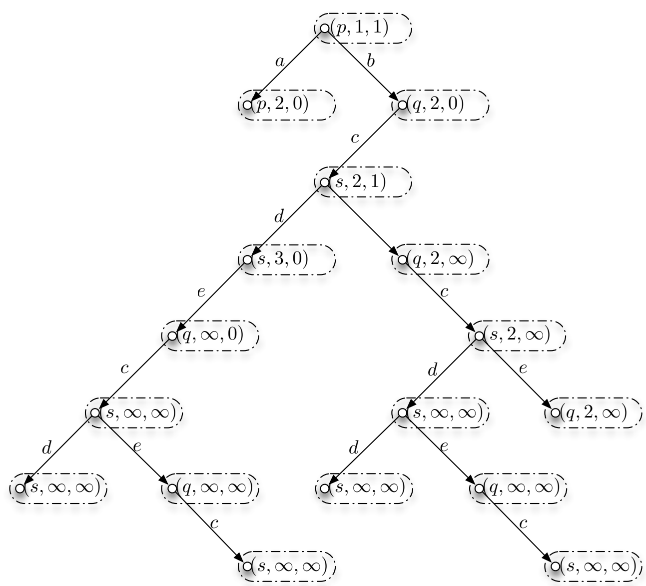
1. Bases of the generalized transition tree contain only single generalized configurations. The graph of the generalized transition tree is therefore isomorphic to its quotient tree. This is not necessarily the case for the Karp and Miller tree for VASS-SJ, where bases can have size greater than one.

2. VASS do not admit sub-increasing sequences. Therefore, point 2) of the construction above does not occur.

Karp and Miller trees for VASS-SJ tend to be too large to be used as an example in a A4 paper page. Since our purpose is to demonstrate the notion of Karp and Miller tree, covering graph, paths in this graph and corresponding generalized transition sequences, the specific case of classical VASS is good enough. The example is taken from [Reu88].
Consider the following VASS S, with three states p, q, s and five regular transitions a, b, c, d, e.



The corresponding Karp and Miller tree for the initial configuration $(p, 1, 1)$ is the following.



3.3 Constructibility of the Karp and Miller Tree

This subsection is devoted to proving that the Karp and Miller tree defined above is finite, and that it can be effectively constructed. Proposition 3.10 is a first, intermediate technical result in this direction.

Proposition 3.10. *The following statements are true:*

1. *Let $(u_n) \in (\mathbb{N} \uplus \infty)^m$, $n \in \mathbb{N}$, be an infinite sequence of m-tuples for some $m \in \mathbb{N}$. Then, there exists an infinite sub-sequence (u'_n), $n \in \mathbb{N}$ of (u_n) that is increasing for the order relation $\leq$.*
2. *Let $u_i \in (\mathbb{N} \uplus \infty)^{m_i}$, and $u_j \in (\mathbb{N} \uplus \infty)^{m_j}$, for $m_i, m_j \in \mathbb{N}$, be two tuples. We define $u_i \leq u_j$ if and only if $m_i \leq m_j$, and $u_i \leq u'_j$, where u'_j is the projection of u_j on its m_i first coordinates. Let (m_n), $n \in \mathbb{N}$, be an infinite sequence of natural numbers, and $(u_n) \in (\mathbb{N} \uplus \infty)^{m_n}$, $n \in \mathbb{N}$, be an infinite sequence of tuples. Then, there exists an infinite sub-sequence (u'_n), $n \in \mathbb{N}$ of u_n that is increasing for the order relation $\leq$.*
3. *Let $S = (G, T, m)$, with $G = (Q, A)$ be a VASS-SJ. Let $(G, \mathcal{B})$, with $\mathcal{B} = (B_n)$, $n \in \mathbb{N}$, be an infinite generalized transition sequence on S. Then, there exists an infinite sub-sequence (B'_n), $n \in \mathbb{N}$ of $\mathcal{B}$ that is increasing for the order relation $\preceq$ in $(G, \mathcal{B})$.*

Proof.

1. Let $(u_n) \in (\mathbb{N} \uplus \infty)^m$, $n \in \mathbb{N}$, be an infinite sequence of m-tuples for some $m \in \mathbb{N}$. Then, two cases arise:
   - The sequence (u_n) is bounded by some $u \in \mathbb{N}^m$. In that case, there exists $u' \leq u$ such that u' has infinitely many occurrences in (u_n). The restriction of (u_n) to these occurrences is an infinite sequence that is (non-strictly) increasing.
   - The sequence (u_n) is unbounded. By induction on m:
     (a) if $m = 1$, if ∞ has infinitely many occurrences in (u_n): the restriction of (u_n) to these occurrences is an infinite sequence that is (non-strictly) increasing. if ∞ has only finitely many occurrences in (u_n): there exists an infinite unbounded sub-sequence $(u'_n) \in \mathbb{N}$, $n \in \mathbb{N}$ with values in $\mathbb{N}$. Then,
       i. if the sequence (u'_n) is bounded by some $u \in \mathbb{N}^m$, it admits as above an infinite sequence that is (non-strictly) increasing, and
       ii. if the sequence (u'_n) is unbounded, it admits an infinite sub-sequence that is (strictly) increasing.
     (b) Assume the result holds for $m-1$. Consider by induction an infinite subsequence (u'_n), $n \in \mathbb{N}$ of (u_n) such that its projection on the first $m-1$ coordinates is increasing, and apply the result for $m = 1$ on the last coordinate of (u'_n).
2. The proof is in three steps:
   (a) the sequence (m_n) contains an infinite sub-sequence $(m^{(1)})_n$, $n \in \mathbb{N}$, that is increasing for the order relation $\leq$. Denote by $(u^{(1)})_n$, $n \in \mathbb{N}$, the corresponding infinite sub-sequence of (u_n).
   (b) For every tuple $u^{(1)}_i$ in $(u^{(1)}_n)$, there exists an infinite number of indexes $j > i$ such that $u^{(1)}_i \leq u^{(1)}_j$. Indeed, it suffices to apply Proposition 3.10, 1), on the infinite sequence of projections of the elements of $u^{(1)}_n$, for $n > i$, on their m_i first coordinates.
   (c) (u'_n) is then built as follows: $u'_1 = u^{(1)}_1$, and, for any $i \in \mathbb{N}$, for any $u'_i = u^{(1)}_j$ for some $j \geq i$, u'_{j+1} is chosen arbitrarily among the (infinitely many) elements $u^{(1)}_k$, $k > j$ with $u^{(1)}_j \leq u^{(1)}_k$. It is then clear that (u'_n) is infinite, and increasing for the relation $\leq$.
3. The proof is by induction on the number of states of S.
   - Assume S has only one state q. Then, every base B_i, $i \in \mathbb{N}$, is a set of labelled vertices $(q, x_1, \cdots, x_n)^k$, where the $x_j \in \mathbb{N} \uplus \infty$ for all j, and $k \in \mathbb{N} \uplus \infty$. Assume B_i has n_i vertices, and define a tuple $V_i \in (\mathbb{N} \uplus \infty)^{n_i(m+1)} = (v_1, \cdots, v_{n_i})$, where the $v_t = (x_1, \cdots, x_n, k)$ are the encodings of the labels $(q, x_1, \cdots, x_n)^k$ of the vertices of B_i: the tuple V_i encodes the base B_i. Note

that different tuples may encode the same base, depending on the order applied on the vertices of the base. Yet, clearly, for two bases B_i and B_j with $i < j$, and two tuples V_i and V_j encoding B_i and B_j respectively, $V_i \leq V_j$ implies $B_i \preceq B_j$. The result follows now from Proposition 3.10, 2) on any sequence (V_n), $n \in \mathbb{N}$ of tuples encoding the bases (B_n), $n \in \mathbb{N}$.

- Assume the result holds for t states, and that S has $t+1$ states $q_1, \cdots, q_{t+1}$. For a base B_i of S, in the sequence, we define C_i to be the base of S consisting only of the vertices of B_i labelled with states $q_1, \cdots, q_t$. Apply the induction hypothesis on (C_n), $n \in \mathbb{N}$: $C_n^{(1)}$, $n \in \mathbb{N}$ is an infinite subsequence of (C_n), increasing for the order relation $\preceq$.

  Let $B_n^{(1)}$, $n \in \mathbb{N}$ be the corresponding subsequence of (B_n). As above, for any base $B_i^{(1)}$ in this sequence, with n_i vertices labelled with state q_{t+1}, let $V_i^{(1)} \in (\mathbb{N} \uplus \infty)^{n_i(m+1)} = (v_1, \cdots, v_{n_i})$, where the $v_t = (x_1, \cdots, x_n, k)$ are the encodings of the labels $(q_{t+1}, x_1, \cdots, x_n)^k$ of vertices of $B_i^{(1)}$. Then, as above, for two bases $B_i^{(1)}$ and $B_j^{(1)}$ with $i < j$, and two tuples $V_i^{(1)}$ and $V_j^{(1)}$ encoding their vertices labelled with q_{t+1}, $V_i^{(1)} \leq V_j^{(1)}$ implies $B_i^{(1)} \preceq B_j^{(1)}$. The result follows now from Proposition 3.10, 2) on any sequence $(V_n^{(1)})$, $n \in \mathbb{N}$ of tuples encoding the vertices of the bases $B_i^{(1)}$ labelled with state q_{t+1}.

$\square$

We are now in a position to prove the finiteness and constructibility of the Karp and Miller tree.

Theorem 3.11. *Let $S = (G, T, m)$, with $G = (Q, A)$ be a VASS-SJ and B be a base of a configuration $\mathcal{C}$ of S. The Karp and Miller tree $\mathfrak{T}$ on (S, B) is finite, and can be effectively constructed.*

Proof. The proof of the finiteness is by contradiction.
Let $\mathfrak{T} = (V, E, \mathcal{B}_\mathfrak{T})$, with $G_\mathfrak{T} = (V, E)$. By construction, every node in the tree $G_\mathfrak{T}/\mathcal{B}_\mathfrak{T}$ has a finite number of sons. Assume by contradiction $G_\mathfrak{T}/\mathcal{B}_\mathfrak{T}$ is infinite: then, by Koenig's Lemma, $G_\mathfrak{T}/\mathcal{B}_\mathfrak{T}$ has an infinite branch. Denote by $(G, \mathcal{B})$ the corresponding infinite generalized transition sequence, where $\mathcal{B} = (B_n)$, $n \in \mathbb{N}$.
By Proposition 3.10, 3), there exists an infinite sub-sequence $(B_n^{(1)})$, $n \in \mathbb{N}$, of $\mathcal{B}$, that is increasing for the order relation $\preceq$ in $(G, \mathcal{B})$. For a base B, denote by $\mathcal{F}(B)$ the set of vertices of B labelled with finite multiplicity, and $\mathcal{I}(B)$ the set of vertices of B labelled with ∞ multiplicity.
In a first step, let us prove by contradiction that the sizes of the bases $B_i^{(1)}$ in this sequence are bounded. Assume that the sizes of these bases are not bounded: Then, there exists an infinite subsequence $(B_n^{(2)})$, $n \in \mathbb{N}$, of $(B_n^{(1)})$, of bases of strictly increasing sizes.
Since $(B_n^{(2)})$, $n \in \mathbb{N}$, is also increasing for the order relation $\preceq$, for each $i \in \mathbb{N}$, there exists a base $E_{i+1}^{(2)}$ and a vertex w_{i+1} labelled with ∞ multiplicity such that

- $B_{i+1}^{(2)} = E_{i+1}^{(2)} \uplus \{w_{i+1}\}$, and
- $B_i^{(2)} \preceq E_{i+1}^{(2)}$.

By construction of $\mathfrak{T}$, each of these vertices w_{i+1} in $\mathcal{I}(B_{i+1}^{(2)})$, has been created by the application of a Step (1) of a single lifting, from a v_{i+1} in $\mathcal{F}(B_{i+1}^{(2)})$ with augmentation multiplicity. Since the sequence (w_n), $n \in \mathbb{N}$, is infinite, it contains an infinite increasing subsequence $(w_n^{(3)})$, $n \in \mathbb{N}$. Since each construction step of $\mathfrak{T}$ contains a normalization of the bases, the sequence $(w_n^{(3)})$, $n \in \mathbb{N}$ is strictly increasing.
Then $(v_n^{(4)})$, $n \in \mathbb{N}$ contains an infinite sub-sequence $(v_n^{(5)})$, $n \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$, $(v_i^{(5)} < v_j^{(5)})$ and there exists an edge $(v_i^{(5)}, v_{i+1}^{(5)})$ in a welding graph $G_{B_i^{(5)} \preceq B_{i+1}^{(5)}}$.
Indeed, fix a sequence of welding graphs $G_{B_k^{(4)} \preceq B_{k+1}^{(4)}}$ for all $k \geq 1$. Then, for all $i < j$, define $G_{i,j}$ as the composition of the graphs $G_{B_i^{(4)} \preceq B_{i+1}^{(4)}}, \cdots, G_{B_{j-1}^{(4)} \preceq B_j^{(4)}}$, and G_i as the closure of the $G_{i,j}$'s. Define $V_i^{(4)} \subseteq B_i^{(4)}$

as the set of vertices v in $B_i^{(4)}$ such that there exists a path from v to some $v_k^{(4)}$ in G_i. For each $i \in \mathbb{N}$, two cases arise:

1. there exists a vertex $v \in V_i^{(4)}$, and a path from v in G_i passing through infinitely many vertices $v_k^{(4)}$. This infinitely many vertices $v_k^{(4)}$ is exactly the sequence $(v_n^{(5)})$, $n \in \mathbb{N}$.

2. No path starting from $V_i^{(4)}$ passes through infinitely many vertices $v_k^{(4)}$. Since vertices of multiplicity one have out-degree and in-degree one in these welding graphs, there exists infinitely many vertices $t_k^{(4)}$ isolated in the welding graphs $G_{B_{k-1}^{(4)} \preceq B_k^{(4)}}$, such that there exists $j > k$ and a path from $t_k^{(4)}$ to $v_j^{(4)}$ in G_i. As above, the sequence $t_n^{(4)}$, n$\in \mathbb{N}$ contains an infinite, strictly increasing subsequence. Therefore, there exists $k \in \mathbb{N}$, with $v_i^{(4)} < t_k^{(4)}$ and a path from $t_k^{(4)}$ to some $v_j^{(4)}$ in G_i. In the welding graph $G_{B_i^{(4)} \preceq B_k^{(4)}}$ induced by $G_{i,k}$ for the relation $B_i^{(4)} \preceq B_k^{(4)}$, there exists an edge $(v_i^{(4)}, z_k)$ for some vertex z_k in $B_k^{(4)}$. The graph obtained from $G_{B_i^{(4)} \preceq B_k^{(4)}}$ by removing this edge $(v_i^{(4)}, z_k)$ and adding an edge $(v_i^{(4)}, t_k^{(4)})$, is clearly a welding graph for the relation $B_i^{(4)} \preceq B_k^{(4)}$. Its composition with the graphs $G_{B_k^{(4)} \preceq B_{k+1}^{(4)}}, \cdots, G_{B_{j-1}^{(4)} \preceq B_j^{(4)}}$ induces a welding graph for the relation $B_i^{(4)} \preceq B_j^{(4)}$, with an edge $(v_i^{(4)}, v_j^{(4)})$. It follows that, for each i, there exists $j > i$ and an edge $(v_i^{(4)}, v_j^{(4)})$ in a welding graph $G_{B_i^{(4)} \preceq B_j^{(4)}}$. Moreover, $(v_i^{(4)} < v_j^{(4)})$. The construction of the sequence $(v_n^{(5)})$, $n \in \mathbb{N}$ follows inductively.

Now, for each $i \in \mathbb{N}$, denote by $V_i^{(5)} \subseteq B_i^{(5)}$ the set of vertices of $B_i^{(5)}$ such that, for all $v \in V_i^{(5)}$, there exists some $v_k^{(5)}$, $k > i$, with a path from v to $v_k^{(5)}$ in G. Two cases arise:

1. There exists finitely many sets $V_i^{(5)}$ such that $v_i^{(5)} \in V_i^{(5)}$. Then, there exists infinitely many sets $V_i^{(5)} \subsetneq B_i^{(5)}$. It follows that there exists an infinite subsequence $V_n^{(6)}$, $n \in \mathbb{N}$, increasing for the order relation $\preceq$. Since, for all i, $V_i^{(6)} \subsetneq B_i^{(6)}$, there exist sub-increasing generalized transition sequences in the corresponding branch of $\mathfrak{T}$, which contradicts Case 2) of its definition.

2. There exists an infinite subsequence $V_n^{(6)}$, $n \in \mathbb{N}$ such that $v_i^{(6)} \in V_i^{(6)}$ for all i. Then, for each such $v_i^{(6)}$, there exists $v_k^{(6)}$, $k > j$, such that:

   - $v_k^{(6)} > v_i^{(6)}$,
   - there exists an edge $(v_i^{(6)}, v_k^{(6)})$ in some welding graph, and
   - there exists a path from $v_i^{(6)}$ to $v_k^{(6)}$ in G.

   It follows that each of these vertices $v_i^{(6)}$ has an augmentation coordinate, which is replaced by ∞ in the corresponding lifting. Since the dimension m of S is finite, it contradicts the infinite length of this sequence.

Therefore, the sizes of the bases $B_i^{(1)}$ are bounded.
Now, again, Two cases arise:

1. for every state q of S, no infinite sequence $(v_n^{(1)})$, $n \in \mathbb{N}$ of vertices in $B_n^{(1)}$ labelled with state q. contains a strictly increasing infinite subsequence. Then, all single configurations of the $B_n^{(1)}$ are bounded. Since the sequence $(B_n^{(1)})$ is infinite, and the $B_i^{(1)}$'s are bounded in size, it follows that there exists $i < j$ such that $B_i^{(1)} = B_j^{(1)}$, which contradicts case 1) of the definition of the Karp and Miller tree.

2. There exists a state q of S, and an infinite sequence $(v_n^{(1)})$, $n \in \mathbb{N}$ of vertices in $B_n^{(1)}$ labelled with state q. containing a strictly increasing infinite subsequence $(v_n^{(7)})$, $n \in \mathbb{N}$. Then, $(v_n^{(7)})$, $n \in \mathbb{N}$ contains an

infinite sub-sequence $(v_n^{(8)})$, $n \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$, there exists an edge $(v_i^{(8)}, v_{i+1}^{(8)})$ in a welding graph $G_{B_i^{(8)} \preceq B_{i+1}^{(8)}}$.

Indeed, as above, fix a sequence of welding graphs $G_{B_k^{(7)} \preceq B_{k+1}^{(7)}}$ for all $k \geq 1$. Then, for all $1 < j$, define G_j as the composition of the graphs $G_{B_1^{(7)} \preceq B_2^{(7)}}, \cdots, G_{B_{j-1}^{(7)} \preceq B_j^{(7)}}$, and G_1 as the closure of the G_j's. Define $V_1^{(7)} \subseteq B_1^{(7)}$ as the set of vertices v in $B_1^{(7)}$ such that there exists a path from v to some $v_k^{(7)}$ in G_i. Since the bases $B_i^{(7)}$ have bounded sizes, there exists a bounded number of isolated vertices in G_1. Therefore, there exists a vertex $v \in V_1^{(7)}$, and a path from v in G_1 passing through infinitely many vertices $v_k^{(7)}$. This infinitely many vertices $v_k^{(7)}$ is exactly the sequence $(v_n^{(8)})$, $n \in \mathbb{N}$.

Now, for each $i \in \mathbb{N}$, denote by $V_i^{(8)} \subseteq B_i^{(8)}$ the set of vertices of $B_i^{(8)}$ such that, for all $v \in V_i^{(8)}$, there exists some $v_k^{(8)}$, $k > i$, with a path from v to $v_k^{(8)}$ in G. Two cases arise:

(a) There exists finitely many sets $V_i^{(8)}$ such that $v_i^{(8)} \in V_i^{(8)}$. Then, there exists infinitely many sets $V_i^{(8)} \subsetneq B_i^{(8)}$. It follows that there exists an infinite subsequence $V_n^{(9)}$, $n \in \mathbb{N}$, increasing for the order relation $\preceq$. Since, for all i, $V_i^{(9)} \subsetneq B_i^{(9)}$, there exist sub-increasing generalized transition sequences in the corresponding branch of $\mathfrak{T}$, which contradicts Case 2) of its definition.

(b) There exists an infinite subsequence $V_n^{(8)}$, $n \in \mathbb{N}$ such that $v_i^{(8)} \in V_i^{(8)}$ for all i. Then, for each such $v_i^{(8)}$, there exists $v_k^{(8)}$, $k > j$, such that:

- $v_k^{(8)} > v_i^{(8)}$,
- there exists an edge $(v_i^{(8)}, v_k^{(8)})$ in some welding graph, and
- there exists a path from $v_i^{(8)}$ to $v_k^{(8)}$ in G.

It follows that each of these vertices $v_i^{(8)}$ has an augmentation coordinate, which is replaced by ∞ in the corresponding lifting. Since the dimension m of S is finite, it contradicts the infinite length of this sequence.

It follows by contradiction that the generalized transition sequence $(G, \mathcal{B})$ is finite, and, by Koenig's Lemma, so is $\mathfrak{T}$. The constructibility of $\mathfrak{T}$ follows from its constructive definition. $\square$

4 Coverability

In the previous section, we have seen how to abstract increasing positive transition sequences of a VASS-SJ as generalized transition sequences, and how to synthesize all possible increasing generalized transition sequence as a finite data structure, the Karp and Miller tree. This section here is more about doing the reverse operation: starting from the Karp and Miller tree, how can we re-build all corresponding positive transition sequences of the VASS-SJ? A first tool in this direction is a notion of covering graph, derived from the Karp and Miller Tree in a very classical fashion.

4.1 The Covering Graph

Definition 4.1. Covering graph
Let S be a VASS-SJ and $\mathcal{C}$ be a configuration of S with base B. Let $\mathfrak{T} = (G, \mathcal{B})$ be the Karp and Miller tree on (S, B). The covering graph $G_{\mathfrak{T}}$ is obtained from the quotient tree $\mathfrak{T}/\mathcal{B}$ by the following procedure:
For each final base (base with out-degree 0) B_j of a generalized configuration G_j in $\mathcal{B}$, if there exists a base B_i in $\mathcal{B}$ such that:

1. B_i is base of the same generalized configuration B_j, and

2. there exists a path from B_i to B_j in $\mathfrak{T}/\mathcal{B}$,

Then,

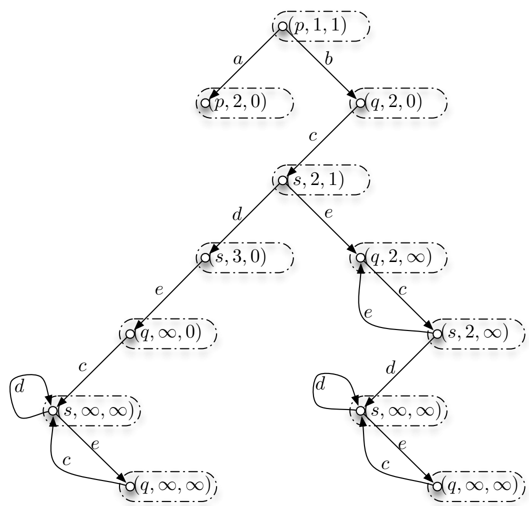
1. B_j is identified with B_i in $G_{\mathfrak{T}}$, and

2. Every edge e with destination vertex B_j in $\mathfrak{T}/\mathcal{B}$, is replaced by g copies of it, with destination vertex B_i, in $G_{\mathfrak{T}}$, where g is the number of different welding graphs for the relation $B_j \preceq B_i$. every copy of e is then labelled with a different welding graph for the relation $B_j \preceq B_i$.

Note that, if such a base B_i exists, it is unique by Point 1) of the construction of $\mathfrak{T}$.

Note the the definition above is only about the quotient tree $\mathfrak{T}/\mathcal{B}$: we do not identify different bases in $\mathfrak{T}$. Doing so would mean identifying every vertex in B_j with one vertex in B_i, i.e. choosing one welding graph for the relation $B_i \preceq B_j$, and loosing the possibility to represent all other possible welding graphs. Of course, this is not an issue when we restrict ourselves to the particular case of VASS, since in that case $\mathfrak{T}/\mathcal{B}$ is isomorphic to $\mathfrak{T}$.

Example 4.2. The covering graph corresponding to the VASS of Example 3.9 is the following. Again, this example is taken from [Reu88]. In this example, we have omitted the welding graphs labeling edges going upwards: since all bases are reduced to single generalized configurations, these welding graphs are reduced to a single edge.



Definition 4.3. Path in the covering graph, and Generalized transition sequence associated to it

Let S be a VASS-SJ and $\mathcal{C}$ be a configuration of S with base B. Let $\mathfrak{T} = (G, \mathcal{B})$ be the Karp and Miller tree on (S, B), and $G_{\mathfrak{T}}$ be its covering graph.
Let $p = B_1 \to \cdots \to B_n$ be a path in $G_{\mathfrak{T}}$. Then, there exists one generalized transition sequence $(H, \mathcal{D})$, with $\mathcal{D} = D_1, \cdots, D_n$ in S, where the D_i are bases of $\mathfrak{T}$, associated to the path p. $(H, \mathcal{D})$ is inductively defined as follows.

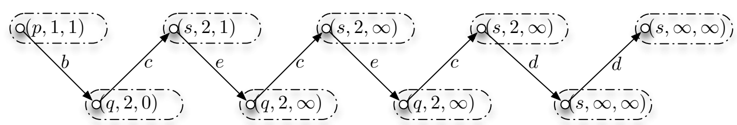
- If $p = B_1$, $(H, \mathcal{D})$ is reduced to the base B_1, and
- if $p = B_1 \to \cdots \to B_{n-1} \to B_n$, with $(H_{n-1}, \mathcal{D}_{n-1})$, where $\mathcal{D}_{n-1} = D_1, \cdots, D_{n-1}$, associated to the subpath $B_1 \to \cdots \to B_{n-1}$, two cases arise:
  1. B_n does not appear in $B_1 \to \cdots \to B_{n-1}$. Then, there exists an edge (D_{n-1}, B_n) in $\mathfrak{T}/\mathcal{B}$, and $(H, \mathcal{D})$ is obtained by extending $(H_{n-1}, \mathcal{D}_{n-1})$ with the corresponding transition firing from D_{n-1} to B_n in $\mathfrak{T}$.

2. B_n appears in $B_1 \to \cdots \to B_{n-1}$. In that case, B_n has been obtained in the construction of $G_{\mathfrak{T}}$ by merging two bases B_i and B_j of $\mathfrak{T}/\mathcal{B}$, such that there exists a path from B_i to B_j in $\mathfrak{T}/\mathcal{B}$. Then, there exists an edge (D_{n-1}, B_j) in $\mathfrak{T}/\mathcal{B}$. Let $(H', \mathcal{D}')$ be the generalized transition sequence obtained by extending $(H_{n-1}, \mathcal{D}_{n-1})$ with the corresponding transition firing from D_{n-1} to B_j in $\mathfrak{T}$. The generalized transition sequence $(H, \mathcal{D})$, with $D_n = B_i$, is obtained by composing $(H', \mathcal{D}')$ with the generalized transition sequence reduced to B_i, with respect to the welding graph for the relation $B_j \preceq B_i$ labeling the edge $B_{n-1} \to B_n$ in p.

Example 4.4. In the covering graph of Example 4.2, consider the following path, starting from the initial base $\{(p,1,1)\}$:

$$p = \{(p,1,1)^1\} \to_b \to_c \to_e \to_c \to_e \to_c \to_d \to_d \{(s,\infty,\infty)^1\}.$$

The generalized transition sequence associated to it is the following:



Note that, in the general case of a VASS-SJ, the unfolding of the loops in the covering graph would be obtained by composing the loops with respect to some given welding graphs: here, the welding between the fifth and the sixth bases is performed with respect to a welding graph for the relation $\{(q,2,\infty)^1\} \preceq \{(q,2,\infty)^1\}$, and the the welding between the eighth and the ninth bases is performed with respect to a welding graph for the relation $\{(s,\infty,\infty)^1\} \preceq \{(s,\infty,\infty)^1\}$. In both cases, the two bases of the corresponding welding graph, while identified in the covering graph, correspond to two different vertices in the Karp and Miller tree. In this example based on a simple VASS, the welding graphs are trivial (reduced to only one edge), yet, this is not necessarily the case for VASS-SJ.

The following proposition shows that our constructions of the Karp and Miller tree, and the corresponding covering graph, cover indeed all possible executions of the given VASS-SJ, from the given initial base. Since our constructions are restricted to generalized transition sequences with no sub-increasing subsequence, the possible executions of the VASS-SJ we consider are only the one with no sub-increasing subsequence. Lemma 2.23 ensures that we do not loose any expressivity with this restriction.

Proposition 4.5. *Let S be a VASS-SJ and $\mathcal{C}$ be a configuration of S with base B. Let $\mathfrak{T} = (G, \mathcal{B})$ be the Karp and Miller tree on (S, B), and $G_{\mathfrak{T}}$ be its covering graph. Let $(H, \mathcal{D})$ be a non-negative transition sequence, where $\mathcal{D} = D_1, \cdots, D_k$ with $D_1 = B$. Assume $(H, \mathcal{D})$ contains no sub-increasing subsequence. Then, there exists a unique path p in $G_{\mathfrak{T}}$ of length k, such that $(H, \mathcal{D})$ is compatible with the generalized transition sequence associated to p.*

Proof. by induction on k,. If $k = 1$, the result is straightforward. Assume the result holds for k, and let $(H, \mathcal{D})$ be a non-negative transition sequence, where $\mathcal{D} = D_1, \cdots, D_{k+1}$ with $D_1 = B$, containing no sub-increasing subsequence. Let $(H_k, \mathcal{D}_k)$ be the restriction of $(H, \mathcal{D})$ to the bases $D_1, \cdots, D_k$, and let $p_k = B_1 \to \cdots \to B_k$ be the corresponding path in $G_{\mathfrak{T}}$. Since D_k is compatible with B_k, and $D_k \hookrightarrow_t D_{k+1}$ for some transition t, there exists a base B'_{k+1} such that D_{k+1} is compatible with B_{k+1}, and $B_k \hookrightarrow_t B'_{k+1}$. Two cases arise:

1. the vertex B_k in $G_{\mathfrak{T}}$ has not been obtained by the merging of different vertices of $\mathfrak{T}/\mathcal{B}$. Let B_i be the corresponding base in $\mathcal{B}$, and $(G_i, \mathcal{B}_i)$ the restriction of $\mathfrak{T}$ to the path from B to B_i in $\mathfrak{T}/\mathcal{B}$. Then Step 1) of the construction of $\mathfrak{T}$ does not apply to B_i, and, by step 3) of the construction of $\mathfrak{T}$, there exists a base B_{k+1} in $\mathfrak{T}$ obtained by lifting and normalizing the extension of $(G_i, \mathcal{B}_i)$ with the transition firing $B_k \hookrightarrow_t B'_{k+1}$. Then, $B'_{k+1} \preceq B_{k+1}$, and D_{k+1} is compatible with B_{k+1} (possibly with compatibility cap 0). Therefore, the generalized transition sequence corresponding to the path $p_{k+1} = B_1 \to \cdots \to B_k \to B_{k+1}$ is compatible with $(H, \mathcal{D})$.

2. the vertex B_k in $G_{\mathfrak{T}}$ has been obtained by the merging of different vertices of $\mathfrak{T}/\mathcal{B}$. Let $B_i \in \mathfrak{T}/\mathcal{B}$ be the uppermost such vertex (i.e. the one such that there exists a path to all others in $\mathfrak{T}/\mathcal{B}$). Step 1) of the construction of $\mathfrak{T}$ does not apply to B_i and the same as above applies.

□

4.2 Elementary Loops in the covering graph

The following lemma allows us to introduce the notion of loops in the covering graph, and, in particular, the notion of augmentation loops, which induce some coordinate or multiplicity increase.

Lemma 4.6. *Let S be a VASS-SJ and $\mathcal{C}$ be a configuration of S with base B. Let $\mathfrak{T} = (G, \mathcal{B})$ be the Karp and Miller tree on (S, B), and $G_{\mathfrak{T}}$ be its covering graph. Let $p = B_1 \to \cdots \to B_n$ be a path in $G_{\mathfrak{T}}$, and assume that*

1. *$B_1 \preceq B_n$, and*
2. *for all $1 < i < B_n$, $B_i \not\preceq B_n$.*

Then, there exists a loop $p' = B'_1 \to \cdots \to B'_n$ in $G_{\mathfrak{T}}$, such that

1. *for all $i = 1, \cdots, n$, $B_i \preceq B'_i$,*
2. *for all $i = 1, \cdots, n-1$, the transition fired in $B'_i \to B'_{i+1}$ is the same as in $B_i \to B_{i+1}$ and*
3. *there exists a path from B_n to B'_1 in $G_{\mathfrak{T}}$.*

Similarly, let $(H, \mathcal{D})$, with $\mathcal{D} = D_1, \cdots, D_n$, be the generalized transition sequence associated to p in $\mathfrak{T}$, and $(H', \mathcal{D}')$, with $\mathcal{D}' = D'_1, \cdots, D'_n$ be the generalized transition sequence associated to p' in $\mathfrak{T}$. Then,

1. *for all $i = 1, \cdots, n$, $D_i \preceq D'_i$,*
2. *for all $i = 1, \cdots, n-1$, the transition fired in $D'_i \to D'_{i+1}$ is the same as in $D_i \to D_{i+1}$ and*
3. *there exists a path from D_n to D'_1 in $\mathfrak{T}/\mathcal{B}$.*

Proof. Let us prove the lemma for p and p'.
In a first step, let us prove by induction on k the following property: Let $p = B_1 \to \cdots \to B_k$ be a path in $G_{\mathfrak{T}}$, and B'_1 a base in $G_{\mathfrak{T}}$ such that $B_1 \preceq B'_1$. Then, there exists a path $p' = B'_1 \to \cdots \to B'_n$ in $G_{\mathfrak{T}}$, such that, for all $i = 1, \cdots, k$, $B_i \preceq B'_i$.
For $k = 1$, the result is straightforward. Assume the property above holds for k, and let $(H, \mathcal{D})$, with $\mathcal{D} = D_1, \cdots, D_k, D_{k+1}$, be the generalized transition sequence associated to $p = B_1 \to \cdots \to B_k \to B_{k+1}$ in $\mathfrak{T}$. Then, the restriction $(H_k, \mathcal{D}_k)$ of $(H, \mathcal{D})$ to $D_1, \cdots, D_k$, is clearly associated to the restriction p_k of p to $B_1 \to \cdots \to B_k$, and the induction hypothesis applies. Let $p'_k = B'_1, \to, B'_k$ be the corresponding path, with $B_k \preceq B'_k$. Denote by $(H'_k, \mathcal{D}'_k)$, with $\mathcal{D}'_k = D'_1, \cdots, D'_k$ the generalized transition sequence associated to p'_k in $\mathfrak{T}$. Then, $D_k \preceq D'_k$. Moreover, $D_k \hookrightarrow D_{k+1}$ in $(H, \mathcal{D})$. Two cases arise:

1. the vertex B'_k in $G_{\mathfrak{T}}$ has not been obtained by the merging of different vertices of $\mathfrak{T}/\mathcal{B}$. Let $(G_k, \mathcal{B}_k)$ be the restriction of $\mathfrak{T}$ to the path from B to D'_k in $\mathfrak{T}/\mathcal{B}$. Then Step 1) of the construction of $\mathfrak{T}$ does not apply to D'_k, and, by step 3) of the construction of $\mathfrak{T}$, there exists a base D'_{k+1} in $\mathfrak{T}$ obtained by lifting and normalizing the extension of $(G_k, \mathcal{B}_k)$ with the transition firing $D'_k \hookrightarrow_t D''_{k+1}$. Then, $D_{k+1} \preceq D'_{k+1}$, and p' is the path associated to the extension of $(H'_k, \mathcal{D}'_k)$ with the transition firing $D'_k \hookrightarrow_t D'_{k+1}$ in $\mathfrak{T}$.
2. the vertex B'_k in $G_{\mathfrak{T}}$ has been obtained by the merging of different vertices of $\mathfrak{T}/\mathcal{B}$. Let $D_i \in \mathfrak{T}/\mathcal{B}$ be the uppermost such vertex (i.e. the one such that there exists a path to all others in $\mathfrak{T}/\mathcal{B}$). Step 1) of the construction of $\mathfrak{T}$ does not apply to D_i and the same as above applies.

Assume $p = B_1 \to \cdots \to B_n$ is a path in $G_{\mathfrak{T}}$, such that

1. $B_1 \preceq B_n$, and
2. for all $1 < i < B_n$, $B_i \not\preceq B_n$.

We define inductively:

- $p^1 = p$, with, for all $j = 1, \cdots, n$, $B_j^1 = B_j$, and,
- for $i > 1$, $p^i = B_1^i \to \cdots \to B_n^i$ is the path obtained by the property above, with $p^{i-1} = B_1^{i-1} \to \cdots \to B_n^{i-1}$, and $B_1^i = B_n^{i-1}$. Then, clearly, $B_1^i \preceq B_n^i$.

Since $\mathfrak{T}$ is finite, the sequence of paths p^i is stationary after some index i: therefore there exists such a path p^i with $B_1^i = B_n^i$. □

Definition 4.7. Elementary (Augmentation) Loop

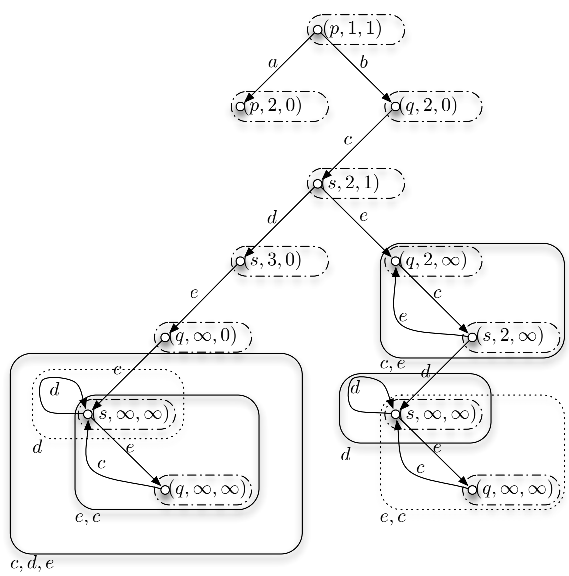
Let S be a VASS-SJ and $\mathcal{C}$ be a configuration of S with base B. Let $\mathfrak{T} = (G, \mathcal{B})$ be the Karp and Miller tree on (S, B), and $G_{\mathfrak{T}}$ be its covering graph. Let $p = B_1 \to \cdots \to B_n$ be a path in $G_{\mathfrak{T}}$, such that that

1. $B_1 \preceq B_n$, and
2. for all $1 < i < n$, $B_i \not\preceq B_n$.

The loop $p' = B'_1 \to \cdots \to B'_n$ in $G_{\mathfrak{T}}$ associated to p by Lemma 4.6 is an *elementary loop.* if, moreover, $B_1 \precneqq B_n$, the loop $p' = B'_1 \to \cdots \to B'_n$ in $G_{\mathfrak{T}}$ associated to p by Lemma 4.6 is an *elementary augmentation loop.*
Similarly, a path $p = B_1 \to \cdots \to B_n$ in $G_{\mathfrak{T}}$ is *elementary* if and only if, for all $1 < i \leq B_n$, $B_i \not\preceq B_n$.

Note that, according to the definition above, elementary augmentation loops are exactly loops induced by the liftings in the construction of $\mathfrak{T}$. Note also that every loop in the covering graph can be obtained by composing elementary loops. Finally, note that every loop in $G_{\mathfrak{T}}$, with no sub-loop, is elementary.

Example 4.8. Consider the covering graph of Example 4.2. The elementary loops are indicated by the large boxes, labelled with the corresponding sequence of transition labels, as follows.



The elementary augmentation loops are the ones indicated by the large black boxes:

- $(q,\infty,\infty) \to_c (s,\infty,\infty) \to_d (s,\infty,\infty) \to_e (q,\infty,\infty)$, wich corresponds to the path $(q,2,0) \to_c (s,2,1) \to_d (s,3,0) \to_e (q,\infty,0)$,
- $(s,\infty,\infty) \to_e (q,\infty,\infty) \to_c (s,\infty,\infty)$, wich corresponds to the path $(s,3,0) \to_e (q,\infty,0) \to_c (s,\infty,\infty)$,
- $(q,2,\infty) \to_c (s,2,\infty) \to_e (q,2,\infty)$, wich corresponds to the path $(q,2,0) \to_c (s,2,1) \to_e (q,2,\infty)$, and
- $(s,\infty,\infty) \to_d (s,\infty,\infty)$, which corresponds to the path $(s,2,\infty) \to_d (s,\infty,\infty)$.

The elementary non-augmentation loops are the ones indicated by the large dotted boxes:

- $(s,\infty,\infty) \to_d (s,\infty,\infty)$, which corresponds to the path $(s,\infty,\infty) \to_d (s,\infty,\infty)$, and
- $(s,\infty,\infty) \to_e (q,\infty,\infty) \to_c (s,\infty,\infty)$, which corresponds to the path $(s,\infty,\infty) \to_e (q,\infty,\infty)$.

4.3 The Coverability Result

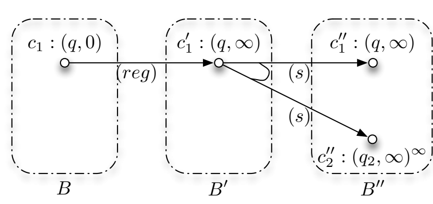
Our aim here is to prove the constructibility, for any compatibility cap N and for any vertex v in the Karp and Miller tree $\mathfrak{T}$, of a non-negative transition sequence from the initial base to a base compatible with v, with the given compatibility cap N. The construction of this transition sequence is, of course, based on repetitions of elementary augmentation loops. In the classical case of VASS, it is enough to repeat these loops a number of time linear in N. In the general case of VASS-SJ, it is no longer the case: some loops need to be repeated an order of time greater than others. The following definitions allow us to precise these number of repetition needed.

Definition 4.9. inherited ∞ coordinates and multiplicities
Let $S = (H,T,m,v)$ be a VASS-SJ, and $(G,\mathcal{B})$ with $\mathcal{B} = B_1,\cdots,B_f$ be a simply connected generalized transition sequence on S. The set of *inherited* ∞ coordinates and multiplicities in $(G,\mathcal{B})$ is inductively defined as follows:

1. Let $v \in B_1$ be a vertex labelled with $(q,x_1,\cdots,x_m)^k$. Any coordinate $x_j = \infty$ is *inherited* in $(G,\mathcal{B})$. Similarly, if $k = \infty$, it as an *inherited* ∞ multiplicity in $(G,\mathcal{B})$.
2. for $i = 1,\cdots,f-1$, let $w \in B_{i+1}$ labelled with $(r,y_1,\cdots,y_m)^l$. Let $y_j = \infty$. This ∞ coordinate is *inherited* in $(G,\mathcal{B})$ if and only if there exists $v \in B_i$ labelled with $(q,x_1,\cdots,x_m)^k$, and an edge (v,w) in G, such that x_j is an *inherited* ∞ coordinate. Similarly, if $l = \infty$, it is an *inherited* ∞ multiplicity if and only if there exists $v \in B_i$ labelled with $(q,x_1,\cdots,x_m)^k$, and an (id) edge (v,w) in G, such that k is an *inherited* ∞ multiplicity.

Let $(G,\mathcal{B})$ with $\mathcal{B} = B_1,\cdots,B_f$, be a generalized transition sequence, and B_i, B_j, with $i < j$ be two bases in $\mathcal{B}$. We say that an ∞ coordinate or multiplicity in B_j is *inherited from* B_i in $(G,\mathcal{B})$, if it is inherited in the restriction of $(G,\mathcal{B})$ to $B_i,\cdots,B_j$.

Example 4.10. Let S be a VASS-SJ, B_1 be a base, and assume the Karp and Miller tree of S on initial base B_1 contains the following branch, where the regular transition considered is $(q,+1,q)$.



Then, the ∞ coordinates and multiplicities in B'' are not inherited in the generalized transition sequence. Yet, the ∞ coordinates in B'' are inherited from B' in the sequence.

Definition 4.11. Degree of an elementary augmentation elementary loop

Let S be a VASS-SJ and $\mathcal{C}$ be a configuration of S with base B. Let $\mathfrak{T} = (G, \mathcal{B})$ be the Karp and Miller tree on (S, B), and $G_{\mathfrak{T}}$ be its covering graph. The degrees of the elementary augmentation loops of $G_{\mathfrak{T}}$ is the smallest integer function on the loops such that the following holds.
For any path $p^1 = B_1^1 \to \cdots \to B_{n_1}^1$ in $\mathfrak{T}$, such that that

1. $B_1^1 \precneqq B_{n_1}^1$, and
2. for all $1 < i < n_1$, $B_i \not\preceq B_{n_1}^1$,

For any path $p^2 = B_1^2 \to \cdots \to B_{n_2}^2$ in $\mathfrak{T}$, such that that

1. $B_1^2 \precneqq B_{n_2}^2$,
2. for all $1 < i < n_2$, $B_i^2 \not\preceq B_{n_2}^2$, and
3. there exists a path from $B_{n_1}^1$ to $B_{n_1}^2$ in $\mathfrak{T}/\mathcal{B}$.

$$deg(l^1) \geq deg(l^2),$$

where l^1 (respectively l^2) is the elementary augmentation loop associated to p^1 (resp. p^2).
If, moreover,

1. the single lifting associated to p^1 introduces an ∞ coordinate x_j in some vertex $v \in B_{n_1}^1$, and
2. the single lifting associated to p^2 introduces a new vertex $w \in B_{n_2}^2$, with ∞ multiplicity, and with ∞ coordinate x_j inherited from $v \in B_{n_1}^1$, then

$$deg(l^1) > deg(l^2).$$

Example 4.12. Consider the branch of a Karp and Miller tree of Example 4.10. The path $B \to B'$ is associated to an elementary augmentation loop l in the covering graph, and the path $B' \to B''$ is associated to an elementary augmentation l' in the covering graph. Then, $deg(l) > deg(l')$.
Now, if we want to build a non-negative transition sequence from B to a base B_f, compatible with B'' with compatibility cap N, we need to repeat the loop l $N.(N+1)$ times, and then the loop l' N times. It is clear from this example that the degree of the elementary augmentation loop is related to the number of repetition of this loop needed to obtain the desired non-negative transition sequence.

The following lemma is a key step towards our coverability result.

Lemma 4.13. *Let S be a VASS-SJ. Let $(G_{i,j}, \mathcal{B}_{i,j})$, with $\mathcal{B}_{i,j} = B_i, \cdots B_j$, be a simply connected positive generalized transition sequence in S. Assume also that, for $k = 1, \cdots, j-1$, all ∞ coordinates and multiplicities in B_k are inherited in $(G_{i,j}, \mathcal{B}_{i,j})$. Let $s_{\max}$ be the number of split transitions fired in $(G_{i,j}, \mathcal{B}_{i,j})$ on vertices labelled with inherited ∞ coordinates, $d_{\max}$ be the maximal absolute value of any coordinate decrease of any regular transition fired in $(G_{i,j}, \mathcal{B}_{i,j})$ on vertices labelled with corresponding inherited ∞ coordinate, and $j_{\max}$ be the number of non-(id) transitions fired in $(G_{i,j}, \mathcal{B}_{i,j})$ on vertices labelled with inherited ∞ multiplicities.*
Let $N \in \mathbb{N}$, and let D_i be a base. Assume D_i si compatible with B_i, with compatibility cap $\max\{((j-i)d_{\max} + N).(s_{\max}+1), (j_{\max}+1).N\}$.
Then, there exists a positive transition sequence $(P_{i,j}, \mathcal{D}_{i,j})$, where $\mathcal{D}_{i,j} = D_i, \cdots, D_j$, compatible with $(G_{i,j}, \mathcal{B}_{i,j})$, such that D_j is compatible with B_j with compatibility cap N.

Proof. Let $w \in B_i$ labelled with $(q, y_1, \cdots, y_m)$, with $y_t = \infty$. Then, for any $k = i+1, \cdots, j$, any $w' \in B_k$ such that there exists a path $v, \cdots, w'$ in $G_{i,j}$, w' is labelled with $(q', y_1', \cdots, y_m')$, with $y_t' = \infty$.
Let $N \in \mathbb{N}$, and D_i, with corresponding welding function $f_{D_i \to B_i}$, be a set of vertices satisfying the premises of the Lemma. Let $P_{i,j}$ be a transition sequence derived from D_i and $G_{i,j}$ by firing the transitions of $G_{i,j}$, such that the firing of any split transition splits the coordinates evenly. Then, the maximal decrease of the t^{th} coordinate of any vertex v' below v is less than one can obtain by

- performing $s_{\max}$ even split transition firings, and then

- preforming $j-i$ regular transition firing with a $d_{\max}$ decrease of the t^{th} coordinate.

Similarly, the maximal decrease of the number of vertices in D_i pre-image of a vertex with ∞ inherited multiplicity in B_i by $f_{D_i \to B_i}$ is less than one can obtain by performing $j_{\max}$ non-(id) transitions on them. Therefore $P_{i,j}$ satisfies the property of the Lemma. Moreover, since $G_{i,j}$ is non-negative and the only non-inherited ∞ coordinates and multiplicities in $G_{i,j}$ can be in B_j, $P_{i,j}$ is non-negative. $\square$

We are now in a position to state and prove our coverability result.

Theorem 4.14. *Let S be a VASS-SJ and $\mathcal{C}$ be a configuration of S with base B_1. Let $\mathfrak{T} = (G, \mathcal{B})$ be the Karp and Miller tree on (S, B_1), and $G_{\mathfrak{T}}$ be its covering graph.*
Let $B_i \in \mathcal{B}$. Then, for any $N \in \mathbb{N}$, there exist a path $\mathcal{P}_i^N$ in $G_{\mathfrak{T}}$ with initial base B_1 and final base B_l, and a non-negative transition sequence $(P_i^N, \mathcal{D}_i^N)$ of S compatible with $\mathcal{P}_i^N$ with initial base $D_1 = B_1$ and final base D_l, such that D_l is compatible with B_l with compatibility cap N.
Moreover, let $G_1, \cdots, G_k$, where for $l = 1, \cdots, k$, $G_l = ((G_{i_l,j_l}, \mathcal{B}_{i_l,j_l})$, be the sequence of generalized transitions sequences in $\mathfrak{T}$ corresponding to the elementary augmentation loops in $\mathcal{P}_i^N$, ordered by decreasing degree. Let $(G_i^N, \mathcal{D}_i^N)$ be the generalized transition sequence associated to $\mathcal{P}_i^N$. Then,

$$(G_i^N, \mathcal{D}_i^N) = p_1.G_1^{q_1(N)}.p_2.G_2^{q_2(N)}.\cdots.p_k.G_k^{q_k(N)}.p_{k+1},$$

where

- *$p_1, \cdots, p_{k+1}$ are generalized transition sequences corresponding to simple sub-paths of $\mathcal{P}_i^N$, without any occurrence of any loop,*

- *L_i^k stands for the k^{th} composition of the loop L_i with itself (with welding graph clear from context), and*

- *$q_i, \cdots, q_k$ are integer polynomials of degree $\deg(l_i)$, where l_i is the elementary augmentation loop in $G_{\mathfrak{T}}$ associated to G_i.*

Proof. By induction on the depth of B_i in $\mathfrak{T}/\mathcal{B}$. If $B_i = B_1$, the result is straightforward.
Assume now that the result holds for all vertices above B_i in $\mathfrak{T}/\mathcal{B}$, and let B_{i-1} be the parent vertex of B_i in $\mathfrak{T}/\mathcal{B}$. Denote by B_{i-1}, B_i the generalized transition sequence consisting in the restriction of $\mathfrak{T}$ to $B_{i-1} \uplus B_i$. Let $s \in \{0,1\}$ be the number of split transitions fired in B_{i-1}, B_i on vertices labelled with inherited ∞ coordinates, d be the maximal absolute value of any coordinate decrease of any regular transition fired in B_{i-1}, B_i on vertices labelled with corresponding inherited ∞ coordinate, and $j \in \{0,1\}$ be the number of non-(id) transitions fired in B_{i-1}, B_i on vertices labelled with inherited ∞ multiplicities.
Two cases arise:

- There exist no elementary augmentation loop l_i in $G_{\mathfrak{T}}$, with corresponding generalized transition sequence ending on B_i in $\mathfrak{T}$.

  Apply the induction hypothesis on B_{i-1} with $N' = \max\{(d+N).(s+1), (j+1).N\}$: There exists a path $\mathcal{P}_{i-1}^{N'}$ and a generalized transition sequence

  $$(G_{i-1}^{N'}, \mathcal{D}_{i-1}^{N'}) = p_1.G_1^{q_1(N')}.p_2.G_2^{q_2(N')}.\cdots.p_k.G_k^{q_k(N')}.p_{k+1}$$

  Define now
  $$(G_i^N, \mathcal{D}_i^N) = (G_{i-1}^{N'}, \mathcal{D}_{i-1}^{N'}).B_{i-1} \uplus B_i,$$
  and apply Lemma 4.13 on the expansion of $B_{i-1} \uplus B_i$ corresponding to this composition: then, the result holds for $(G_i^N, \mathcal{D}_i^N)$.

- There exist an elementary augmentation loop l_i in $G_{\mathfrak{T}}$, with corresponding generalized transition sequence ending on B_i in $\mathfrak{T}$. Then, by construction of $\mathfrak{T}$, any other elementary augmentation loop with corresponding generalized transition sequence ending on B_i has the same degree as l_i.

  Let now $(G_1, \mathcal{B}_1), \cdots, (G_l, \mathcal{B}_l)$ be the generalized transition sequences in $\mathfrak{T}$, ending on B_i, associated to elementary loops in $G_{\mathfrak{T}}$. Let $W_1, \cdots, W_l$ be the corresponding welding graphs. For any $t = 1, \cdots, l$, let $s^t_{\max}$ be the maximal number of split transitions fired in G_t on vertices labelled with inherited ∞ coordinates, $d^t_{\max}$ be the maximal absolute value of any coordinate decrease of any regular transition fired in G_t on vertices labelled with corresponding inherited ∞ coordinate, $j^t_{\max}$ be the number of non-(id) transitions fired in G_t on vertices labelled with inherited ∞ multiplicities, and l^t be the length of G_t.

  Apply the induction hypothesis on B_{i-1} with

  $$N' = \max\{\left(N.\Sigma_{t=1}^{l} l^t.d^t_{\max}\right).\left(1 + \Sigma_{t=1}^{l} s^t_{\max}\right), N.\left(1 + \Sigma_{t=1}^{l} j^t_{\max}\right)\}:$$

  There exists a path $\mathcal{P}_{i-1}^{N'}$ and a generalized transition sequence

  $$(G_{i-1}^{N'}, \mathcal{D}_{i-1}^{N'}) = p_1.G_1^{q_1(N')}.p_2.G_2^{q_2(N')}.\cdots.p_k.G_k^{q_k(N')}.p_{k+1}$$

  Let now $L_1 = ((G_{j,i}, \mathcal{B}_{j,i}), G^1_{B_j \preceq_\infty B_i}), \cdots, L_k = ((G_{j,i}, \mathcal{B}_{j,i}), G^k_{B_j \preceq_\infty B_i})$ be the elementary augmentation loops of $\mathfrak{T}$ with final base B_i.

  Define now

  $$(G_i^N, \mathcal{D}_i^N) = (G_{i-1}^{N'}, \mathcal{D}_{i-1}^{N'}).B_{i-1} \uplus (G_1, \mathcal{B}_1)^N.\cdots.(G_l, \mathcal{B}_l)^N,$$

  where the $(G_t, \mathcal{D}_t)$ are composed with respect to the welding graphs W_t.

  Then, Lemma 4.13 and a simple induction on N shows that the result holds for $(G_i^N, \mathcal{D}_i^N)$.

$\square$

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