On the complexity of the Eulerian closed walk with precedence path constraints problem

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Abstract

The Eulerian closed walk problem in a digraph is a well-known polynomial-time solvable problem. In this paper, we show that if we impose the feasible solutions to fulfill some precedence constraints specified by paths of the digraph, then the problem becomes NP-complete. We also present a polynomial-time algorithm to solve this variant of the Eulerian closed walk problem when the paths are arc-disjoint. We also give necessary and sufficient conditions for the existence of feasible solutions in this polynomial-time solvable case.

Keywords: Eulerian closed walk, precedence path constraints, NP-completeness, polynomial-time algorithm.

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1 Introduction

In this paper, we consider a variant of the famous Eulerian Closed Walk Problem (ECWP) which consists of finding an Eulerian closed walk starting from a fixed vertex and fulfilling some precedence constraints on the arcs, specified by a partial order on arcs, the latter being defined by a set of paths. Let $D = (V, A)$ be a loopless Eulerian digraph and let $v_0 \in V$ be a specified vertex. A *walk* of *length* k is a sequence $P = (a_1, \ldots, a_k)$ of k arcs of A with $k \geq 1$, $a_i = (u_i, v_i)$ for all $i = 1, \ldots, k$ and $v_i = u_{i+1}$ for $i = 1, \ldots, k-1$. The vertex u_1 (respectively v_k) is called the *starting* (respectively *ending*) *vertex* of P. A *path* is a walk so that the vertices u_1, v_i, $i = 1, 2, \ldots, k$ are all different. A *closed walk* is a walk having $v_k = u_1$. A closed walk P is *Eulerian* if each arc of D appears exactly once in P. Given a walk P composed of distinct arcs, we write $a \prec_P a'$ if the arc a precedes the arc $a' \neq a$ in P, that is, if P traverses a before a'. Moreover if we consider a path $Q = (a_1, a_2, \ldots, a_k)$ of D, $k \geq 1$, we say that the walk P *respects* the path Q if $a \prec_Q a'$ and P contains a' imply that P also contains a and $a \prec_P a'$. (Remark that two adjacent arcs of Q are not necessarily adjacent in P.) We now precisely define the problem we consider hereafter. Let $D = (V, A)$ be a loopless Eulerian digraph and let $v_0 \in V$ be a specified vertex. (Note that D is not necessarily simple, that is, it may have multiple arcs.) The required partial order on A is given by a set $K = \{Q_1, Q_2, \ldots, Q_q\}$ of paths of D, $q \geq 1$. The *Eulerian Closed Walk with Precedence Path Constraints Problem (ECWPPCP)* consists of finding an Eulerian closed walk P of D whose starting vertex is v_0 and which respects all the paths of K, that is, for $i = 1, 2, \ldots, q$, if $a \prec_{Q_i} a'$ then $a \prec_P a'$ for all $a, a' \in A$. An instance of the ECWPPCP then is defined by the (ordered) triple (D, v_0, K).

Studying the ECWPPCP was originally motivated by the so-called *Single-vehicle Preemptive Pickup and Delivery Problem (SPPDP)* [2]. In this vehicle routing problem with a single vehicle having limited capacity, each demand may be temporarily unloaded elsewhere than its destination and picked up later.
Given a closed walk P corresponding to the vehicle route and the set K of the demand paths, Kerivin et al. [2] showed that (P, K) corresponds to a feasible solution to the SPPDP if the capacity constraints are satisfied and P respects the paths of K. To avoid carrying too much information (and then variables when formulating the SPPDP as a mixed-integer linear program), a natural question is whether or not one can get rid of the sequence of arcs of the ve-

hicle route, that is, can we represent a solution by an ordered pair (D, K), where D corresponds to the digraph induced by the set of arcs traversed by the vehicle, and determine in polynomial time if (D, K) is a feasible solution? Since, given (D, K), it is easy to check if the capacity constraints are satisfied, and since the digraph induced by the set of arcs of a closed walk is Eulerian, the problem of determining whether or not (D, K) corresponds to a feasible solution to the SPPDP is nothing but determining if the instance (D, v_0, K) of the ECWPPCP, where v_0 corresponds to the depot of the vehicle, admits a feasible solution.

To the best of our knowledge, the ECWPPCP has not been considered yet. However, a close-related problem, called the *Eulerian Superpath Problem (ESP)* has been considered by Pevzner et al. [4]. This problem has the same input as the ECWPPCP, except that each path in K is specified by a sequence of adjacent vertices, instead of a set of arcs. The ESP consists of determining an Eulerian closed walk starting at v_0 and having all the paths specified in K as subpaths (whereas in the ECWPPCP, the Eulerian closed walk may not contain paths of K as subpaths ; it must just respect these paths). Pevzner et al. [4] proved that the ESP is NP-complete by reducing the DNA fragment assembly problem, known for being NP-hard [1], to it. They also pointed out that the ESP can be solved in polynomial time whenever the digraph D is simple. Note that despite looking alike, the ESP and ECWPPCP are quite different; for instance, as we will see in Section 2, the ECWPPCP remains NP-complete even when D is a simple digraph.

In the next section, we prove the NP-completeness of the ECWPPCP for the general case. Section 3 is dedicated to the polynomial-time solvable case of ECWPPCP where the demand paths are arc-disjoint.

2 NP-completeness of the ECWPPCP

In this section, we prove the NP-completeness of the ECWPPCP. To do so, we use a polynomial reduction from the *Directed Hamiltonian Circuit of indegrees and outdegrees exactly two Problem (2DHCP)*, which can be stated as follows. Let $D_H = (V_H, A_H)$ be a digraph having all its vertices of indegree and outdegree two, that is, $\deg^{\text{in}}_{G_H}(v) = \deg^{\text{out}}_{G_H}(v) = 2$ for all $v \in V_H = \{v_1, v_2, \ldots, v_n\}$ with $n \geq 2$. The 2DHCP consists of asserting whether or not D_H contains a Hamiltonian circuit, that is, a closed walk traversing all the vertices of V_H exactly once. The 2DHCP is known to be NP-complete [3]. We remark that the proof given by Plesnik [3] was devised for planar digraphs with indegrees and outdegrees at most two, yet by considering some additional arcs, we can easily extend this result to planar digraphs (with possible multiple arcs) with indegrees and outdegrees two.

The first step of our reduction is the construction of an instance (D, v_0, K) of the ECWPPCP from D_H. We construct D as follows. With vertex v_1 of V_H, we associate six vertices $v_1^1, v_1^2, v_1^3, v_1^4, w_1, w_2$, together with the following arc set A_1 composed of the ten following arcs $(v_1^1, v_1^3), (v_1^3, v_1^2), (v_1^2, v_1^4)$, $(v_1^1, w_1), (w_1, w_2), (w_2, v_1^2)$, $(v_1^4, w_1), (w_1, v_1^2)$, $(v_1^3, w_2), (w_2, v_1^3)$.
For any $i \in \{2, 3, \dots, n\}$, we associate, with v_i, four vertices $v_i^1, v_i^2, v_i^3, v_i^4$, together with the arc set A_i defined by the eight following arcs $(v_i^1, v_i^3), (v_i^3, v_i^2)$, (v_i^2, v_i^4), $(v_i^4, w_1), (w_1, v_i^2)$, $(v_i^3, w_2), (w_2, v_i^3)$. Let

$$V = \{v_i^j : i = 1, 2, \dots, n \text{ and } j = 1, 2, 3, 4\} \cup \{w_1, w_2\}.$$

We now consider the arcs of D_H in the following manner. With every arc (v_i, v_j) in A_H, we associate the arc (v_i^2, v_j^1). Let

$$A = \left(\bigcup_{i=1}^{n} A_i \right) \cup \{(v_i^2, v_j^1) : (v_i, v_j) \in A_H\}.$$

The digraph $D = (V, A)$ obtained by our construction is clearly Eulerian. To obtain an instance of the ECWPPCP, we set v_0 equal to v_1^2 and we define the path set K as follows. For all $i = 1, 2, \dots, n$, we consider the paths $P_i = ((v_i^4, w_1), (w_1, w_2), (w_2, v_i^3), (v_i^3, v_i^2))$ and $Q_i = ((v_i^4, w_1), (w_1, v_i^2))$. The path set K is then equal to

$$K = \{P_i : i = 1, 2, \dots, n\} \cup \{Q_i : i = 1, 2, \dots, n\} \cup \{\big((v_1^1, w_1), (w_1, w_2), (w_2, v_1^2)\big)\}.$$

The instance of the ECWPPCP obtained from D_H is then (D, v_1^2, K). We show in the next lemma the relation between D_H and (D, v_1^2, K).

Lemma 2.1 *D_H has a Hamiltonian circuit if and only if D has an Eulerian closed walk starting at v_1^2 and respecting the precedence path constraints specified by K.*

Proof. ($\Rightarrow$) Let C_H be a Hamiltonian circuit of D_H. Without loss of generality, we suppose that $C_H = ((v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1))$. We construct an Eulerian closed walk C of D starting from v_1^2 and respecting the paths of K in several steps. Let C_1 be the arc sequence obtained by substituting arc

(v_i, v_{i+1}) of C_H by the path $\big((v_i^2, v_i^4), (v_i^4, w_1), (w_1, v_i^2), (v_i^2, v_{i+1}^1), (v_{i+1}^1, v_{i+1}^2)\big)$ for any $i = 1, 2, \ldots, n-1$, and arc (v_n, v_1) by the path $((v_n^2, v_n^4), (v_n^4, w_1), (w_1, v_n^2), (v_n^2, v_1^1), (v_1^1, w_1), (w_1, w_2))$. C_1 clearly corresponds to a walk starting at v_1^2, since C_H is a closed walk of D_H. Note that neither the arcs (w_2, v_i^3), (v_i^3, w_2), for $i = 1, 2, \ldots, n$, nor arc (w_2, v_1^2) appears in C_1. Since for $i = 1, 2, \ldots, n$, the paths Q_i and $((v_i^1, w_1), (w_1, w_2))$ are subpaths of C_1, the latter straightforwardly respects K.
Consider now the digraph $D^* = (V^*, A^*)$ induced by the remaining arcs of D except arc (w_2, v_1^2), that is, by $A \setminus \{C_1 \cup \{(w_2, v_1^2)\}\}$. We have $V^* = V \setminus (\{v_i^4 : i = 1, 2, \ldots, n\} \cup \{w_1\})$ and $\deg^{\text{in}}_{D^*}(v) - \deg^{\text{out}}_{D^*}(v) = 0$ for all $v \in V^*$. Moreover, D^* is weakly connected, that is its underlying graph is connected, since it contains the arcs (w_2, v_i^3), (v_i^1, v_i^3) and (v_i^3, v_i^2) for $i = 1, 2, \ldots, n$, which implies that D^* is Eulerian. Furthermore, the digraph $D^* - w_2 = D^*[V^* \setminus \{w_2\}]$ may not be weakly connected, yet each vertex in $V^* \setminus \{w_2\}$ clearly has equal indegree and outdegree. Let $D_1^*, D_2^*, \ldots, D_p^*$, $p \geq 1$, be the strongly connected components of $D^* - w_2$ which obviously are Eulerian. Since $D^* - w_2$ is not weakly connected only if $D_H - C_H$ is not weakly connected either, then for any $k = 1, 2, \ldots, p$, there must exist $i_k \in \{1, 2, \ldots, n\}$ so that $v_{i_k}^3$ is a vertex of D_k^*. Consider any strongly connected component D_k^*, $k = 1, 2, \ldots, p$. Any Eulerian closed walk B_k^* of D_k^* starting at $v_{i_k}^3$ can be transformed into a closed walk B_k of D^* starting at w_2 in the following manner. The first arc of B_k is $(w_2, v_{i_k}^3)$. All the arcs of B_k^* are then added to B_k sequentially. If the head of the added arc is a vertex v_i^3, for some $i \in \{1, 2, \ldots, n\}$, then the path $((v_i^3, w_2)(w_2, v_i^3))$ is added to B_k before moving to the next arc of B_k^*. Once we have dealt with all the arcs of B_k^*, we complete B_k by adding $(v_{i_k}^3, w_2)$. Since any B_k, for $k = 1, 2, \ldots, p$, starts at vertex w_2, the concatenation $(B_1, B_2, \ldots, B_p)$ of these Eulerian closed walks clearly forms an Eulerian closed walk C_2 of D^*.
Let C be the concatenation of C_1, C_2 and arc (w_2, v_1^2). Note that C is composed of all the arcs in A. Moreover, since C_1 starts at v_1^2 and ends at w_2 which is the starting vertex of closed walk C_2, C is an Eulerian closed walk of D starting at v_1^2. For any $i = 1, 2, \ldots, n$, the paths $((v_i^4, w_1), (w_1, w_2))$ and $((w_2, v_i^3), (v_i^3, w_2))$ are subpaths of C_1 and C_2, respectively. Recalling that all the paths Q_i, for $i = 1, 2, \ldots, n$, are subpaths of C_1, we can conclude that C respects all the paths of K.

($\Leftarrow$) Let C be an Eulerian closed walk of D starting at v_1^2 and respecting all the paths of K. Due to the definition of K, arc (w_1, w_2) appears in C before all the arcs of D leaving w_2, that is, $(w_1, w_2) \prec_C a$ for all $a \in \delta^{\text{out}}(w_2)$. Moreover, since we have $\deg^{\text{out}}(v) = \deg^{\text{in}}(v)$ for all $v \in V$, we deduce that arc (w_1, w_2)

appears in C before any other entering arc of w_2, that is, $(w_1, w_2) \prec_C a$ for all $a \in \delta^{\text{in}}(w_2) \setminus \{(w_1, w_2)\}$. Let $\overline{C}$ be the walk obtained from C by only considering the arcs of D preceding (w_1, w_2) in C. By considering all the paths P_i of K, we deduce that $(w_1, w_2) \prec_C (v_i^3, v_i^2)$ for all $i = 1, 2, \ldots, n$ which implies, with the previous results, that $\overline{C}$ does not contain any vertex in $\{v_i^3 : i = 1, 2, \ldots, n\}$. Moreover, due to the vertex degree conditions, $\overline{C}$ contains arc (v_i^4, w_1) for all $i = 1, 2, \ldots, n$. Since w_1 is not the starting vertex of $\overline{C}$, by considering Q_i, we deduce that (v_1^1, w_1) is the last arc of $\overline{C}$ and every path Q_i, $i = 1, 2, \ldots, n$ is a subpath of $\overline{C}$. As every vertex v_i^4 has only one entering arc, namely (v_i^2, v_i^4), removing closed walks $((v_i^2, v_i^4), (v_i^4, w_1), (w_1, v_i^2))$ for all $i = 1, 2, \ldots, n$, and arc (v_1^1, w_1) leads to a walk $\tilde{C}$ starting at v_1^2, ending at v_1^1, and containing all the vertices in $\{v_i^2 : i = 1, 2, \ldots, n\}$. Since $\tilde{C}$ does not contain any vertex in $\{v_i^3 : i = 1, 2, \ldots, n\}$, we clearly have $\delta_D^{\text{in}}(v_i^2) \cap \tilde{C} = \{(v_i^1, v_i^2)\}$ for all $i = 2, 3, \ldots, n$. Therefore, $\tilde{C}$ contains all the vertices in $\{v_i^1, v_i^2 : i = 1, 2, \ldots, n\}$ exactly once. Consequently, $\tilde{C}$ is an alternate sequence of arcs of $\{(v_i^2, v_j^2) : (v_i, v_j) \in A_H\}$ and of $\{(v_i^1, v_i^2) : i = 2, 3, \ldots, n\}$ so that every arc from both sets appears once. Therefore, contracting the vertices v_i^1 and v_i^2 into vertex v_i, for $i = 1, 2, \ldots, n$, transforms $\tilde{C}$ into a Hamiltonian circuit of D_H. □

Theorem 2.2 *The Eulerian closed walk with precedence path constraints problem is NP-complete.*

Proof. Clearly the problem is in NP. Moreover, the construction from an instance of the NP-complete 2DHCP into an instance of the ECWPPCP can be performed in polynomial time. Therefore, the NP-completeness of the Eulerian closed walk with precedence path constraints problem directly follows from Lemma 2.1. □

We remark that, on the contrary of the ESP, the ECWPPCP remains NP-hard if the digraph D is simple. To prove this, one has just to modify the construction of (D, v_0, K) from D_H by sequentially replacing, in D and every path of K, each multiple arc (u, v) by the two arcs (u, w) and (w, v) where w is a new vertex with indegree and outdegree one.

3 A polynomial-time solvable case

Throughout this section, we consider an instance (D, v_0, K) of the ECWP-PCP where K is composed of arc-disjoint paths. We prove that, in this case, the ECWPPCP can be solved in polynomial time. From now on, we say that arc $a \in A$ is a *predecessor* of an arc $a' \in A \setminus \{a\}$ if there exists a path Q

of K with $a \prec_Q a'$. (Note that, since K is composed of arc-disjoint paths, given two distinct arcs, at most one is the predecessor of the other.) We now define particular subdigraphs of D. Let $D' = (V', A')$ be an Eulerian digraph induced by an arc subset $A' \subseteq A$. A vertex $v \in V'$ is said D'*-impregnable* if either it is different from v_0 and incident with no arc in A', or all its leaving arcs in A' have a predecessor in $\delta^{\text{in}}_{D'}(v)$. The subdigraph D' is then called *impregnable* (with respect to (D, v_0, K)) if all its vertices are D'-impregnable. By definition, the impregnable Eulerian subdigraphs are composed of arcs that cannot appear in a closed walk of D starting from v_0 and respecting the paths of K. Therefore, if (D, v_0, K) contains an impregnable Eulerian subdigraph, then the ECWPPCP associated with instance (D, v_0, K) has not a feasible solution.

Given an instance (D, v_0, K) of the ECWPPCP so that v_0 is not D-impregnable, one can easily construct a closed walk C of D starting from v_0 and respecting K as follows. (Remark that C may not be Eulerian.) Since v_0 is not D-impregnable, there exists an arc leaving v_0, say (v_0, v_1), with no predecessor. Therefore, the walk $C = ((v_0, v_1))$ respects K. Moreover, as long as the ending vertex of C, say v, is different from v_0, we can extend C by pushing back to C a new arc leaving v. Indeed, since D is Eulerian, we have $|\delta^{\text{out}}(v) \setminus C| = |\delta^{\text{in}}(v) \setminus C| + 1$. Since K is composed of arc-disjoint paths, each arc of $\delta^{\text{in}}(v) \setminus C$ is the predecessor of at most one of $\delta^{\text{out}}(v) \setminus C$, which implies that there exists an arc (v, w) of $A \setminus C$ with no predecessor in $\delta^{\text{in}}(v) \setminus C$. therefore, the walk $(C, (v, w))$ respects K. By iteratively pushing back arcs to C until reaching v_0, we obtain a closed walk C starting at v_0 and respecting K.

By definition, if (D, v_0, K) contains an impregnable Eulerian subdigraph, then it admits no feasible solution. Suppose now that (D, v_0, K) does not contain any impregnable Eulerian subdigraph. Using the previous routine, we now construct a feasible solution to the ECWPPCP as follows. We first compute a closed walk C of (D, v_0, K). If C is Eulerian, then it corresponds to a feasible solution to the ECWPPCP. Otherwise, let $\overline{D} = (\overline{V}, \overline{A})$ be the digraph induced by the arc set $A \setminus C$. It is clear that each connected component of $\overline{D}$ corresponds to an Eulerian subdigraph of D. Let $D' = (V', A')$ be any of these connected components. By hypothesis, D' is not impregnable. Therefore, let $v_0' \in V'$ be the last vertex appearing in C that is not D'-impregnable and K' the restriction of K on D'. (D', v_0', K') corresponds to an instance of the ECWPPCP. Using the previous routine, we can construct a closed walk C' of D' starting at v_0' and respecting K'. Inserting C' into C after the last time v_0' appears in C leads to a new closed walk C'' starting at v_0. Suppose now that C'' does not respect K. Since C and C' respect K, this implies that there exists an arc a' in C' that has a predecessor in C, say a, which appears after a' in C''. We suppose, without loss of generality, that $a = (u, v)$ and $a' = (v, w)$ are incident. Vertex v is not D'-impregnable because it is incident to at least one arc of C and a' has only one predecessor in $\delta^{\text{in}}(v)$, namely a, which does not belong to D'. Since v appears after v'_0 in C, this contradicts the fact that v'_0 is the last vertex of C that is not D'-impregnable. Thus, C'' respects K. By iteratively inserting closed walks into the current one, we finally obtain a solution to the ECWPPCP. (Otherwise, the digraph induced by the remaining arcs contains at least one impregnable Eulerian subdigraph.) These results lead to the following theorem.

Theorem 3.1 *The ECWPPCP can be solved in polynomial time if the paths are arc-disjoint. Moreover, (D, v_0, K) admits a feasible solution if and only if it does not contain any impregnable Eulerian subdigraph.* □

4 Conclusion

In this paper, we introduced a new variant of the Eulerian closed walk problem where some precedence constraints are specified by a set of paths K. We first proved that this problem is NP-complete. We also presented a polynomial-time algorithm to solve the problem if K is composed of arc-disjoint paths.

References

[1] Kececioglu, J. D. and Meyers, E. W., "Combinatorial algorithms for DNA sequence assembly", Algorithmica **13** (1995) 7–51.

[2] Kerivin, H. L. M., Lacroix, M. and Mahjoub, R., "Models for the single-vehicule preemptive pickup and delivery problem", preprint (2008).

[3] Plesnik, J., "The NP-Completeness of the Hamiltonian Cycle Problem in Planar Digraphs with Degree Bound Two", Information Processing Letters **8(4)** (1979) 199–201.

[4] Pevzner, P. A., Tang, H. and Waterman, M. S., "An Eulerian path approach to DNA fragment assembly", Proceedings of the 5th Annual International Conference on Computational Molecular Biology (RECOMB-01), Montreal, Canada (2001) 256–267.